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On Divisibility of Some Sums of Binomial Coefficients Arising From Collection Formulas

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In this paper we establish a series of identities for sums of binomial coefficients to prove their divisibility by prime n. These sums arise from exponents of commutators in collection formula for $(xy)^n$ with some restrictions on variables of the commutators.

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Introduction

In 1932, P. Hall proved the formula (known as Hall's collection formula) in [1] that makes it possible to investigate interdependently the power and commutator structures of $p-$groups. The following statement was proved by P. Hall in the article on the theory of $p-$groups [1]. For any two elements x and y of any group G let the formally distinct complex commutators $R_1, R_2, \dots$ of x and y be arranged in order of increasing weights (the order among the commutators of the same weight is arbitrary). Then for any natural n the following formula holds:

$$(xy)^n = x^n y^n {R_3}^{f_3(n)} \dots {R_i}^{f_i(n)} \dots, \tag{1}$$

where $f_i(n) = a_1\binom{n}{1} + a_2\binom{n}{2} + \dots + a_n\binom{n}{w}$, w is the weight of R_i, and non-negative coefficients a_k depend only on R_i but not on n. To compute the exponents is a difficult problem. In that regard, research is conducted in two lines. On the one hand, an explicit form of the exponents is sought (see, for example, [2]), on the other hand, a series of explicit collection formulas [3, 4] was found with some restrictions on the group.

In connection with the research on regularity of Sylow p-subgroups of group $GL_n(\mathbb{Z}_{p^m})$ (problem 8.3 [5]), the following theorem was proved in [6]. Here we use the abridged notation for commutators $[y,x] = y^{-1}x^{-1}yx$, $[y,{}_i x] = [[y,{}_{i-1}x],x]$, $i = 1, 2, \dots$.

Theorem. *Let G be a group, $x, y \in G$, any commutator of x and y equals 1 if it includes more than two y's, then we have*

$$(xy)^n = x^n y^n \prod_{u=1}^{n-1} [y,{}_u x]^{\binom{n}{u+1}} \prod_{u=1}^{n-1} [y,{}_u x, y]^{n\binom{n}{u+1} - \binom{n+1}{u+2}} \prod_{n-1 \geqslant u > v \geqslant 1} [[y,{}_u x],[y,{}_v x]]^{F_n(u,v)+G_n(u,v)}, \tag{2}$$

where

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$$F_n(u,v) = \sum_{m=1}^{n-1}\sum_{k=1}^{v}\sum_{i=v-k}^{n-m-k}\binom{n-m-i-1}{k-1}\binom{i}{u-k+1}\binom{i}{v-k}, \tag{3}$$

$$G_n(u,v) = \sum_{m=1}^{n-2}\sum_{k=m+1}^{n-1}\binom{m}{v}\binom{k}{u}. \tag{4}$$

For any collection formula, there is a divisibility problem for powers of commutators with weights less than n, specifically for prime n. According to Hall's collection formula (1), these commutators have powers divisible by n. From the properties of binomial coefficients it obviously follows that powers $\binom{n}{u+1}$ and $n\binom{n}{u+1} - \binom{n+1}{u+2}$ in (2) are divisible by prime n if $u+1 < n$ and $u+2 < n$, respectively. But the divisibility of $F_n(u,v) + G_n(u,v)$ is not obvious.

The aim of this paper is to transform $F_n(u,v)$ and $G_n(u,v)$ into such expressions that it will be easy to see the divisibility of $F_n(u,v)$ and $G_n(u,v)$ individually.

Let us recall that classical binomial coefficient

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!}, & \text{if } n \geqslant k \geqslant 0; \\ 0, & \text{if } n < k \text{ or } k < 0 \end{cases}$$

is defined for integers $n \geqslant 0$, k and satisfies the following recurrence relation:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}. \tag{5}$$

Working with the sums $F_n(u,v)$ and $G_n(u,v)$, it is useful to extend the domain of the binomial coefficient to all integers n and k so that (5) remains valid. To set such extension we define $\binom{n}{k}$ for each negative n with any single integer k [7]. In this paper the following extension is used: $\binom{i}{i} = 1$ for all integers i. In what follows, we will use these extended coefficients without additional qualifications. We also agree that sums of the form $\sum_{i=a}^{b} c_i$ are equal to zero if $b < a$.

Our main result is the following

Theorem 1. *For any integers $n \geqslant 1$, $u \geqslant 0$, $v \geqslant 1$ we have*

$$F_n(u,v) + G_n(u,v) =$$
$$= \sum_{k=1}^{v}\sum_{s=0}^{v-k}\binom{u-k+1+s}{v-k}\binom{v-k}{s}\binom{n}{u+s+2} + \sum_{i=0}^{v+1}(-1)^i\binom{n+i}{u+i+1}\binom{n}{v-i+1}.$$

In particular, if n is prime and $u+v+2 < n$, then $F_n(u,v) + G_n(u,v)$ is divisible by n.

1. Auxiliary statements

Before we start working with $F_n(u,v)$ and $G_n(u,v)$ we will prove some statements about the extended binomial coefficient. Those statements are valid and well known for the classical binomial coefficient. The following statement will be used most often.

Lemma 1. *If n, k are integers and $n < k$, then $\binom{n}{k} = 0$.*

Proof. If $n < k$ then $k = n+s$, where $s > 0$. We use the induction on s to prove that $\binom{n}{n+s} = 0$ for any integer n. For $s = 1$ and any n according to (5), we have $\binom{n}{n+1} = \binom{n+1}{n+1} - \binom{n}{n} = 0$. Let $\binom{n}{n+s} = 0$ for some s and any n, then $\binom{n}{n+s+1} = \binom{n+1}{n+s+1} - \binom{n}{n+s} = \binom{n+1}{(n+1)+s} = 0$. □

Lemma 2. *For any integers n and a we have the summation formula*

$$\sum_{i=a}^{n} \binom{i}{a} = \binom{n+1}{a+1}. \tag{6}$$

Proof. If $n < a$ then both sides of (6) are equal to zero. Further on, we fix an arbitrary integer a and use the induction on $n \geqslant a$. For $n = a$, the formula (6) is valid because $\binom{a}{a} = \binom{a+1}{a+1} = 1$. Assume that (6) is valid for some $n \geqslant a$, then we get

$$\sum_{i=a}^{n+1} \binom{i}{a} = \sum_{i=a}^{n} \binom{i}{a} + \binom{n+1}{a} = \binom{n+1}{a+1} + \binom{n+1}{a} = \binom{n+2}{a+1}.$$

□

Lemma 3. *For any non-negative integers n, b, a we have the summation formula*

$$\sum_{i=b}^{n-a} \binom{n-i}{a}\binom{i}{b} = \binom{n+1}{a+b+1}. \tag{7}$$

Proof. If $n - a < b$ then both sides of (7) are equal to zero.

Now assume that $n - a \geqslant b$. Let us transform $\binom{n-i}{a}\binom{i}{b}$ using (5).

$$\binom{n-i}{a}\binom{i}{b} = \binom{n-i}{a}\binom{i+1}{b+1} - \binom{n-i}{a}\binom{i}{b+1} =$$
$$= \binom{n-i}{a}\binom{i+1}{b+1} - \binom{n-i+1}{a}\binom{i}{b+1} + \binom{n-i}{a-1}\binom{i}{b+1}.$$

By induction on integer $s \geqslant 0$ we obtain

$$\binom{n-i}{a}\binom{i}{b} = \binom{n-i}{a-s}\binom{i}{b+s} + \sum_{j=1}^{s} \left(\binom{n-i}{a-j+1}\binom{i+1}{b+j} - \binom{n-i+1}{a-j+1}\binom{i}{b+j} \right). \tag{8}$$

Suppose $s = n - b + 1$ ($n - b + 1 \geqslant a + 1 > 0$), then $\binom{i}{b+s} = \binom{i}{n+1} = 0$ because $i < n + 1$. Therefore, if we extend the summation in (7) to n ($\binom{n-i}{a} = 0$ for $i > n - a$) and substitute (8) into (7), we get

$$\sum_{i=b}^{n-a} \binom{n-i}{a}\binom{i}{b} = \sum_{i=b}^{n} \sum_{j=1}^{n-b+1} \left(\binom{n-i}{a-j+1}\binom{i+1}{b+j} - \binom{n-i+1}{a-j+1}\binom{i}{b+j} \right) =$$
$$= \sum_{j=1}^{n-b+1} \left(\binom{n-b}{a-j+1}\binom{b+1}{b+j} - \binom{n-b+1}{a-j+1}\binom{b}{b+j} \right) +$$
$$+ \sum_{j=1}^{n-b+1} \left(\binom{n-b-1}{a-j+1}\binom{b+2}{b+j} - \binom{n-b}{a-j+1}\binom{b+1}{b+j} \right) +$$
$$\cdots$$
$$+ \sum_{j=1}^{n-b+1} \left(\binom{1}{a-j+1}\binom{n}{b+j} - \binom{2}{a-j+1}\binom{n-1}{b+j} \right) +$$
$$+ \sum_{j=1}^{n-b+1} \left(\binom{0}{a-j+1}\binom{n+1}{b+j} - \binom{1}{a-j+1}\binom{n}{b+j} \right).$$

After collecting similar terms we have

$$\sum_{i=b}^{n-a}\binom{n-i}{a}\binom{i}{b} = -\sum_{j=1}^{n-b+1}\binom{n-b+1}{a-j+1}\binom{b}{b+j} + \sum_{j=1}^{n-b+1}\binom{0}{a-j+1}\binom{n+1}{b+j}.$$

On the right-hand side of the obtained equality, the first sum is equal to zero because $\binom{b}{b+j} = 0$ for $j \geqslant 1$. In the second sum $\binom{0}{a-j+1} \neq 0$ only for $j = a+1$, so all terms except one (for $j = a+1$) are equal to zero. The sum always includes this term because $1 \leqslant a+1 \leqslant n-b+1$. Finally, we get

$$\sum_{i=0}^{n-a}\binom{n-i}{a}\binom{i}{b} = \binom{0}{0}\binom{n+1}{a+b+1} = \binom{n+1}{a+b+1}.$$

□

2. Transformation of the sum $F_n(u, v)$

Let us simplify the expression $F_n(u, v)$ as follows. At first, we extend the summation over i to $n-k-1$ (added terms are equal to zero because $\binom{n-m-i-1}{k-1} = 0$ for $i > n-m-k$), then we change the order of summation.

$$F_n(u,v) = \sum_{k=1}^{v}\sum_{i=v-k}^{n-k-1}\sum_{m=1}^{n-1}\binom{n-m-i-1}{k-1}\binom{i}{u-k+1}\binom{i}{v-k}.$$

Now we replace $n-m-i-1$ by m.

$$F_n(u,v) = \sum_{k=1}^{v}\sum_{i=v-k}^{n-k-1}\sum_{m=-i}^{n-i-2}\binom{m}{k-1}\binom{i}{u-k+1}\binom{i}{v-k}.$$

The starting value of m can be changed to $k-1$ because $-i \leqslant 0$, $k-1 \geqslant 0$ and $\binom{m}{k-1} = 0$ for $m < k-1$. So, we apply the formula (6) to the summation over m and, finally, get

$$F_n(u,v) = \sum_{k=1}^{v}\sum_{i=v-k}^{n-k-1}\binom{n-i-1}{k}\binom{i}{u-k+1}\binom{i}{v-k}. \qquad (9)$$

Further on, we will consider the summation over i, so it will be useful to introduce the notation:

$$f_n(u,v,k) = \sum_{i=v-k}^{n-k}\binom{n-i}{k}\binom{i}{u-k}\binom{i}{v-k}.$$

Lemma 4. *For any nonnegative integers n, u, v, k the following relation holds:*

$$f_n(u,v,k) = f_n(u+1,v,k+1) + f_n(u,v,k+1) + f_n(u,v+1,k+1). \qquad (10)$$

Proof. If $n < v$ then both sides of (10) are equal to zero.

Now assume that $n \geqslant v$. Replacing $\binom{n-i}{k}$ on the difference $\binom{n-i+1}{k+1} - \binom{n-i}{k+1}$ in $f_n(u,v,k)$ and taking into account that

$$-\sum_{i=v-k}^{n-k}\binom{n-i}{k+1}\binom{i}{u-k}\binom{i}{v-k} = -\sum_{i=v-k}^{n-k-1}\binom{n-i}{k+1}\binom{i}{u-k}\binom{i}{v-k} =$$

$$= -\sum_{i=v-k}^{n-k-1}\binom{n-i}{k+1}\binom{i+1}{u-k}\binom{i}{v-k} + \sum_{i=v-k}^{n-k-1}\binom{n-i}{k+1}\binom{i}{u-k-1}\binom{i}{v-k} =$$

$$= -\sum_{i=v-k}^{n-k-1}\binom{n-i}{k+1}\binom{i+1}{u-k}\binom{i}{v-k} + f_n(u,v+1,k+1),$$

we get

$$f_n(u,v,k) = \sum_{i=v-k}^{n-k}\binom{n-i+1}{k+1}\binom{i}{u-k}\binom{i}{v-k} - \sum_{i=v-k}^{n-k-1}\binom{n-i}{k+1}\binom{i+1}{u-k}\binom{i}{v-k} +$$
$$+ f_n(u,v+1,k+1).$$

Because

$$-\sum_{i=v-k}^{n-k-1}\binom{n-i}{k+1}\binom{i+1}{u-k}\binom{i}{v-k} =$$

$$= -\sum_{i=v-k}^{n-k-1}\binom{n-i}{k+1}\binom{i+1}{u-k}\binom{i+1}{v-k} + \sum_{i=v-k}^{n-k-1}\binom{n-i}{k+1}\binom{i+1}{u-k}\binom{i}{v-k-1} =$$

$$= -\sum_{i=v-k+1}^{n-k}\binom{n-i+1}{k+1}\binom{i}{u-k}\binom{i}{v-k} + \sum_{i=v-k}^{n-k-1}\binom{n-i}{k+1}\binom{i+1}{u-k}\binom{i}{v-k-1},$$

we can write:

$$f_n(u,v,k) = \sum_{i=v-k}^{n-k}\binom{n-i+1}{k+1}\binom{i}{u-k}\binom{i}{v-k} - \sum_{i=v-k+1}^{n-k}\binom{n-i+1}{k+1}\binom{i}{u-k}\binom{i}{v-k} +$$
$$+ \sum_{i=v-k}^{n-k-1}\binom{n-i}{k+1}\binom{i+1}{u-k}\binom{i}{v-k-1} + f_n(u,v+1,k+1).$$

If $n \geqslant v+1$ we see that the difference of the first two sums is equal to $\binom{n-(v-k)+1}{k+1}\binom{v-k}{u-k}\binom{v-k}{v-k}$. This remains true if $n = v$ (then the second sum equals zero, while the first consists of only one term). Therefore

$$f_n(u,v,k) = \sum_{i=v-k}^{n-k-1}\binom{n-i}{k+1}\binom{i+1}{u-k}\binom{i}{v-k-1} +$$
$$+\binom{n-(v-k)+1}{k+1}\binom{v-k}{u-k}\binom{v-k}{v-k} + f_n(u,v+1,k+1) =$$
$$= \sum_{i=v-k}^{n-k-1}\binom{n-i}{k+1}\binom{i}{u-k}\binom{i}{v-k-1} + \sum_{i=v-k}^{n-k-1}\binom{n-i}{k+1}\binom{i}{u-k-1}\binom{i}{v-k-1} +$$
$$+\binom{n-(v-k)+1}{k+1}\binom{v-k}{u-k}\binom{v-k}{v-k} + f_n(u,v+1,k+1).$$

Now we transform two sums in the obtained equality. If $n-1 \geqslant v$ then, firstly:

$$\sum_{i=v-k}^{n-k-1}\binom{n-i}{k+1}\binom{i}{u-k}\binom{i}{v-k-1} =$$

$$= \sum_{i=v-k-1}^{n-k-1} \binom{n-i}{k+1}\binom{i}{u-k}\binom{i}{v-k-1} - \binom{n-(v-k)+1}{k+1}\binom{v-k-1}{u-k}\binom{v-k-1}{v-k-1} =$$

$$= f_n(u+1,v,k) - \binom{n-(v-k)+1}{k+1}\binom{v-k-1}{u-k}\binom{v-k-1}{v-k-1},$$

secondly:

$$\sum_{i=v-k}^{n-k-1} \binom{n-i}{k+1}\binom{i}{u-k-1}\binom{i}{v-k-1} =$$

$$= \sum_{i=v-k-1}^{n-k-1} \binom{n-i}{k+1}\binom{i}{u-k-1}\binom{i}{v-k-1} - \binom{n-(v-k)+1}{k+1}\binom{v-k-1}{u-k-1}\binom{v-k-1}{v-k-1} =$$

$$= f_n(u,v,k) - \binom{n-(v-k)+1}{k+1}\binom{v-k-1}{u-k-1}\binom{v-k-1}{v-k-1}.$$

It can easily be checked that the obtained equalities are also valid if $n-1=v-1$. Finally, we get

$$f_n(u,v,k) = f_n(u+1,v,k+1) + f_n(u,v,k+1) + f_n(u,v+1,k+1) +$$
$$+ \binom{n-(v-k)+1}{k+1}\binom{v-k}{u-k}\binom{v-k}{v-k} - \binom{n-(v-k)+1}{k+1}\binom{v-k-1}{u-k}\binom{v-k-1}{v-k-1} -$$
$$- \binom{n-(v-k)+1}{k+1}\binom{v-k-1}{u-k-1}\binom{v-k-1}{v-k-1}.$$

Since $\binom{v-k}{v-k} = \binom{v-k-1}{v-k-1} = 1$ and $\binom{v-k}{u-k} - \binom{v-k-1}{u-k} - \binom{v-k-1}{u-k-1} = 0$, we obtain (10). □

In the following two lemmas we generalize the relation (10) as follows. The sum $f_n(u,v,k)$ will be expressed in terms such as $f_n(x,y,y)$. These sums can be minimized by the formula (7), so this will be sufficient to prove the divisibility.

For integers n, s, l, we introduce the notation for trinomial coefficient [8]

$$\binom{n}{s,l} = \binom{n}{n-s}\binom{n-s}{l}.$$

It can easily be checked that $\binom{n}{s,l} = \binom{n}{l,s}$ if $n \geqslant 0$. Besides, by (5) it follows that for all integers n, s, l we have the recurrence relation

$$\binom{n-1}{s-1,l} + \binom{n-1}{s,l} + \binom{n-1}{s,l-1} = \binom{n}{s,l}. \quad (11)$$

Lemma 5. *For any non-negative integers p, n, u, v, k the following relation holds:*

$$f_n(u,v,k) = \sum_{s=0}^{p}\sum_{l=0}^{p-s} \binom{p}{s,l} f_n(u+s, v+l, k+p). \quad (12)$$

Proof. The proof is by induction on p. For $p=0$ there is nothing to prove. Suppose (12) is valid for some p; then we apply (10) to $f_n(u+s, v+l, k+p)$ and get three sums on the right-hand side of (12). In each of these sums we will replace the indices.

In the first sum s is replaced by $s-1$.

$$\sum_{s=0}^{p}\sum_{l=0}^{p-s} \binom{p}{s,l} f_n(u+s+1, v+l, k+p+1) = \sum_{s=0}^{p+1}\sum_{l=0}^{p-s+1} \binom{p}{s-1,l} f_n(u+s, v+l, k+p+1).$$

As before, the start value of s remains zero to make the further transformations easier (for $s = 0$ we have $\binom{p}{s-1,l} = \binom{p}{p-(s-1)}\binom{p-(s-1)}{l} = \binom{p}{p+1}\binom{p+1}{l} = 0$).

In the second sum we increase the finish values of s and l to $p+1$ and $p-s+1$, respectively (if $s = p+1$ or $l = p-s+1$, then $\binom{p}{s,l} = 0$).

$$\sum_{s=0}^{p}\sum_{l=0}^{p-s}\binom{p}{s,l}f_n(u+s, v+l, k+p+1) = \sum_{s=0}^{p+1}\sum_{l=0}^{p-s+1}\binom{p}{s,l}f_n(u+s, v+l, k+p+1).$$

In the third sum we replace l by $l-1$ (the starting value of l remains zero) and increase the finish value of s to $p+1$.

$$\sum_{s=0}^{p}\sum_{l=0}^{p-s}\binom{p}{s,l}f_n(u+s, v+l+1, k+p+1) = \sum_{s=0}^{p+1}\sum_{l=0}^{p-s+1}\binom{p}{s,l-1}f_n(u+s, v+l, k+p+1).$$

If $s = p+1$ or $l = 0$, then $\binom{p}{s,l-1} = 0$, so the last equality is valid.

Finally, we obtain

$$f_n(u, v, k) = \sum_{s=0}^{p+1}\sum_{l=0}^{p-s+1}\binom{p}{s-1,l}f_n(u+s, v+l, k+p+1)+$$
$$+\sum_{s=0}^{p+1}\sum_{l=0}^{p-s+1}\binom{p}{s,l}f_n(u+s, v+l, k+p+1) + \sum_{s=0}^{p+1}\sum_{l=0}^{p-s+1}\binom{p}{s,l-1}f_n(u+s, v+l, k+p+1).$$

To conclude the proof, it remains to note that $\binom{p}{s-1,l} + \binom{p}{s,l} + \binom{p}{s,l-1} = \binom{p+1}{s,l}$ according to (11). □

Lemma 6. *For any non-negative integers n, p, q, u, v, k we have the relation*

$$f_n(u, v, k) = \sum_{l=1}^{p}\sum_{s=0}^{p-l}\binom{p+q}{s, l+q}f_n(u+s, v+l+q, k+p+q)+$$
$$+\sum_{l=0}^{q}\sum_{s=0}^{p}\binom{p+l-1}{p-1}\binom{p}{s}f_n(u+s, v+l, k+p+l). \quad (13)$$

Proof. The proof is by induction on q. For $q = 0$ the equality (13) takes the form of (12) (with changed summation order). Suppose (13) holds true for some $q \geqslant 0$, then we transform the first sum on the right-hand side of (13). The application of (10) to $f_n(u+s, v+l+q, k+p+q)$ divides the sum into three sums. We transform two of them to the form of the third one:

$$\sum_{l=1}^{p}\sum_{s=0}^{p-l}\binom{p+q}{s, l+q}f_n(u+s, v+l+q+1, r+1),$$

for convenience, the temporary notation $r = k+p+q$ will be used.

In the first sum we separate terms for $l = 1$, then, in the remaining expression, replace s by $s-1$, l by $l+1$. The starting value of s and the final value of l remain zero and p, respectively (for $s = 0$ or $l = p$ we have $\binom{p+q}{s-1,l+q+1} = 0$).

$$\sum_{l=1}^{p}\sum_{s=0}^{p-l}\binom{p+q}{s, l+q}f_n(u+s+1, v+l+q, r+1) = \sum_{s=0}^{p-1}\binom{p+q}{s, q+1}f_n(u+s+1, v+q+1, r+1)+$$

$$+\sum_{l=2}^{p}\sum_{s=0}^{p-l}\binom{p+q}{s,l+q}f_n(u+s+1,v+l+q,r+1)=\sum_{s=0}^{p-1}\binom{p+q}{s,q+1}f_n(u+s+1,v+q+1,r+1)+$$
$$+\sum_{l=1}^{p}\sum_{s=0}^{p-l}\binom{p+q}{s-1,l+q+1}f_n(u+s,v+l+q+1,r+1).$$

In the second sum we also separate terms for $l=1$ and, in the remaning expression, replace l by $l+1$. The final values of l and s remain p and $p-l$, respectively (for $l=p$ or $s=p-l$ we have $\binom{p+q}{s,l+q+1}=0$).

$$\sum_{l=1}^{p}\sum_{s=0}^{p-l}\binom{p+q}{s,l+q}f_n(u+s,v+l+q,r+1)=\sum_{s=0}^{p-1}\binom{p+q}{s,q+1}f_n(u+s,v+q+1,r+1)+$$
$$+\sum_{l=2}^{p}\sum_{s=0}^{p-l}\binom{p+q}{s,l+q}f_n(u+s,v+l+q,r+1)=\sum_{s=0}^{p-1}\binom{p+q}{s,q+1}f_n(u+s,v+q+1,r+1)+$$
$$+\sum_{l=1}^{p}\sum_{s=0}^{p-l}\binom{p+q}{s,l+q+1}f_n(u+s,v+l+q+1,r+1).$$

According to (11), $\binom{p+q}{s-1,l+q+1}+\binom{p+q}{s,l+q+1}+\binom{p+q}{s,l+q}=\binom{p+q+1}{s,l+q+1}$, so summing up two transformed sums and the third one, we get

$$\sum_{l=1}^{p}\sum_{s=0}^{p-l}\binom{p+q}{s,l+q}f_n(u+s,v+l+q,r)=\sum_{l=1}^{p}\sum_{s=0}^{p-l}\binom{p+q+1}{s,l+q+1}f_n(u+s,v+l+q+1,r+1)+$$
$$+\sum_{s=0}^{p-1}\binom{p+q}{s,q+1}f_n(u+s+1,v+q+1,r+1)+\sum_{s=0}^{p-1}\binom{p+q}{s,q+1}f_n(u+s,v+q+1,r+1).$$

In the second sum we replace s by $s-1$ but, as before, we sum from $s=0$ ($\binom{p+q}{s-1,q+1}=0$ for $s=0$).

$$\sum_{s=0}^{p-1}\binom{p+q}{s,q+1}f_n(u+s+1,v+q+1,k+p+q+1)=\sum_{s=0}^{p}\binom{p+q}{s-1,q+1}f_n(u+s,v+q+1,k+p+q+1).$$

In the third sum we increase the final value of s to p ($\binom{p+q}{s,q+1}=0$ for $s=p$).

$$\sum_{s=0}^{p-1}\binom{p+q}{s,q+1}f_n(u+s,v+q+1,r+1)=\sum_{s=0}^{p}\binom{p+q}{s,q+1}f_n(u+s,v+q+1,r+1).$$

From the definition of the trinomial coefficient and (5) it follows that

$$\binom{p+q}{s-1,q+1}+\binom{p+q}{s,q+1}=\binom{p+q}{p-1}\binom{p-1}{s-1}+\binom{p+q}{p-1}\binom{p-1}{s}=\binom{p+q}{p-1}\binom{p}{s},$$

so, finally we get

$$\sum_{l=1}^{p}\sum_{s=0}^{p-l}\binom{p+q}{s,l+q}f_n(u+s,v+l+q,r)=\sum_{l=1}^{p}\sum_{s=0}^{p-l}\binom{p+q+1}{s,l+q+1}f_n(u+s,v+l+q+1,r+1)+$$

$$+\sum_{s=0}^{p}\binom{p+q}{p-1}\binom{p}{s}f_n(u+s,v+l+q+1,r+1).$$

The substitution of the obtained expression into (13) completes the proof. □

Theorem 2. *For any non-negative integers n,u,v,k such that $k\leqslant v$ or $k\leqslant u$ we have*

$$f_n(u,v,k)=\sum_{s=0}^{v-k}\binom{u-k+s}{v-k}\binom{v-k}{s}\binom{n+1}{u+s+1}. \tag{14}$$

Proof. Let us consider the case $k\leqslant v$ and $k\leqslant u$. Suppose $p=v-k\geqslant 0$, $q=u-k\geqslant 0$ in (13). Then, since by definition

$$f_n(u+s,v+l+u-k,v+u-k)=\sum_{i=l}^{n-(v+u-k)}\binom{n-i}{v+u-k}\binom{i}{s-(v-k)}\binom{i}{l}=0$$

because $\binom{i}{s-(v-k)}=0$ for $i\geqslant l\geqslant 1$ and $s-(v-k)<0$, we obtain

$$f_n(u,v,k)=\sum_{l=0}^{u-k}\sum_{s=0}^{v-k}\binom{v-k+l-1}{v-k-1}\binom{v-k}{s}f_n(u+s,v+l,v+l). \tag{15}$$

Moreover, by definition, we have

$$f_n(u+s,v+l,v+l)=\sum_{i=0}^{n-(v+l)}\binom{n-i}{v+l}\binom{i}{u+s-(v+l)} \tag{16}$$

because $\binom{i}{0}=1$ for $i\geqslant 0$.

Note that if $u+s-(v+l)<0$, i.e. $l>u+s-v$, then all terms in $f_n(u+s,v+l,v+l)$ equal zero because $\binom{i}{u+s-(v+l)}=0$ for $i\geqslant 0$. So, after changing the order of the sums in (15) and removing zero terms in the summation over l, we get

$$f_n(u,v,k)=\sum_{s=0}^{v-k}\sum_{l=0}^{u+s-v}\binom{v-k+l-1}{v-k-1}\binom{v-k}{s}f_n(u+s,v+l,v+l).$$

Finally, we remove zero terms in (16) and apply the formula (7) to it.

$$f_n(u+s,v+l,v+l)=\sum_{i=u+s-(v+l)}^{n-(v+l)}\binom{n-i}{v+l}\binom{i}{u+s-(v+l)}=\binom{n+1}{u+s+1}.$$

Thus,

$$f_n(u,v,k)=\sum_{s=0}^{v-k}\sum_{l=0}^{u+s-v}\binom{v-k+l-1}{v-k-1}\binom{v-k}{s}\binom{n+1}{u+s+1}=$$

$$=\sum_{s=0}^{v-k}\sum_{l=v-k-1}^{u+s-k-1}\binom{l}{v-k-1}\binom{v-k}{s}\binom{n+1}{u+s+1}.$$

The application of the formula (6) to the summation over l gives us (14).

Let us consider other cases. If $k>v$ and $k\leqslant u$, then left-hand and right-hand sides of (14) are equal to zero. Indeed, all terms in $f_n(u,v,k)$ are equal to zero because $\binom{i}{v-k}=0$ for $i\geqslant 0$

and $\binom{i}{u-k} = 0$ for $i < 0$. If $k \leqslant v$ and $k > u$, then $f_n(u, v, k)$ is equal to zero again because $\binom{i}{u-k} = 0$ for $i \geqslant v - k \geqslant 0$. The right-hand side is equal to zero because $\binom{u+s-k}{v-k} = 0$ for $s \leqslant v - k$. □

Corollary 1. *For any natural n and non-negative integers u, v we have*

$$F_n(u,v) = \sum_{k=1}^{v}\sum_{s=0}^{v-k}\binom{u-k+1+s}{v-k}\binom{v-k}{s}\binom{n}{u+s+2}. \tag{17}$$

In particular, if $u + v + 1 < n$ then $F_n(u, v)$ is divisible by prime n.

Proof. Applying (14) to $f_{n-1}(u+1, v, k)$ in (9), we obtain (17) (condition $k \leqslant v$ is satisfied). If n is prime, $u + v + 1 < n$, then $u + s + 2 \leqslant u + v + 1 < n$, so the binomial coefficient $\binom{n}{u+s+2}$ is divisible by n for any s. □

3. Transformation of the sum $G_n(u, v)$

Lemma 7. *For any natural n and integers $u \geqslant 0$, $v \geqslant 1$ we have*

$$G_n(u,v) = \sum_{i=1}^{n-1}\binom{i}{v+1}\binom{i}{u}. \tag{18}$$

Proof. We change the order of summation in $G_n(u, v)$.

$$G_n(u,v) = \sum_{m=1}^{n-2}\sum_{k=m+1}^{n-1}\binom{m}{v}\binom{k}{u} = \sum_{m=2}^{n-1}\sum_{k=1}^{m-1}\binom{k}{v}\binom{m}{u}.$$

Because $v \geqslant 1$ we remove zero terms in the summation over k and apply the formula (6).

$$\sum_{m=2}^{n-1}\sum_{k=1}^{m-1}\binom{k}{v}\binom{m}{u} = \sum_{m=2}^{n-1}\sum_{k=v}^{m-1}\binom{k}{v}\binom{m}{u} = \sum_{m=2}^{n-1}\binom{m}{v+1}\binom{m}{u}.$$

Adding to the last sum the term $\binom{1}{v+1}\binom{1}{u} = 0$ we get (18). □

Theorem 3. *For any natural n and non-negative integers $u \geqslant 1$, $v \geqslant 1$ we have*

$$\sum_{i=1}^{n-1}\binom{i}{u}\binom{i}{v} = \sum_{i=0}^{v}(-1)^i\binom{n+i}{u+i+1}\binom{n}{v-i}. \tag{19}$$

Proof. Let us transform $\binom{i}{u}\binom{i}{v}$ using (5).

$$\binom{i}{u}\binom{i}{v} = \binom{i+1}{u+1}\binom{i}{v} - \binom{i}{u+1}\binom{i}{v} =$$

$$= \binom{i+1}{u+1}\binom{i+1}{v} - \binom{i+1}{u+1}\binom{i}{v-1} - \binom{i}{u+1}\binom{i}{v}.$$

By induction on integer $s \geqslant 0$ we obtain

$$\binom{i}{u}\binom{i}{v} = (-1)^s\binom{i+s}{u+s}\binom{i}{v-s} + \sum_{j=0}^{s-1}(-1)^j\left(\binom{i+j+1}{u+j+1}\binom{i+1}{v-j} - \binom{i+j}{u+j+1}\binom{i}{v-j}\right).$$

Suppose $s = v+1$, then $\binom{i}{v-s} = \binom{i}{-1} = 0$ because $i \geqslant 1$, so we get

$$\binom{i}{u}\binom{i}{v} = \sum_{j=0}^{v}(-1)^j\left(\binom{i+j+1}{u+j+1}\binom{i+1}{v-j} - \binom{i+j}{u+j+1}\binom{i}{v-j}\right).$$

Therefore,

$$\sum_{i=1}^{n-1}\binom{i}{u}\binom{i}{v} = \sum_{i=1}^{n-1}\sum_{j=0}^{v}(-1)^j\left(\binom{i+j+1}{u+j+1}\binom{i+1}{v-j} - \binom{i+j}{u+j+1}\binom{i}{v-j}\right) =$$

$$= \sum_{j=0}^{v}(-1)^j\left(\binom{j+2}{u+j+1}\binom{2}{v-j} - \binom{j+1}{u+j+1}\binom{1}{v-j}\right) +$$

$$+ \sum_{j=0}^{v}(-1)^j\left(\binom{j+3}{u+j+1}\binom{3}{v-j} - \binom{j+2}{u+j+1}\binom{2}{v-j}\right) +$$

$$\dots$$

$$+ \sum_{j=0}^{v}(-1)^j\left(\binom{n-1+j}{u+j+1}\binom{n-1}{v-j} - \binom{n-2+j}{u+j+1}\binom{n-2}{v-j}\right) +$$

$$+ \sum_{j=0}^{v}(-1)^j\left(\binom{n+j}{u+j+1}\binom{n}{v-j} - \binom{n-1+j}{u+j+1}\binom{n-1}{v-j}\right).$$

After collecting similar terms we have

$$\sum_{i=1}^{n-1}\binom{i}{u}\binom{i}{v} = -\sum_{j=0}^{v}(-1)^j\binom{j+1}{u+j+1}\binom{1}{v-j} + \sum_{j=0}^{v}(-1)^j\binom{n+j}{u+j+1}\binom{n}{v-j},$$

where the first sum is equal to zero if $u \geqslant 1$ or $v \geqslant 1$. In the second case we have

$$\sum_{j=0}^{v}(-1)^j\binom{j+1}{u+j+1}\binom{1}{v-j} = (-1)^{v-1}\binom{v}{u+v}\binom{1}{1} + (-1)^v\binom{v+1}{u+v+1}\binom{1}{0} = 0.$$

Thus, we obtain (19). □

Corollary 2. *For any integers $n \geqslant 1, u \geqslant 0$, $v \geqslant 1$ we have*

$$G_n(u,v) = \sum_{i=0}^{v+1}(-1)^i\binom{n+i}{u+i+1}\binom{n}{v-i+1}. \tag{20}$$

In particular, if $u+v+2 < n$ then $G_n(u,v)$ is divisible by prime n.

Proof. Because $v+1 \geqslant 2$ we apply (19) to (18) in order get (20).

Suppose n is prime, $u+v+2 < n$. The binomial coefficient $\binom{n}{v-i+1}$ is divisible by n for any $0 \leqslant i \leqslant v$, and $\binom{n+i}{u+i+1}$ is divisible by n for $i = v+1$. □

Corollaries 1 and 2 are a proof of Theorem 1.

Remark 1. If we impose the conditions $n-1 \geqslant u > v \geqslant 1$ on the expression $F_n(u,v)+G_n(u,v)$ in Theorem 1, then (1) contains only classical binomial coefficients, i.e. it does not depend on the extension of binomial coefficient.

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О делимости некоторых сумм биномиальных коэффициентов, возникающих в собирательных формулах

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В данной работе установлен ряд тождеств для сумм биномиальных коэффициентов, чтобы доказать их делимость на простое n. Эти суммы возникают в степенях коммутаторов из собирательной формулы для $(xy)^n$ *при некоторых ограничениях на вхождение переменных в коммутаторы.*

Ключевые слова: делимость, суммы биномиальных коэффициентов, собирательные формулы.