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Power Series Nonextendable Across the Boundary of their Convergence Domain

Aleksandr D. Mkrtchyan*
Faculty of Mathematics and Mechanics,
Yerevan State University,
Alex Manoogian, 1, Yerevan, 375010
Armenia
Institute of Mathematics and Computer Science,
Siberian Federal University,
Svobodny, 79, Krasnoyarsk, 660041
Russia

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In the article we construct a new power series in a single variable nonextendable through the boundary circle of the convergence disk. This series refines the known Fredholm's example.

Using this series we construct a double power series that does not admit an analytic continuation across the boundary of its convergence domain.

Keywords: power series, analitic continuation, infinitely differentiate, Dirichlet series.

The problem of describing the relations between singularities power series in one variable and their coefficients attracted mathematicians' attention already at the end of 19th century. Remarkable results were obtained in the first half of the 20th century which allowed thinking that the development in this direction was almost completed. Many obtained results touch upon the question about series non extendable analytically across the boundary of their convergence domain and these results are connected with the names of famous Hungarian mathematicians Sego and Polya (see, for example, articles [1, 2] and also the list of their articles in the book of L.Bieberbach [3]). Examples of series that are non extendable analytically across the boundary of their convergence domain we can find in the text-books about theory functions of complex variables. These examples deal with the so-called "strong lacunar" series, in other words, having "many" monomials with zero coefficients. Such series, for instance, are

$$\sum_{n=0}^{\infty} z^{n!}, \qquad \sum_{n=0}^{\infty} z^{2^n}, \qquad \sum_{n=0}^{\infty} z^{n^n}.$$

In 1891, Fredholm [4] gave examples of "moderate lacunar" non extendable series, moreover, these series represented infinitely differentiable function in the closure of the convergence disk. These series depend on a parameter a, and they have the following form

$$\sum_{n=0}^{\infty} a^n z^{n^2}, \qquad 0 < a < 1.$$

Here n^2 has a power order 2 respective to the summation index n, therefore we say that Fredholm's series have the lacunarity order 2.

*alex0708@bk.ru

A more general result on a non extendable series in terms of lacunarity belongs to Fabry (see [3] or [5]). It claims that, if the sequence of natural numbers m_n increases faster than n (i.e. $n = o(m_n)$), then there is series

$$\sum_{n=0}^{\infty} a_n z^{m_n},$$

converging in the unit disk and not extending across his boundary.

The purpose of this work is to construct a lacunar scale of power series oinone variable that are not extendable across the convergence boundary and represent infinitely differentiable functions in the closed disk. Furthermore, they should include Fredholm‘s series. Besides, we construct examples of double power series that converge in bidisk and do not extend across its boundary.

One of the main results is given by Theorem 1.1 . It demnstrates that Fredholm‘s example may be refined to the power order of lacunarity from 2 to 1+ε. The precise formulation is the following: if the increasing sequence of natural numbers n_k satisfies the inequality $n_k \geqslant \text{const} * k^{1+\varepsilon}$ with $\varepsilon > 0$, then the power series

$$\sum_{k=0}^{\infty} a^k z^{n_k}, \qquad 0 < |a| < 1$$

is not extending across the unit circle boundary and represents infinitely differentiable function in the closed disk.

Also, we give examples of double power series that are not extendable outside of the unit bidisk $U^2 = \{(z_1, z_2) : |z_1|<1, |z_2|<1\}$ and represent infinitely differentiable function in $\bar{U}^2 \setminus T^2$ where $T^2 = \{(z_1, z_2) : |z_1| = 1, |z_2| = 1\}$.

1. Generalization of the Fredholm result

Theorem 1.1. *If the increasing sequence of natural numbers n_k satisfies the inequality $n_k \geqslant$ const $* k^{1+\varepsilon}$ with $\varepsilon > 0$, then the power series*

$$\sum_{k=0}^{\infty} a^k z^{n_k}, \qquad 0 < |a| < 1 \tag{1}$$

are not extendable across the boundary circle and represents infinitely differentiable function in the closed disk.

Proof. Consider the following series

$$\varphi(t, u) = \sum_{k=0}^{\infty} e^{n_k t + ku}, \quad \text{where} \quad t, u \in \mathbb{C}. \tag{2}$$

Its terms exponentially decrease in the product $\Pi \times \bar{\Pi}$ of subspaces $\Pi = \{u : \text{Re}\, u < 0\}$ and $\bar{\Pi} = \{t : \text{Re}\, t \leqslant 0\}$. These series converge uniformaly on compact subsets of $\Pi \times \bar{\Pi}$, and therefore $\varphi(t, u)$ is holomorphic in the product $\Pi \times \Pi$ of open subspaces. This property is preserved for all derivatives with respect to the variable t of these series. Consequently the function $\varphi(t, u)$ is infinitely differentiable in $\bar{\Pi}$ for each fixed $u \in \Pi$.

Introduce the following notation

$$F(-t) = \sum_{k=0}^{\infty} e^{ku_0} e^{-n_k(-t)} = \sum_{k=0}^{\infty} e^{n_k t + ku_0} = \varphi(t, u_0) \tag{3}$$

for $t \in \bar{\Pi}$ and for each fixed $u_0 \in \Pi$. Here, the function $F(-t)$ is represented by a Dirichlet series

$$\sum_{k=0}^{\infty} a_k e^{-\lambda_k t}$$

with exponential indexes $\lambda_k = n_k$ and coefficiens $a_k = e^{ku_0}$.

Let us compute the value

$$L = \overline{\lim_{k\to\infty}} \frac{\ln k}{\lambda_k}.$$

We obtain

$$L = \overline{\lim_{k\to\infty}} \frac{\ln k}{\lambda_k} = \overline{\lim_{k\to\infty}} \frac{\ln k}{n_k} \leqslant \overline{\lim_{k\to\infty}} \frac{\ln k}{k^{1+\varepsilon}} = 0.$$

Therefore, the abscissa of convergence for the series (3) we can be found as follows

$$r = \overline{\lim_{k\to\infty}} \frac{\ln |e^{ku_0}|}{n_k} = \overline{\lim_{k\to\infty}} \frac{\ln e^{kRe\ u_0}}{n_k} = \overline{\lim_{k\to\infty}} \frac{k \operatorname{Re} u_0}{n_k} \leqslant \overline{\lim_{k\to\infty}} \frac{k \operatorname{Re} u_0}{k^{1+\varepsilon}} = 0.$$

Now we demonstrate that for the function $F(-t)$ is perform the conditions of Polya‘s theorem [6]:

If $0 < n_k \uparrow \infty, \quad \lim\limits_{k\to\infty} \dfrac{k}{n_k} = 0, \qquad n_{k+1} - n_k \geqslant h > 0$ *and series*

$$F(z) = \sum_{k=1}^{\infty} a_k e^{-n_k z}$$

has a finite abscissa of convergence $-\infty < r < +\infty$, then the line of convergence $\operatorname{Re} z = r$ is the natural boundary for the function $F(z)$.

Indeed,

$$0 < n_k \uparrow \infty, \quad \lim_{k\to\infty} \frac{k}{n_k} \sim \lim_{k\to\infty} \frac{k}{k^{1+\varepsilon}} \to 0$$

and

$$n_{k+1} - n_k \sim (k+1)^{1+\varepsilon} - k^{1+\varepsilon} = k^{1+\varepsilon}\left(\left(1+\frac{1}{k}\right)^{1+\varepsilon} - 1\right) = k^{1+\varepsilon}\left((1+\varepsilon)\frac{1}{k} + o(\frac{1}{k})\right) \xrightarrow[k\to\infty]{} \infty$$

Consequently, the function $F(-t)$ is not analitically extendable. Then, denoting $a = e^u$ (fixed) and $z = e^t$, from (2) we get (1), as desired. □

Theorem 1.2. *For an arbitrary pair of natural numbers $p > q$, the series*

$$f(z) = \sum_{\nu=0}^{\infty} a^{\nu^q} z^{\nu^p}, \quad 0 < a < 1, \tag{4}$$

is not expendable across of the unit disk boundary $|z| < 1$ and represents an infinitely differentiable function in the closed disk.

Proof. We can prove this theorem directly, without referring to the Polya‘s theorem.

We consider the following series

$$\varphi(t,u) = \sum_{\nu=0}^{\infty} e^{\nu^p t + \nu^q u}, \quad \text{where} \quad t, u \in \mathbb{C}. \tag{5}$$

Its terms are exponentially decreasing in the product $\Pi \times \bar{\Pi}$ of subspaces $\Pi = \{u : \text{Re } u < 0\}$ and $\bar{\Pi} = \{t : \text{Re } t \leqslant 0\}$. The series converges uniformly on the compact subsets of $\Pi \times \bar{\Pi}$, therefore $\varphi(t,u)$ is holomorphic in the product of open subspaces $\Pi \times \Pi$. Besides, the function φ holomorphic in $u \in \Pi$ for any fixed $t_0 \in \bar{\Pi}$.
We consider the Taylor expansion of φ

$$\varphi(t,u) = \sum_{k=0}^{\infty} \frac{\partial^k \varphi}{\partial u^k}(t,u_0)\frac{(u-u_0)^k}{k!}, \tag{6}$$

with the centre $u_0 \in \Pi$, regarding $t \in \bar{\Pi}$ as a parameter. In view of (5), we have

$$\frac{\partial^k \varphi}{\partial u^k}(t,u) = \sum_{\nu=0}^{\infty} (\nu^q)^k e^{\nu^p t + \nu^q u}.$$

Substituting this expression in (6), we obtain

$$\sum_{k=0}^{\infty} \left(\sum_{\nu=0}^{\infty} (\nu^q)^k e^{\nu^p t + \nu^q u_0} \right) \frac{(u-u_0)^k}{k!}. \tag{7}$$

We demonstrate that the series (7) has a finite convergence radius for any fixed t_0 from boundary $\partial\, \bar{\Pi}$ (i.e. Re $t_0 = 0$).

The series (5) diverges if Re $u \geqslant 0$ and Re $t_0 = 0$, because its general term

$$|e^{\nu^p t_0 + \nu^q u}| = |e^{\nu^p t_0}||e^{\nu^q u}| = (e^{Re\ u})^{\nu^q}$$

does not tend to 0. Besides, the series (5) can be considered as a power series in the variable $w = e^u$. Using these facts, we obtain that function $\varphi(t_0,u)$ has a singularity point $\hat{u}$ such that $Re\ \hat{u} = 0$. Hence, the series (7) has a finite convergence circle.

By using these facts and the Cauchy-Hadamar formula, we obtain that there is a sequence k_l with the following property

$$\left| \sum_{\nu=0}^{\infty} (\nu^q)^{k_l} e^{\nu^p t_0 + \nu^q u_0} \right| \sim \frac{k_l!}{\rho_l^k} \quad \text{with} \quad k_l \to \infty, \tag{8}$$

where ρ is the convergence radius of the series (5), which depends on the choice of points $u_0 \in \Pi$ and $t_0 \in \Pi$.

Assume that the function $\varphi(t,u_0)$ extends analytically with respect to t from $\bar{\Pi}$ across some boundary point $t_0 \in \partial\, \bar{\Pi}$ for some fixed $u_0 \in \Pi$. We denote by $\tilde{\varphi}(t,u_0)$ the analytic continuation of the function $\varphi(t,u_0)$. Its Taylor series is the following:

$$\tilde{\varphi}(t,u_0) = \sum_{k=0}^{\infty} \frac{\partial^k \tilde{\varphi}}{\partial t^k}(t_0,u_0)\frac{(t-t_0)^k}{k!} = \sum_{k=0}^{\infty} \frac{\partial^k \varphi}{\partial t^k}(t_0,u_0)\frac{(t-t_0)^k}{k!}. \tag{9}$$

Taking into account (5), we have

$$\frac{\partial^k \varphi}{\partial t^k}(t,u) = \sum_{\nu=0}^{\infty} (\nu^p)^k e^{\nu^p t + \nu^q u}.$$

Substituting this expression in (9), we obtain

$$\tilde{\varphi}(t,u_0)=\sum_{k=0}^{\infty}\left(\sum_{\nu=0}^{\infty}(\nu^p)^k e^{\nu^p t_0+\nu^q u_0}\right)\frac{(t-t_0)^k}{k!}=\sum_{k=0}^{\infty}\left(\sum_{\nu=0}^{\infty}(\nu^q)^{\frac{p}{q}k} e^{\nu^p t_0+\nu^q u_0}\right)\frac{(t-t_0)^k}{k!}. \quad (10)$$

We investigate the convergence radius of this series by the Cauchy–Hadamard theorem. In the sequence

$$\sqrt[k]{\frac{1}{k!}\left|\sum_{\nu=0}^{\infty}(\nu^q)^{\frac{p}{q}k} e^{\nu^p t_0+\nu^q u_0}\right|}$$

we consider the subsequence taking $k_q = qk_l$:

$$\sqrt[qk_l]{\frac{1}{(qk_l)!}\left|\sum_{\nu=0}^{\infty}(\nu^q)^{pk_l} e^{\nu^p t_0+\nu^q u_0}\right|}.$$

Using the estimate (8), we obtain

$$\sqrt[qk_l]{\frac{1}{(qk_l)!}\frac{(pk_l)!}{\rho^{pk_l}}}=\sqrt[qk_l]{\frac{(pk_l)!}{(qk_l)!}}\,\rho^{\frac{p}{q}}.$$

By Stirling's formula

$$\sqrt[qk_l]{\frac{(pk_l)^{pk_l-\frac{1}{2}}e^{-pk_l}}{(qk_l)^{qk_l-\frac{1}{2}}e^{-qk_l}}}\,\rho^{\frac{p}{q}}\sim\frac{k_l^{\frac{p}{q}}}{k_l}\xrightarrow[k_l\to\infty]{}\infty \quad \text{with} \quad p>q.$$

Thus, the series (9) has empty convergence domain.

It follows that the series (5) does not continue analitically with respect to t across the point $t_0 \in \partial\,\bar{\Pi}$.

Denoting $a = e^u$ (fixed) and $z = e^t$, from (5) we obtain (4). The theorem was proved. □

2. Double power series not extendable across the unit bidisk

Theorem 2.1. *If the support A of the double power series*

$$\sum_{(k_1,k_2)\in A} {z_1}^{k_1}{z_2}^{k_2} \quad (11)$$

is of type

$$A=\{(k_1,k_2)\in {Z_+}^2 : k_2\geqslant {k_1}^{1+\varepsilon}\}\cup\{(k_1,k_2)\in {Z_+}^2 : k_1\geqslant {k_2}^{1+\varepsilon}\} \quad \textit{with} \quad \varepsilon>0,$$

then the double series (11) is not extendable across of the boundary of bidisk

$$U^2=\{(z_1,z_2):\ |z_1|<1,\ |z_2|<1\}$$

and represents an infinitely differentiable function in $\bar{U}^2\setminus T^2$ where

$$T^2=\{(z_1,z_2):|z_1|=1,|z_2|=1\}.$$

Proof. We can present the power series (11) by the sum of two series:

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} {z_1}^{k_1}{z_2}^{k_2+[{k_1}^{1+\varepsilon}]} + \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} {z_1}^{k_1+[{k_2}^{1+\varepsilon}]}{z_2}^{k_2} =$$

$$= \sum_{k_2=0}^{\infty} {z_2}^{k_2} \sum_{k_1=0}^{\infty} {z_1}^{k_1}{z_2}^{[{k_1}^{1+\varepsilon}]} + \sum_{k_1=0}^{\infty} {z_1}^{k_1} \sum_{k_2=0}^{\infty} {z_2}^{k_2}{z_1}^{[{k_2}^{1+\varepsilon}]} =$$

$$= \frac{1}{1-z_2}\sum_{k_1=0}^{\infty} {z_1}^{k_1}{z_2}^{n_{k_1}} + \frac{1}{1-z_1}\sum_{k_2=0}^{\infty} {z_2}^{k_2}{z_1}^{n_{k_2}}.$$

Here $[{k_j}^{1+\varepsilon}]$ means the integer part of the number ${k_j}^{1+\varepsilon}$. According to the Theorem 1.1 the series

$$\sum_{k_1=0}^{\infty} {z_1}^{k_1}{z_2}^{n_{k_1}}, \tag{12}$$

considered respective the variable z_2, converges in the unit disk and is not extending across the boundary circle, when $0 < |z_1| < 1$.

Using the change of variables $e^u = z_1$ and $e^t = z_2$, we rewrite (12) as an exponential series $\sum\limits_{k_1=0}^{\infty} e^{k_1 u}e^{n_{k_1} t}$ that represents an infinitely differentiable function in $\{(u,t) : \mathrm{Re}\, u \leqslant 0,\ \mathrm{Re}\, t \leqslant 0\} \setminus \{(u,t) : \mathrm{Re}\, u = 0,\ \mathrm{Re}\, t = 0\}$. Consequently, the series (13) represent an infinitely differentiable function in $\bar{U}^2 \setminus T^2$.

Similar properties one gets for the series

$$\sum_{k_2=0}^{\infty} {z_2}^{k_2}{z_1}^{n_{k_2}}$$

converges in the unit disk, does not continue with respect to the variable z_1 with $0 < |z_2| < 1$ and represent infinitely differentiable function in the $\bar{U}^2 \setminus T^2$. Therefore, we obtain the require statement for series (11). □

Proposition 1. *Let K be the sector with integer generate vectors $m_1 = (m_{11}, m_{12})$ and $m_2 = (m_{21}, m_{22})$. Then the series*

$$f(z) = \sum_{k \in N^2 \cap K} {z_1}^{k_1}{z_2}^{k_2} \tag{13}$$

represents the rational function

$$f(z) = \frac{P(z)}{(1-{z_1}^{m_{11}}{z_2}^{m_{12}})(1-{z_1}^{m_{21}}{z_2}^{m_{22}})} \quad \text{with} \quad P(z) = 1 + \sum_{\alpha \in (N^2 \cap\, \mathrm{int}\, D)} z^{\alpha} \tag{14}$$

where intD is the interior of parallelogram D with verteces $(0,0)$, m_1, m_2 and $m_1 + m_2$.

Proof. We can cover the all integer points of K by semigroup $L = \{l_1 m_1 + l_2 m_2,\ l_i \in Z_{\geqslant}\ i = 1,2\}$ and its shifts $L_j = a_j + L$ where a_j runs over $N^2 \cap \mathrm{int} D$,. Thus, we have

$$\sum_{k \in N^2 \cap K} {z_1}^{k_1}{z_2}^{k_2} = \sum_{k \in L} {z_1}^{k_1}{z_2}^{k_2} + \sum_{k \in L_1} {z_1}^{k_1}{z_2}^{k_2} + \ldots + \sum_{k \in L_p} {z_1}^{k_1}{z_2}^{k_2}.$$

where p is the cardinality of $N^2 \cap \mathrm{int}\, D$

$$\sum_{k \in L} {z_1}^{k_1}{z_2}^{k_2} = \sum_{l_1,l_2 \geqslant 0} z^{l_1 m_1 + l_2 m_2} = \sum_{l_1,l_2 \geq 0} (z^{m_1})^{l_1}(z^{m_2})^{l_2} = \tag{15}$$

$$= \frac{1}{(1-z^{m_1})(1-z^{m_1})} = \frac{1}{(1-{z_1}^{m_{11}}{z_2}^{m_{12}})(1-{z_1}^{m_{21}}{z_2}^{m_{22}})},$$

and

$$\sum_{k\in L_j} {z_1}^{k_1}{z_2}^{k_2} = {z_1}^{a_{j1}}{z_2}^{a_{j2}} \sum_{k\in L} {z_1}^{k_1}{z_2}^{k_2}, \quad a_j = (a_{j1}, a_{j2}).$$

Therefore we obtain

$$\sum_{k\in N^2\cap K} {z_1}^{k_1}{z_2}^{k_2} = \frac{1+{z_1}^{a_{11}}{z_2}^{a_{12}}+...+{z_1}^{a_{p1}}{z_2}^{a_{p2}}}{(1-{z_1}^{m_{11}}{z_2}^{m_{12}})(1-{z_1}^{m_{21}}{z_2}^{m_{22}})}$$

as desired. □

In conclusion, we are interested the following question. Consider the series (13) for general (not necessary integer) generators m_1 and m_2. Is it true, that this series either is not extandable across of the boundary of converge domain, or represents a rational function of type

$$f(z) = \frac{P(z)}{(1-{z_1}^{m_{11}}{z_2}^{m_{12}})(1-{z_1}^{m_{21}}{z_2}^{m_{22}})}, \tag{16}$$

where $P(z)$ is polynomial?

We can interpret this fact as a two-dimentional analogue of Segö's theorem [1] (see also [3]) on series with the finite number of different Taylor coefficients. In the multivariate case, the interest of studing series of form (13) arises in the thermodynamics of several hamiltonians [7].

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О степенных рядах, непродолжимых через границу области сходимости

Александр Д. Мкртчян

Мы построили пример степенных рядов, которые не продолжаются через границу своей области сходимости и представляют бесконечно дифференцируемую функцию в замыкании круга. Он усиливает известный пример Фредгольма. Используя полученный результат, мы строим двумерные степенные ряды, которые не продолжаются через единичный бикруг.

Ключевые слова: степенные ряды, аналитическое продолжение, бесконечная дифференцируемость, ряды Дирихле.