УДК 517.55

On an Analog of Descartes' Rule of Signs and the Budan-Fourier Theorem for Entire Functions

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Received 06.11.2017, received in revised form 10.12.2017, accepted 20.01.2018

It is proven the analog of Descartes' rule of signs and the Budan-Fourier theorem for entire functions.

Keywords: entire function, Descartes' rule, Budan-Fourier theorem.
DOI: 10.17516/1997-1397-2018-11-3-317-321.

To determine the number of real roots of a real polynomial on a segment of the real axis one frequently chooses to apply the classical Sturm method. However this method is rather cumbersome. Usually when solving this problem Hermite's theorem is used in combination with Descartes' rule of signs and the Budan-Fourier theorem (see, e.g., [1]).

Theorem 1 (Descartes)**.** *Let $P(x)$ be a real polynomial of degree n*

$$P(x) = a_0x^n + a_1x^{n-1} + \dots + a_n.$$

Denote by $W(a_0, a_1, \dots, a_n)$ the number of sign changes in the sequence of coefficients of the polynomial $a_0, a_1, \dots, a_n$ (zero coefficients are not counted). If all the roots of $P(x)$ are real then the number of positive roots (counted with multiplicities) is equal to $W(a_0, a_1, \dots, a_n)$, otherwise it is less than $W(a_0, a_1, \dots, a_n)$ by an even number.

Theorem 2 (Budan-Fourier)**.** *Let $P(x)$ be a real polynomial of degree n. If all the roots of P on the segment $[a, b]$ are real then their number is equal to the difference*

$$W\big(P(a), P'(a), \dots, P^{(n)}(a)\big) - W\big(P(b), P'(b), \dots, P^{(n)}(b)\big),$$

otherwise it is less than this difference by an even number.

Here it is assumed that $P(a) \neq 0$, $P(b) \neq 0$. If some derivatives at a or b are zero then they are not taken into account in W.

The proofs of these theorems are based on the application of Rolle's theorem.

In chemical kinetics there arise systems of non-algebraic equations (see, e.g, [2, 3, 4]). By eliminating unknowns from such systems, we obtain an entire transcendental function (see [5]). Therefore, there naturally arises a problem of an analogue of Descartes' rule of signs for the number of positive roots of an entire function, and of an analogue of the Budan-Fourier theorem for entire functions.

Suppose that an entire function $f(z)$ with real Taylor coefficients has the form

$$f(z) = 1 + b_1z + \dots + b_nz^n + \dots . \tag{1}$$

Denote the number of sign changes in the sequence

$$1, b_1, \dots b_n, \dots . \tag{2}$$

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by W_-, and the number of sign non-changes in (2) by W_+. If there are zero elements in the sequence (2) then we simply omit them leaving in (2) only non-zero elements. For an entire transcendental function f the numbers W_-, W_+ can be finite or infinite.

Proposition 1. *If W_- is finite then f has finitely many positive roots.*

Proof. Let W_- be finite and equal to m and let the sequence of coefficients have no sign changes after some b_s. Then the sequence of coefficients of the derivative $f^{(s)}$ has no sign changes. Either they are all negative, or all are positive. We assume that they are all positive (recall that the zero coefficients are removed). Then the function $f^{(s)}$ is positive on the positive part of the real axis.

If the function f has an infinite number of positive roots, then by Rolle's theorem its derivative f' also has an infinite number of positive roots, and so on. Therefore, the function $f^{(s)}$ must also have an infinite number of positive roots, which is impossible.

Rolle's theorem also shows that the number of positive roots (counted with multiplicities) does not exceed $m + 1$. □

Corollary 1. *If the number of positive roots of f is infinite, then the number of sign changes W_- is also infinite.*

If the number of sign changes W_- is infinite, then the number of positive roots can be finite.

Example 1. Let $f(z) = 2 - \sin z$. Then the number of sign changes W_- is infinite, but the function $f(z)$ has no real roots.

For entire functions there is a refinement of the previous statements, namely, an analogue of Descartes' rule of signs for polynomials (Theorem 1). It can be found in [6, Division 5, Chapter 1, Sec. 4].

Theorem 3. *Suppose that for an entire function $f(z)$ of the form (1) with real Taylor coefficients the number of positive roots (counted with multiplicities) is equal to N_+, and the number of sign changes in the sequence of Taylor coefficients of f is equal to W_-. If the number W_- is finite, then N_+ is also finite and the difference $W_- - N_+$ is a nonnegative even number.*

Now we discuss the question for which functions f the number of sign changes coincides with the number of positive roots.

Let all roots of f be real and the Hadamard expansion of the function $f(z)$ has the form

$$f(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{\beta_j}\right), \tag{3}$$

where β_j are real.

This property have, for example, entire functions of not higher than the first order and of minimal type (see [7, Chapter 7]).

Theorem 4. *If for a function of the form (3) the number of sign changes W_- in the sequence of Taylor coefficients is equal to m, then the number of positive roots (counted with multiplicities) is also equal to m. If the sequence of Taylor coefficients of the function has an infinite number of sign changes, then the number of positive roots of $f(z)$ is infinite.*

Proof. Let the number W_- be finite and equal to m. The number of positive roots of the function is equal to k (it can be zero) and by Theorem 3 we have $k \leqslant m$. Let the roots be $\alpha_1, \ldots, \alpha_k$. If the number of negative roots is also finite, then f is a polynomial and the theorem follows from Descartes' rule of signs (Theorem 1). Let the number of negative roots $f(z)$ be infinite and they be $-\gamma_1, \ldots, -\gamma_s, \ldots$.

Consider the polynomial

$$P_s(z)=\prod_{j=1}^{s}\Big(1+\frac{z}{\gamma_j}\Big)\prod_{j=1}^{k}\Big(1-\frac{z}{\alpha_j}\Big).$$

Then the number of sign changes in the sequence of coefficients of the polynomial $P_s(z)$ for each s is equal to k by Descartes' Theorem 1. We denote its coefficients by $b_0^{(s)},\dots,b_s^{(s)}, b_{s+1}^{(s)}=$ $=0,\dots(b_0^{(s)}=1)$. Since the polynomials $P_s(z)$ converge uniformly to the function $f(z)$ as $s\to\infty$, for each fixed j the sequence $b_j^{(s)}\to b_j$. Therefore, for each sign change of the function $f(z)$, the corresponding coefficients of the polynomial $P_s(z)$ also change a sign for sufficiently large s. Therefore, $m\leqslant k$, and $m=k$. Let the number of sign changes W_- be infinite, then two cases are possible: the number of negative zeros is either finite or infinite. Let the number of negative zeros of $f(z)$ be finite and equal to p. If the number of positive roots is finite and equal to k, then

$$f(z)=\prod_{j=1}^{p}\Big(1+\frac{z}{\gamma_j}\Big)\prod_{j=1}^{k}\Big(1-\frac{z}{\alpha_j}\Big),$$

and by Theorem 1 of Descartes the number of sign changes is k, which contradicts the assumption,

If the number of negative roots is infinite and the number of positive roots is finite and equal to k, then by an argument above for sufficiently large s the number of sign changes for the polynomial $P_s(z)$ is equal to k and it is not less than the number of sign changes for the function f which contradicts the assumption. □

The condition for all roots of an entire function $f(z)$ to be real are studied in [8]. For polynomials they turn into the classical Hermite theorem (see, for example, [9]).

If the function f does not admit the form (3), then Theorem 4 is not true.

Example 2. Consider $f(z)=(1-z)e^{-z}$. Its order is equal to 1, the number of sign changes in the sequence of its coefficients is infinite but it has only one real root.

A situation is possible where the canonical product of a function can be a function of not higher than the first order and of minimal type, while the function itself is not. In this case, the question arises on factorization of the function, i.e. on isolation of the canonical product. This question was considered in [10].

As a corollary, we obtain a theorem on the number of negative roots of $f(z)$.

Corollary 2. *Suppose that for an entire function $f(z)$ of the form (3) with real coefficients, the number of negative roots (counted with multiplicities) is equal to N_-, and the number of sign changes in the sequence of numbers*

$$1,\,-b_1,\dots,\,(-1)^k b_k,\dots \tag{4}$$

is equal to W_+. If the number W_+ is finite, then the number N is also finite, and $N=N_-$. If the sequence (4) has an infinite number of sign changes, then the number of negative roots of $f(z)$ is infinite.

Proof. This follows in an obvious way from Theorem 4 when z is replaced by $-z$. □

Note that the number W_+ does not need to coincide with the number of sign non-changes in the sequence of Taylor coefficients of the function f, since some Taylor coefficients may vanish. However, if all the coefficients of a function are non-zero, then the number of negative roots of $f(z)$ coincides with the number of sign non-changes in the sequence of the coefficients of the function.

Let us now consider the analogue of the Budan-Fourier theorem for entire functions.

Corollary 3. *Suppose that the function has the form* (3)*, a is a real number and $f(a) \neq 0$. Then the number of real roots of the function f (counted with multiplicities) is equal to the number of sign changes in the sequence*

$$f(a),\ f'(a), \dots,\ f^{(k)}(a), \dots \tag{5}$$

(zero values are not taken into account). This means that if the number of sign changes in (5) *is finite and equal to s, then the number of roots (of the function f lying to the right of a is also equal to s. If the number of sign changes in (5) is infinite, then the number of roots of the function lying to the right of a is also infinite.*

Proof. This also follows from Theorem 4 when $w = z - a$. □

Consider the case when $a, b \in \mathbb{R}$, $a < b$ and $f(a) \neq 0$, $f(b) \neq 0$. Note that if for a function of the form (3) the number of sign changes in the sequence (5) is finite, then the number of sign changes in the sequence

$$f(b),\ f'(b), \dots,\ f^{(k)}(b), \dots \tag{6}$$

is also finite, since to the right of a there are only finitely many roots of f by Corollary 3, so the number of roots lying to the right of b is also finite.

Since an entire function can not have an infinite number of roots on a finite segment $[a, b]$, the converse is also true: if the number of sign changes in the sequence (6) is finite for a function of the form (3), then the number of sign changes in the sequence (5) is also finite.

Thus we obtain the following result.

Corollary 4. *Suppose that for a function of the form* (3) *the number of sign changes in the sequence* (5) *is finite and equal to p, and the number of sign changes in the sequence* (6) *is finite and equal to q, then the number of roots of the function on* $[a, b]$ *(counting multiplicities) is equal to $p - q$.*

If the number of sign changes in (5) or in (6) is infinite, then the second one is also infinite. In this case nothing can be said about the number of sign changes on the segment $[a, b]$: it may be any finite number.

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Об аналоге правила Декарта и теоремы Бюдана-Фурье для целых функций

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Ключевые слова: целая функция, правило знаков Декарта, теорема Бюдана-Фурье.