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Uniqueness of a Solution of an Ice Plate Oscillation Problem in a Channel

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In this paper an initial-boundary value problem for two mathematical models of elastic and viscoelastic oscillations of a thin ice plate in an infinite channel under the action of external load is considered in terms of the linear theory of hydroelasticity. The viscosity of ice is treated in the context of the Kelvin-Voigt model. The joint system of equations for the ice plate and an ideal fluid is considered. Boundary conditions are conditions of clamped edges for the ice plate at the walls of the channel, condition of impermeability for the flow velocity potential and the damping conditions for the oscillations at infinity. The uniqueness theorem for the classical solution of the initial-boundary value problem is proved.

Keywords: Euler equations, viscoelastic oscillations, ice plate, external load, uniqueness.

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1. Formulation of the problem

Oscillations of an ice plate in a channel under the action of external load applied to the ice are considered. The channel has a rectangular cross section with finite depth H , ($-H < z < 0$), and width $2L$, ($-L < y < L$), and it is infinitely long, $-\infty < x < \infty$, (x, y, z) is the Cartesian coordinate system. Fluid in the channel is inviscid and incompressible with density ρ_l . The thickness of the ice plate h_i and rigidity D , $D = Eh_i^3/[12(1 - \nu^2)]$, are constant, where E is the Young modulus and ν is the Poisson ratio. The ice plate is clamped at the channel walls, $y = \pm L$. We denote unbounded domains occupied with the ice plate and the liquid by $\Pi \subset \mathbb{R}^2$ and $\Omega \subset \mathbb{R}^3$, respectively

$$\Pi = \{-\infty < x < \infty, -L < y < L\}, \quad \Omega = \{-\infty < x < \infty, -L < y < L, -H < z < 0\}.$$

Boundaries of these domains are $\Gamma = \partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $G = \partial\Pi = G_1 \cup G_2$, where

$$\Gamma_1 = \Gamma_1^1 \cup \Gamma_1^2 = \{-\infty < x < \infty, y = \pm L, -H < z < 0\} \cup \{-\infty < x < \infty, -L < y < L, z = -H\},$$

$$\Gamma_2 = \{-\infty < x < \infty, -L < y < L, z = 0\}, \quad \Gamma_3 = \{x = \pm\infty, -L < y < L, -H < z < 0\},$$

$$G_1 = \{-\infty < x < \infty, y = \pm L\}, \quad G_2 = \{x = \pm\infty, -L < y < L\}.$$

Let us introduce $\Omega_T = \Omega \times [0, T]$, $\Pi_T = \Pi \times [0, T]$, where $t \in [0, T]$ is time. The problem is formulated in terms of the vertical displacement of the ice plate $w(x, y, t)$ and the velocity potential of the flow $\varphi(x, y, z, t)$ in the context of the theory of linear hydroelasticity [1].

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An irrotational flow of an ideal fluid in Ω_T with the velocity potential $\varphi(x, y, z, t)$ is considered. The potential φ satisfies the Laplace equation in the flow domain

$$\Delta\varphi(x, y, z, t) = 0, \quad (x, y, z, t) \in \Omega_T. \quad (1)$$

The boundary conditions of impermeability in Γ_1 are

$$\varphi_y = 0 \quad (y = \pm L), \quad \varphi_z = 0 \quad (z = -H), \quad (2)$$

and the linearised kinematic condition and Bernoulli integral in Γ_2 are

$$\varphi_z(x, y, 0, t) = w_t(x, y, t), \quad p(x, y, 0, t) = -\rho_l\varphi_t(x, y, 0, t) - \rho_lgw(x, y, t), \quad (3)$$

where $p(x, y, 0, t)$ is the hydrodynamic pressure on the ice-fluid interface, g is the gravitational acceleration. We assume that fluid oscillations are damped out far away from the load. This boundary condition in Γ_3 is

$$\varphi(x, y, z, t), \varphi_x(x, y, z, t) \rightarrow 0 \quad (|x| \rightarrow \infty). \quad (4)$$

The ice displacement $w(x, y, t)$ satisfies the equation of a thin viscoelastic plate in the ice plate domain

$$D \left(1 + \tau \frac{\partial}{\partial t} \right) \nabla^4 w + Mw_{tt} = P(x, y, t) + p(x, y, 0, t), \quad (x, y, t) \in \Pi_T. \quad (5)$$

The initial conditions are

$$w(x, y, 0) = w^1(x, y), \quad w_t(x, y, 0) = w^2(x, y), \quad (x, y) \in \Pi.$$

The clamped conditions in G_1 are

$$w = 0, \quad w_y = 0 \quad (y = \pm L), \quad (6)$$

and the damping conditions far away from the load in G_2 are

$$w = 0, \quad w_x = 0 \quad (|x| \rightarrow \infty). \quad (7)$$

The initial-boundary value problem (1)–(7) describes oscillations of an ice plate clamped to channel walls under the action of external load $P(x, y, t)$. Here $\tau = \eta/E$ is the retardation time; η is the viscosity of ice; $\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4}$; $M = \rho_i h_i$ is the mass ratio; ρ_i is the density of ice; $\varphi(x, y, 0, t)$ is the velocity potential at the lowest boundary of the ice plate; the external load is a smooth localized function $P(x, y, t)$.

Let a pair of functions $w(x, y, t)$ and $\varphi(x, y, z, t)$ be a solution of the system of equations (1)–(7) if these functions are defined in Ω_T and Π_T and they have the following properties: (a) functions $w(x, y, t)$ and $\varphi(x, y, z, t)$ satisfy system of equations (1)–(7), initial and boundary conditions and they are continuous in Π_T and Ω_T , respectively; (b) functions $w(x, y, t)$, $w_x(x, y, t)$, $\varphi(x, y, z, t)$ and $\varphi_x(x, y, z, t)$ vanish when $|x| \rightarrow \infty$; (c) all derivatives up to third order of $w(x, y, t)$ exist and they are continuous in Π_T and bounded when $|x| \rightarrow \infty$. The main result of this paper is the following theorem.

Theorem 1. *Problem (1)–(7) has unique classical solution.*

The system of equations (1)–(7) describes forced oscillations of an ice plate clamped to two walls of a channel (parameters of the ice plate and channel may vary). Joint systems of unsteady equations of a thin viscoelastic or elastic plate and fluid were investigated numerically and analytically for various initial-boundary conditions (see, for example, [1–4] etc). There

is an excellent review of the problems with a thin unbounded ice [1]. Unsteady problems of the ice motion under the action of external load described by equations (1)–(5) in unbounded domain were analytically and numerically investigated [3, 4]. Free oscillations of an ice cover described by equations (1)–(4) without taking into account damping and conditions (4) and (7) were studied [5]. Problems of oscillations of an ice plate clamped to a vertical cylinder in a close mathematical formulation were studied [6, 7]. Problems with one wall, in particular, in non-linear formulation were considered [8]. The solution of the problem with two walls for given external load was studied numerically by the normal mode method [9].

Two basic approaches were used to study problems. The first approach is the numerical study of a steady-state solution of the problem in the coordinate system moving together with the load. In this coordinate system the ice displacement and the velocity potential do not depend on time, and solution is obtained using numerical methods. Second approach is to solve unsteady problem, where the ice displacement is represented by the series of integrals using the Fourier transform. These integrals depend on time. Parameters of the ice displacement are determined using asymptotic methods as limiting values of forced unsteady hydroelastic waves which are developed with time. Both approaches do not answer the question of existence of the solution. However the ice displacement and strain distributions caused by the moving load are calculated. Comparing calculated strain distribution with critical value allow one to predict the area where the ice is broken [8, 10].

Solvability of initial-boundary value problems for the Euler equations and unsteady equation of a thin plate were studied in detail separately [11–16]. Initial-boundary value problems for unsteady Euler equations of a potential flow of a homogeneous liquid with a free surface were considered in detail [11]. Solvability of initial-boundary value problems for the Euler equations with a free surface was considered [12]. Uniqueness of the solution of the flow problem with a given vortex for the unsteady Euler equations of an ideal incompressible fluid was proved [13]. Equation (5) determines oscillations of a thin plate within the Kirchhoff-Love hypothesis of the linear theory of elasticity [17]. This equation is known as the Euler-Bernoulli beam equation in one dimensional case with $\tau = 0$. Existence of a classical solution of boundary value problems for the elastic beam equation with a nonlinear right-hand side and fixed edges of the beam was proved [18]. Solvability of the initial-boundary value problem for oscillations of a viscoelastic beam with inhomogeneous boundary conditions was studied using variational methods [19]. Solvability of problems with rigid inclusions and cracks in thin elastic plates within the Kirchhoff-Love hypotheses or Timoshenko hypotheses was studied in detail (see, in example, [14]). Existence of a solution of the bending problem of an elastic plate with rigid inclusion was proved [15]. Unique solvability of the problem of joining of two elastic homogeneous beams, one of which is the Euler-Bernoulli beam and the other is the Timoshenko beam, was proved [16]. In this paper we study the uniqueness of the solution of the joint problem of a plate bending and motion of an ideal fluid. In order to prove the Theorem 1 we used approaches developed before [13].

2. Uniqueness of viscoelastic oscillations

Let us assume that two different non-trivial solutions w_1, φ_1 and w_2, φ_2 of system (1)–(7) exist. Functions $w = w_1 - w_2$ and $\varphi = \varphi_1 - \varphi_2$ satisfy the following initial-boundary value problem

$$D \left(1 + \tau \frac{\partial}{\partial t} \right) \nabla^4 w + M w_{tt} = -\rho_l \varphi_t(x, y, 0, t) - \rho_l g w, \quad (x, y, t) \in \Pi_T, \quad (8)$$

$$\Delta\varphi(x, y, z, t) = 0, \quad (x, y, z, t) \in \Omega_T, \quad (9)$$

$$\varphi_z = w_t \quad (z = 0), \quad \varphi_y = 0 \quad (y = \pm L), \quad \varphi_z = 0 \quad (z = -H), \quad (10)$$

$$w = 0, \quad w_y = 0 \quad (y = \pm L), \quad (11)$$

$$\varphi(x, y, z, t), \varphi_x(x, y, z, t), w(x, y, t), w_x(x, y, t) \rightarrow 0 \quad (|x| \rightarrow \infty). \quad (12)$$

$$w(x, y, 0) = 0, \quad w_t(x, y, 0) = 0. \quad (13)$$

We are to prove that the solution of problem (8)–(13) is nothing but $w = 0$ and $\varphi = 0$.

The solution (φ, w) of problem (8)–(13) has the following properties

$$\int_{\Pi} w(x, y, t) d\Pi = 0, \quad t \in [0, T], \quad (14)$$

$$\int_{\Omega} |\nabla\varphi(x, y, z, t)|^2 d\Omega = \int_{\Pi} \varphi(x, y, 0, t) w_t(x, y, t) d\Pi, \quad t \in [0, T], \quad (15)$$

$$\int_{\Omega} |\nabla\varphi(x, y, z, 0)|^2 d\Omega = 0, \quad \varphi(x, y, z, 0) = 0. \quad (16)$$

Integrating equation (9) in $\Omega_R = \{-R < x < R, -L < y < L, -H < z < 0\}$ and using the Divergence theorem, we obtain

$$\int_{\Omega_R} \Delta\varphi d\Omega_R = \int_{\Gamma_{1R}^1} \varphi_y \Big|_{y=-L}^{y=L} d\gamma + \int_{\Gamma_{1R}^2 \cup \Gamma_{2R}} \varphi_z \Big|_{z=-H}^{z=0} d\gamma + \int_{\Gamma_{3R}} \varphi_x \Big|_{x=-R}^{x=R} d\gamma = 0.$$

Here index R in Γ denotes corresponding boundaries of Ω_R . Using conditions (10)–(12), in the limit $R \rightarrow \infty$ the last equation provides

$$\int_{\Pi} \varphi_z(x, y, 0, t) d\Pi = \int_{\Pi} w_t(x, y, t) d\Pi = \frac{d}{dt} \int_{\Pi} w(x, y, t) d\Pi = 0.$$

Using condition (13), from this equation we arrive at (14). Now we multiply equation (9) by φ and then integrate the result in Ω_R . After applying the Divergence theorem, we obtain

$$\begin{aligned} \int_{\Omega_R} \varphi \Delta\varphi d\Omega_R &= - \int_{\Omega_R} |\nabla\varphi|^2 d\Omega_R + \int_{\Gamma_R} \varphi (\nabla\varphi \cdot \vec{n}) d\Gamma_R = \\ &= - \int_{\Omega_R} |\nabla\varphi|^2 d\Omega_R + \int_{\Gamma_{1R}^1} (\varphi \varphi_y) \Big|_{y=-L}^{y=L} d\gamma + \int_{\Gamma_{1R}^2 \cup \Gamma_{2R}} (\varphi \varphi_z) \Big|_{z=-H}^{z=0} d\gamma + \int_{\Gamma_{3R}} (\varphi \varphi_x) \Big|_{x=-R}^{x=R} d\gamma = 0. \end{aligned}$$

Using conditions (10)–(12), in the limit $R \rightarrow \infty$ we obtain from the last relation

$$- \int_{\Omega} |\nabla\varphi|^2 d\Omega + \int_G \varphi \frac{\partial\varphi}{\partial z}(x, y, 0, t) dG = - \int_{\Omega} |\nabla\varphi|^2 d\Omega + \int_G \varphi(x, y, 0, t) w_t dG = 0.$$

Condition (15) is obviously obtained from this relation. After taking $t = 0$ in (15), we have

$$\int_{\Omega} |\nabla\varphi(x, y, z, 0)|^2 d\Omega = \int_{\Pi} \varphi(x, y, 0, 0) w_t(x, y, 0) d\Pi = 0.$$

Using (13), from this equation we arrive at (16).

Equation (9) provides

$$\frac{\partial}{\partial t} \left(D\tau \nabla^4 w + D \int_0^t \nabla^4 w d\tau + Mw_t + \rho_l \varphi(x, y, 0, t) + \rho_l g \int_0^t w d\tau \right) = 0. \quad (17)$$

The expression in brackets in (17) is equal to zero for $t = 0$ due to conditions (13) and (16). Taking this into account, we obtain

$$D\tau \nabla^4 w + D \int_0^t \nabla^4 w d\tau + Mw_t + \rho_l \varphi + \rho_l g \int_0^t w d\tau = 0. \quad (18)$$

Multiplying equation (18) by $w_t(x, y, t)$ and then integrating the result in $\Pi_R = \{-R < x < R, -L < y < L\}$, we get

$$\begin{aligned} D\tau \int_{\Pi_R} w_t \nabla^4 w \, d\Pi_R + D \int_{\Pi_R} \left(\int_0^t \nabla^4 w d\tau \right) w_t \, d\Pi_R + M \int_{\Pi_R} w_t^2 \, d\Pi_R + \\ + \rho_l \int_{\Pi_R} \varphi w_t \, d\Pi_R + \rho_l g \int_{\Pi_R} \left(\int_0^t w d\tau \right) w_t \, d\Pi_R = 0. \end{aligned} \quad (19)$$

Using (15), the fourth term in the left hand side of (19) is

$$\int_{R \rightarrow \infty} \int_{\Pi_R} \varphi(x, y, 0, t) w_t(x, y, t) \, d\Pi_R = \int_{\Omega} |\nabla \varphi(x, y, 0, t)|^2 \, d\Omega.$$

Let us transform all other terms in (19).

1. The first term in the left hand side of (19) is

$$\begin{aligned} \int_{\Pi_R} w_{xxxx} w_t \, d\Pi_R &= \frac{1}{2} \frac{d}{dt} \int_{\Pi_R} w_{xx}^2 \, d\Pi_R + \int_{-L}^L w_{xxx} w_t \Big|_{x=-R}^{x=R} \, dy - \int_{-L}^L w_{xx} w_{xt} \Big|_{x=-R}^{x=R} \, dy, \\ \int_{\Pi_R} w_{yyyy} w_t \, d\Pi_R &= \frac{1}{2} \frac{d}{dt} \int_{\Pi_R} w_{yy}^2 \, d\Pi_R + \int_{-R}^R w_{yyy} w_t \Big|_{y=-L}^{y=L} \, dx - \int_{-R}^R w_{yy} w_{yt} \Big|_{y=-L}^{y=L} \, dx, \\ \int_{\Pi_R} w_{xxyy} w_t \, d\Pi_R &= \frac{1}{2} \frac{d}{dt} \int_{\Pi_R} w_{xy}^2 \, d\Pi_R + \int_{-L}^L w_{xyy} w_t \Big|_{x=-R}^{x=R} \, dy - \int_{-R}^R w_{xy} w_{xt} \Big|_{y=-L}^{y=L} \, dx. \end{aligned}$$

2. We introduce function $u = \int_0^t w d\tau$, $u_t = w$, $u_{tt} = w_t$. Then the second term in the left hand side of (19) is

$$\begin{aligned} \int_{\Pi_R} u_{xxxx} u_{tt} \, d\Pi_R &= \frac{1}{2} \frac{d^2}{dt^2} \int_{\Pi_R} u_{xx}^2 \, d\Pi_R - \int_{\Pi_R} u_{xxt}^2 \, d\Pi_R + \int_{-L}^L u_{xxx} u_{tt} \Big|_{x=-R}^{x=R} \, dy - \int_{-L}^L u_{xx} u_{xtt} \Big|_{x=-R}^{x=R} \, dy, \\ \int_{\Pi_R} u_{yyyy} u_{tt} \, d\Pi_R &= \frac{1}{2} \frac{d^2}{dt^2} \int_{\Pi_R} u_{yy}^2 \, d\Pi_R - \int_{\Pi_R} u_{yyt}^2 \, d\Pi_R + \int_{-R}^R u_{yyy} u_{tt} \Big|_{y=-L}^{y=L} \, dx - \int_{-R}^R u_{yy} u_{ytt} \Big|_{y=-L}^{y=L} \, dx, \\ \int_{\Pi_R} u_{xxyy} u_{tt} \, d\Pi_R &= \frac{1}{2} \frac{d^2}{dt^2} \int_{\Pi_R} u_{xy}^2 \, d\Pi_R - \int_{\Pi_R} u_{xyt}^2 \, d\Pi_R + \int_{-L}^L u_{xyy} u_{tt} \Big|_{x=-R}^{x=R} \, dy - \int_{-R}^R u_{xy} u_{xtt} \Big|_{y=-L}^{y=L} \, dx. \end{aligned}$$

3. Using function $u = \int_0^t w d\tau$, the last term in the left hand side of (19) is

$$\int_{\Pi_R} \left(\int_0^t w d\tau \right) w_t \, d\Pi_R = \frac{1}{2} \frac{d^2}{dt^2} \int_{\Pi_R} u^2 \, d\Pi_R - \int_{\Pi_R} u_t^2 \, d\Pi_R.$$

Summarizing all relations for the integrals, we obtain

$$\begin{aligned}
 & M \int_{\Pi_R} w_t^2 d\Pi_R + \rho_l \int_{\Omega} \varphi w_t d\Omega + D\tau \frac{1}{2} \frac{d}{dt} \left(\int_{\Pi_R} w_{xx}^2 d\Pi_R + \int_{\Pi_R} w_{yy}^2 d\Pi_R + 2 \int_{\Pi_R} w_{xy}^2 d\Pi_R \right) + \\
 & + D \frac{1}{2} \frac{d^2}{dt^2} \left(\int_{\Pi_R} u_{xx}^2 d\Pi_R + \int_{\Pi_R} u_{yy}^2 d\Pi_R + 2 \int_{\Pi_R} u_{xy}^2 d\Pi_R \right) + \rho_l \frac{1}{2} \frac{d^2}{dt^2} \int_{\Pi_R} u^2 d\Pi_R = \\
 & = I_{\Gamma}^x + I_{\Gamma}^y + D \left(\int_{\Pi_R} u_{xxt}^2 d\Pi_R + \int_{\Pi_R} u_{yyt}^2 d\Pi_R + 2 \int_{\Pi_R} u_{xyt}^2 d\Pi_R \right) + \rho_l \int_{\Pi_R} u_t^2 d\Pi_R, \quad (20)
 \end{aligned}$$

where

$$\begin{aligned}
 I_{\Gamma}^x &= D\tau \left(\int_{-L}^L w_{xx} w_{xt} \Big|_{x=-R}^{x=R} dy - \int_{-L}^L w_{xxx} w_t \Big|_{x=-R}^{x=R} dy - 2 \int_{-L}^L w_{xyy} w_t \Big|_{x=-R}^{x=R} dy \right) + \\
 & + D \left(\int_{-L}^L u_{xx} u_{xtt} \Big|_{x=-R}^{x=R} dy - \int_{-L}^L u_{xxx} u_{tt} \Big|_{x=-R}^{x=R} dy - 2 \int_{-L}^L u_{xyy} u_{tt} \Big|_{x=-R}^{x=R} dy \right), \\
 I_{\Gamma}^y &= D\tau \left(\int_{-R}^R w_{yy} w_{yt} \Big|_{y=-L}^{y=L} dx - \int_{-R}^R w_{yyy} w_t \Big|_{y=-L}^{y=L} dx + 2 \int_{-R}^R w_{xy} w_{xt} \Big|_{y=-L}^{y=L} dx \right) + \\
 & + D \left(\int_{-R}^R u_{yy} u_{ytt} \Big|_{y=-L}^{y=L} dx - \int_{-R}^R u_{yyy} u_{tt} \Big|_{y=-L}^{y=L} dx + 2 \int_{-R}^R u_{xy} u_{xtt} \Big|_{y=-L}^{y=L} dx \right).
 \end{aligned}$$

In the limit $R \rightarrow \infty$ in (20) for $y = \pm L$ we have

$$w = w_y = w_t = w_{yt} = 0,$$

and for $|x| \rightarrow \infty$

$$(w, w_x, w_t, w_{xt}) \rightarrow 0.$$

Additionally, for $t = 0$ we have

$$w = w_t = u = w_x = w_y = w_{xy} = u_x = u_y = u_{xy} = 0.$$

Thus the boundary integrals in (20) are equal to 0, $I_{\Gamma}^x = 0$ and $I_{\Gamma}^y = 0$.

We introduce

$$Y(t) = \int_{\Pi} (w_{xx}^2 + w_{yy}^2 + 2w_{xy}^2) d\Pi, \quad Z(t) = \int_{\Pi} (u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2) d\Pi.$$

Using (15), relation (20) becomes

$$\begin{aligned}
 & M \int_{\Pi} w_t^2 d\Pi + \rho_l \int_{\Omega} |\nabla \varphi|^2 d\Omega + \frac{1}{2} D\tau \frac{d}{dt} Y(t) + \frac{1}{2} D \frac{d^2}{dt^2} Z(t) + \\
 & + \frac{1}{2} \rho_l \frac{d^2}{dt^2} \int_{\Pi} u^2 d\Pi = DY(t) + \rho_l \int_{\Pi} u_t^2 d\Pi. \quad (21)
 \end{aligned}$$

Using relation

$$w^2(x, y, t) = 2 \int_0^t w(x, y, \tau) w_{\tau}(x, y, \tau) d\tau$$

and Holder inequality, we obtain

$$\int_{\Pi} w^2(x, y, t) d\Pi \leq 2 \int_0^t \left(\int_{\Pi} w^2(x, y, \tau) d\Pi \right)^{1/2} \left(\int_{\Pi} w_{\tau}^2(x, y, \tau) d\Pi \right)^{1/2} d\tau.$$

It follows from the last relation that

$$\int_{\Pi} w^2(x, y, t) d\Pi \leq T \int_0^t \left(\int_{\Pi} w_{\tau}^2(x, y, \tau) d\Pi \right) d\tau. \quad (22)$$

Let us introduce

$$\tilde{Y}(t) = M \int_0^t \left(\int_{\Pi} w_{\tau}^2 d\Pi \right) d\tau + \frac{1}{2} D\tau Y(t), \quad C_1 = \max\left(\frac{2}{\tau}, \frac{T\rho_l}{2M}\right),$$

Using (21) and (22), we obtain the following inequality

$$\frac{d}{dt} \tilde{Y}(t) + \frac{d^2}{dt^2} \left(\frac{1}{2} DZ(t) + \frac{1}{2} \rho_l \int_{\Pi} u^2 d\Pi \right) \leq C_1 \tilde{Y}(t).$$

Then we have

$$\begin{aligned} \frac{d}{dt} (\tilde{Y} e^{-C_1 t}) + e^{-C_1 t} \frac{d^2}{dt^2} \left(\frac{1}{2} DZ(t) + \frac{1}{2} \rho_l \int_{\Pi} u^2 d\Pi \right) &= \\ &= \frac{d}{dt} \left(\tilde{Y} e^{-C_1 t} + e^{-C_1 t} \frac{d}{dt} \left(\frac{1}{2} DZ(t) + \frac{1}{2} \rho_l \int_{\Pi} u^2 d\Pi \right) \right) + \\ &\quad + C_1 e^{-C_1 t} \frac{d}{dt} \left(\frac{1}{2} DZ(t) + \frac{1}{2} \rho_l \int_{\Pi} u^2 d\Pi \right) \leq 0. \end{aligned} \quad (23)$$

Upon integrating (23) with respect to t from 0 to t_1 and taking into account that

$$\tilde{Y}(0) = 0, \quad \left. \frac{dZ(t)}{dt} \right|_{t=0} = 0, \quad \left. \frac{d}{dt} \int_{\Pi} u^2 d\Pi \right|_{t=0} = 0,$$

we obtain

$$\begin{aligned} &\tilde{Y} e^{-C_1 t_1} + e^{-C_1 t_1} \frac{d}{dt_1} \left(\frac{1}{2} DZ(t_1) + \frac{1}{2} \rho_l \int_{\Pi} u^2 d\Pi \right) + \\ &+ C_1 e^{-C_1 t_1} \left(\frac{1}{2} DZ(t_1) + \frac{1}{2} \rho_l \int_{\Pi} u^2 d\Pi \right) + C_1^2 \int_0^{t_1} e^{-C_1 t} \left(\frac{1}{2} DZ(t) + \frac{1}{2} \rho_l \int_{\Pi} u^2 d\Pi \right) \leq 0. \end{aligned} \quad (24)$$

The first, third and fourth terms in the left hand side of the last inequality are nonnegative. Thus

$$\frac{d}{dt_1} \left(\frac{1}{2} DZ(t_1) + \frac{1}{2} \rho_l \int_{\Pi} u^2 d\Pi \right) \leq 0.$$

We conclude from the last inequality that $Z(t) = 0$, $u(x, y, t) = 0$. Considering (23), (22) and (21), it is easy to show that $w = 0$, $\nabla\varphi = 0$, $\varphi = 0$.

Theorem 1 is proved.

3. Uniqueness of elastic oscillations

Viscous damping is not well understood. Because of this, damping coefficient $\tau = 0$ is neglected in many studies. We are to prove the analogue of the Theorem 1 for the case $\tau = 0$. In this case system of equation (8)–(13) becomes

$$D\nabla^4 w + M w_{tt} = -\rho_l \varphi_t(x, y, 0, t) - \rho_l g w, \quad (x, y, t) \in \Pi_T, \quad (25)$$

$$\Delta\varphi(x, y, z, t) = 0, \quad (x, y, z, t) \in \Omega_T, \tag{26}$$

$$\varphi_z = w_t \quad (z = 0), \quad \varphi_y = 0 \quad (y = \pm L), \quad \varphi_z = 0 \quad (z = -H), \tag{27}$$

$$w = 0, \quad w_y = 0 \quad (y = \pm L), \tag{28}$$

$$\varphi(x, y, z, t), \varphi_x(x, y, z, t), w(x, y, t), w_x(x, y, t) \rightarrow 0 \quad (|x| \rightarrow \infty). \tag{29}$$

$$w(x, y, 0) = 0, \quad w_t(x, y, 0) = 0. \tag{30}$$

Let us that the solution (φ, w) of problem (25)–(30) satisfies conditions (14)–(16).

The equation (25) gives

$$D\nabla^4 w + Mw_{tt} + \rho_l \varphi_t + \rho_l g w_t = 0. \tag{31}$$

Multiplying equation (31) by $w_t(x, y, t)$ and then integrating the result over $\Pi_R = \{-R < x < R, -L < y < L\}$, we obtain

$$D \int_{\Pi_R} (\nabla^4 w) w_t \, d\Pi_R + M \int_{\Pi_R} w_{tt} w_t \, d\Pi_R + \rho_l \int_{\Pi_R} \varphi_t w_t \, d\Pi_R + \rho_l g \int_{\Pi_R} w w_t \, d\Pi_R = 0. \tag{32}$$

Taking into account Section 2 and integrating by parts the terms of equality (32), we arrive at the identity

$$\begin{aligned} \frac{M}{2} \frac{d}{dt} \int_{\Pi_R} w_t^2 \, d\Pi_R + \rho_l \int_{\Pi_R} \varphi_t w_t \, d\Pi_R + \frac{D}{2} \frac{d}{dt} \int_{\Pi_R} w_{xx}^2 + \\ + w_{yy}^2 + 2w_{xy}^2 \, d\Pi_R + \frac{\rho_l g}{2} \frac{d}{dt} \int_{\Pi_R} w^2 \, d\Pi_R = I_\Gamma, \end{aligned} \tag{33}$$

where the sum of boundary integrals I_Γ is defined in the same way as the sum of boundary integrals I_Γ^x and I_Γ^y in equation (20). Using (27)–(29), in the limit $R \rightarrow \infty$ in (33) we obtain

$$\frac{d}{dt} \left(\frac{M}{2} \int_{\Pi} w_t^2 \, d\Pi + \frac{\rho_l}{2} \int_{\Omega} |\nabla\varphi|^2 \, d\Omega + \frac{D}{2} \int_{\Pi} Y(x, y, t) \, d\Pi + \frac{\rho_l g}{2} \int_{\Pi} w^2 \, d\Pi \right) = 0. \tag{34}$$

For $t = 0$ we have

$$w = w_t = w_{xx} = w_{xy} = w_{yy} = 0. \tag{35}$$

The expression in brackets in (34) for $t = 0$ is equal to zero due to conditions (16) and (35).

Taking this into account, we obtain

$$\frac{M}{2} \int_{\Pi} w_t^2 \, d\Pi + \frac{\rho_l}{2} \int_{\Omega} |\nabla\varphi|^2 \, d\Omega + \frac{D}{2} \int_{\Pi} Y(x, y, t) \, d\Pi + \frac{\rho_l g}{2} \int_{\Pi} w^2 \, d\Pi = 0.$$

All terms in the left hand side of the last relation are nonnegative. Then we can conclude that $w_t = 0, \nabla\varphi(x, y, z, t) = 0, Y(x, y, t) = 0, w(x, y, t) = 0$ and, consequently, $\varphi(x, y, z, t) = 0$.

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Единственность решения задачи о колебаниях ледового покрова в канале

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В работе, в рамках линейной теории гидроупругости, рассматривается начально-краевая задача для математических моделей упругих и вязкоупругих колебаний тонкой ледовой пластины в бесконечном канале, вызванных внешней нагрузкой. Вязкоупругие свойства льда моделируются на основе реологического закона Кельвина-Фойгта. Совместная система уравнений динамики пластины и идеальной жидкости замыкается условиями жесткого защемления для пластины на стенках канала, условиями непротекания для потенциала скорости течения и условиями затухания колебаний на бесконечности. Доказана теорема единственности классического решения поставленной начально-краевой задачи.

Ключевые слова: уравнения Эйлера, вязкоупругие колебания, ледовый покров, внешняя нагрузка, единственность.