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On Sharply Doubly-Transitive Groups

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The question of existence in any sharply doubly-transitive group of a regular abelian normal subgroup is investigated.

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The permutation group G acting on a set Ω ($|\Omega| \geqslant 3$) is called sharply doubly-transitive if for any two ordered pairs of elements from the set Ω there exists a unique element from G, mapping the first pair to the second one. In particular, the only permutation in G stabilizing two points is the identity.

C.Jordan [1] (according to [2]) proved that in a finite sharply doubly-transitive group all the regular permutations together with the identity one form an abelian normal subgroup. It is still not known if it is true in the infinite case. Sharply doubly-transitive groups are closely connected with near-fields and near-domains. Near-fields were introduced by L.E.Dickson in 1905 [3] as algebraic systems $(F, +, \cdot)$ with two binary operations, where

1. $(F, +)$ is an abelian group with the identity element 0,
2. $(F^*, \cdot)$ is a group with the identity element 1 ($F^* = F \setminus \{0\}$),
3. Multiplication distributes over addition on the left, viz. $a \cdot (b + c) = a \cdot b + a \cdot c$.

R.D.Carmichael [4] established that the groups $T_2(F)$ of affine transformations $x \to a + bx$ ($b \neq 0$) of a near-field F are sharply doubly-transitive. H.Zassenhauz [5] classified finite sharply doubly-transitive groups and the corresponding near-fields. M.Holl [6] revealed a relation between these groups and projective planes, and under additional restrictions he generalized the Jornan's theorem on the infinite case (theorem 20.7.1 [7]).

J.Tits [8], Grätzer [9] and H.Karzel [10] introduced a more general, than the near-fields, algebraic systems $(F, +, \cdot)$ later called near-domains, in which $(F, +)$ is a loop, $(F^*, \cdot)$ is a group, and for any elements $a, b, c \in F$ the following conditions hold:

1. $a + b = 0 \to b + a = 0$;
2. $0 \cdot a = 0 \to a \cdot 0 = 0$;

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3. $a \cdot (b + c) = a \cdot b + a \cdot c$;

4. There exists a unique element $d_{a,b} \in F^*$ such that $a + (b + x) = (a + b) + d_{a,b} \cdot x$ for each $x \in F$.

Thus, for every sharply doubly-transitive group T there exists a corresponding near-domain F, for which T is the group of affine transformations $x \to a + bx$ $(b \neq 0)$ [2], and conversly, the group $T_2(F)$ of the affine transformations of a near-domain F is sharply doubly-transitive on F. The existence of a near-domain which is not a near-field is the central problem of the theory of these algebraic systems. In particular, the foresaid problem of the existance of a regular abelian normal subgroup of a sharply doubly-transitive group is equivalent to the problem of coincidence between near-domains and near-fields.

V.D.Mazurov [11] proved that in every sharply doubly-transitive group G of the zero characteristic the stabilizer of any point contains a subgroup, which is isomorphic to the multiplicative group of the field of rational numbers, and in the group G there is a subgroup, which is isomorphic to the affine group of the stabilizer. For sharply doubly-transitive groups of an odd characteristic p the following theorem is proved:

Theorem 1. *Let T be a group of affine transformations of a near-domain F of an odd characteristic p, j be an involution from the stabilizer T_α of a point $\alpha \in F$, b be an element from $T \setminus T_\alpha$, strongly real with respect to j; let $A = C_T(b)$ and $V = N_T(A)$. Then*

1. *The subgroup A is inverted by j and it is also strongly isolated in T;*

2. *The subgroup V acts sharply doubly-transitive on the orbit $\Delta = \alpha^A$; A is an Abelian regular normal subgroup in V, $H = V \cap T_\alpha$ is the stabilizer of a point and $V = A \rtimes H$.*

In the 14th edition of the Kourovka Notebook V.D.Mazurov stated a question 12.48(a) [12] about the existence of a regular Abelian normal subgroup in a sharply doubly-transitive group with a locally finite stabilizer of a point. In a special case the following theorem gives an affirmative answer to this question:

Theorem 2. *Let T be a sharply doubly-transitive subgroup, let a near-domain F be its locally-finite subgroup, containing a regular permutation and let its intersection with a stabilizer of some point be normal in this stabilizer and contain more than two elements. Then the group T has a regular normal abelian subgroup, and a near-domain $F(+, \cdot)$ is a near-field of a non-zero characteristic.*

Results were announced in [13].

Proof of the theorem 1. Due to the conditions of the theorem, the group T is sharply doubly-transitive on F (ex. V.1.2 [2]) and the stabilizer T_α of a point $\alpha \in F$ contains an involution j (ex. V.1.4 [2], a lemma 3.4 [14]). According to ex. V.1.3 [2] and a lemma 3.4 [14], the involution j is unique in T_α, in particular $T_\alpha = C_T(j)$. It means that the group T acts (by conjugation) sharply doubly-transitively on the set J of its involutions, and T_α acts transitively on the set $J \setminus \{j\}$ (a lemma 3.4 from [14]). Hence all the elements vk $(v, k \in J,\ v \neq k)$ are conjugate in T, in particular, all non-identity elements $vk \in J^2$ have the same order which is either infinite or equal to a prime number $p \neq 2$. In the former case the characteristic of the near-domain F and the group T is equal to 0, in the latter case we have $\mathrm{Char}\, F = \mathrm{Char}\, T = p$. Due to the conditions of the theorem only the latter case $\mathrm{Char}\, F = \mathrm{Char}\, T = p$ takes place.

Let $N_j = jJ$ and $N_j^* = N_j \setminus \{1\}$. Since T_α acts transitively on the set $J \setminus \{j\}$, all the elements of the form jk ($k \in J$, $k \neq j$) are conjugate by suitable elements from T_α and T_α acts transitively on the set N_j^*. Due to the properties of the dihedral groups (lemma 2.10 [14]) the set N_j^* coincides with the set of all the elements from $T \setminus T_\alpha$ strongly real with respect to j. Besides, if $c \in N_j^*$ then any non-identity element of a cyclic subgroup $\langle c\rangle$ is contained in N_j^*.

Let $b \in T \setminus T_\alpha$ and $b^j = b^{-1}$. Then from the above reasoning it follows that $b \in N_j^*$ and $j \notin A = C_T(b)$. Clearly $j \in V = N_T(A)$, and for (T, T_α) is Frobenius pair (lemma 3.4 [14]) we have $A \cap T_\alpha = 1$ and $C_A(j) = 1$. Recall that the involution j of the group T is called *finite* if for any $t \in T$ the subgroup $\langle j, j^t\rangle$ is finite, that is equivalent to the finiteness of orders of the elements jj^t. Due to the foresaid, the involution j is finite in the group T and hence it is finite in the subgroup $K = A \rtimes \langle j\rangle$. By lemma 2.20 [14] the subgroup A is Abelian, it is inverted by the involution j, and it is clear that $A \subseteq N_j$. Then, each non-identity permutation from T is either regular or stabilizes only one point from F. It follows that the permutation b is regular and each non-identity element from A is as well regular on F.

Now we prove that the subgroup of A is strongly isolated in T, that is, it contains a centralizer of any not identity element. Let c be any non-identity element from A and $C = C_T(c)$. It is clear that $A \leqslant C$ and, as it is proved above, $c \in T \setminus T_\alpha$ and $c^j = c^{-1}$. It means that the statements true of the subgroup A are also true of C. In particular, the subgroup C is Abelian. Hence, $C \leqslant C_T(b) = A$, and for $c \in A^\#$ is arbitrary it follows that the subgroup of A is strongly isolated in T.

Thus, $A \subseteq N_j$. Since the T_α action by conjugation is transitive on a set N_j^*, for any non-identity elements $b, c \in A$ there exists an element $h \in T_\alpha$ such that $b^h = c$. Therefore, $c \in A \cap A^h$ and for A is strongly isolated in T and A is commutative we have $A, A^h \leqslant C_T(c) = A$ and $A = A^h$. It follows that $h \in T_\alpha \cap N_T(A) = H$ and the action of H on $A^\#$ is transitive.

Next we show that $\Delta = \alpha^A$ is the orbit of the group $V = A \rtimes H$. Indeed, if $\beta \in \Delta$, then $\beta = \alpha^b$ for a suitable element $b \in A$, and for any $h \in H$ we have

$$\beta^h = \alpha^{bh} = \alpha^{h^{-1}bh} = \alpha^c,$$

where $c = b^h \in A$. Thus we have $\beta^h \in \Delta$ for any $\beta \in \Delta$, $h \in H$, and $\Delta = \alpha^V$. Further, if $\beta, \gamma \in \Delta$, and $\beta \neq \alpha \neq \gamma$, $\beta \neq \gamma$, then $\beta = \alpha^b$, $\beta = \alpha^c$, for suitable elements $b, c \in A^\#$. As it is shown above, $b^h = c$ for some $h \in H$. But then $\beta^h = \alpha^{bh} = \alpha^{h^{-1}bh} = \alpha^c = \gamma$ and H acts transitively on $\Delta \setminus \{\alpha\}$. Thus, V is doubly-transitive on Δ and for T acts sharply doubly-transitivity on F, the subgroup V acts sharply doubly-transitive on Δ. Hence, the subgroup A is Abelian, regular on Δ and it is normal in the group V. □

Proof of the theorem 2. Let L be a locally finite subgroup of the group T, b be a regular permutation from L and T_α be the stabilizer of the point $\alpha \in F$ for which a subgroup of $H = L \cap T_\alpha$ is normal in T_α and $|H| > 2$. It is clear that H is the proper subgroup in L and due to lemma 3.4 [14] (L, H) is a Frobenius pair. Since L is locally finite, $L = M \rtimes H$ is a Frobenius group with a kernel M and a complement H, thus the set of the regular permutations from L coincides with $M^\#$. Let $t \in M^\#$. Then $t \notin T_\alpha$ and due to the above reasoning, the subgroup of $S_t = \langle H, H^t\rangle$ is a Frobenius group with a kernel $M \cap S_t$ and a complement H. By conditions of the theorem, H is a normal subgroup T_α and for each element $h \in T_\alpha$ the subgroup $S_{th} = S_t^h = \langle H, H^{th}\rangle$ is a locally finite Frobenius group with a complement H.

Since $|H| > 2$ and T_α can not contain more than one involution (a lemma 3.4 [14]), there exists an element a in H, whose order is > 2. Now we prove that for any element $g \in T \setminus T_\alpha$ the subgroup $L_g = \langle a, a^g\rangle$ is a Frobenius finite group with a complement $H \cap L_g$. Again, let $t \in M^\#$.

As a doubly-transitive group, T is the union of the subgroup $T_s\alpha$ and the double cosets $T_\alpha t T_\alpha$. In particular, the element g is representable in the form $g = rth$, where $r, h \in T_\alpha$. Since $a^r \in H$, $L_{rt} = \langle a, a^{rt}\rangle$ is a subgroup of the group S_t. It follows that $L_g \leqslant S_{th}$ and L_g is a Frobenius finite group with a complement $H \cap L_g$.

Thus, the group T, its proper subgroup T_α and the element $a \in T_\alpha$ satisfy all the conditions of theorem 2.11 from [15]. That theorem gives $T = A \leftthreetimes T_\alpha$.

It is left to make sure that A is a regular Abelian subgroup. Suppose T_α to contain an involution j. As all the involutions in T are conjugate, and j is the only involution in T_α, the set J of all the involutions of the group T is contained in the coset Aj. According [14, lemma 3.4, item 3] the involution j is perfect in G, and as $J \subset A \leftthreetimes \langle j\rangle$, j is perfect in $A \leftthreetimes \langle j\rangle$. Due to lemma 2.19 [14] A is an Abelian group inverted by the involution j, besides, T_α acts transitively on $A^\#$ by conjugation. Due to an obvious inclusion $M \leqslant A$, A is an elementary Abelian p-group and $\mathrm{Char}\, T = p$. The theorem for this case is completely proved.

Now suppose that T_α doesn't contain an involution. By lemma 3.3 [14] the set J of all involutions of the group T is not empty and it is clear that $J \subset A$. Since T is sharply doubly-transitive on F, for any point $\beta \in F$ there exists an involution k such that $\alpha^k = \beta$. Therefore, $T_\alpha t T_\alpha = T_\alpha J$, $J = A^\#$ and A is a group of period 2. Hence A is an Abelian group and Char T=2. □

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О точно дважды транзитивных группах

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Исследуется вопрос о существовании в произвольной точно дважды транзитивной группе регулярной абелевой нормальной подгруппы.

Ключевые слова: группы,точно дважды транзитивные группы, почти поля, почти области.