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Enumerations of Ideals in Niltriangular Subalgebra of Chevalley Algebras

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Let $N\Phi(K)$ be the niltriangular subalgebra of Chevalley algebra over a field K associated with a root system Φ. We consider certain non-associative enveloping algebras for some Lie algebra $N\Phi(K)$. We also study the problem of enumeration of standard ideals in algebra $N\Phi(K)$ over any finite field K; for classical Lie types this is the problem which was written earlier (2001).

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Introduction

Any Chevalley algebra over a field K is characterized by a root system Φ and a Chevalley basis consisting of elements e_r $(r \in \Phi)$ and a base of suitable Cartan subalgebra [1, Sec. 4.2]. We fix a positive root system $\Phi^+ \subseteq \Phi$. The subalgebra $N\Phi(K)$ with the basis $\{e_r \mid r \in \Phi^+\}$ is said to be a *niltriangular subalgebra*. In the present paper we consider the following problem.

(A) *Find the number of standard ideals of Lie algebra $N\Phi(K)$ over any finite field K.*

For classical Lie types this problem has arisen earlier as Problem 1 in [2]. In these cases Problem (A) had been solved recently by G. P. Egorychev, V. M. Levchuk, and the author. *Standard ideals* of a Lie ring $N\Phi(K)$ are distinguished in Sec. 1.

Main Theorem 2.1 in Sec. 2 solves Problem (A) for exceptional Lie types.

Also, we study enveloping algebras of Lie algebras $N\Phi(K)$. According to [3], an algebra $R = (R, +, \cdot)$ (possibly, non-associative) is called *an enveloping algebra* of a Lie algebra L if L is isomorphic to the algebra $R^{(-)} := (R, +, [\ ,\ ])$, $[a, b] := ab - ba$. (See also Lie-admissible algebras [4, 5].) The well-known enveloping algebra R of Lie algebra $N\Phi(K)$ [3, Proposition 1] has also base $\{e_r \mid r \in \Phi^+\}$ and its choice depends on signs of structural constants of Chevalley basis.

The representation [6] of Lie algebra $N\Phi(K)$ of classical Lie types determines uniquely their enveloping algebra R. If $\Phi \neq D_n$, then all ideals of such enveloping ring R are exactly standard ideals of Lie ring $N\Phi(K)$. By [3], it is not true for Lie type D_n $(n \geqslant 4)$ and, as a corollary, for Lie types E_n $(n = 6, 7, 8)$.

As it is shown in the following section, there exist enveloping algebras of type F_4 both having nonstandard ideals and not having them.

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We use standard notation from [1]. Let $ht(r)$ be the height of $r \in \Phi$. The highest root in Φ^+ is denoted by ρ. The Coxeter number h of Φ is defined by $ht(\rho) + 1 = h(\Phi) = h$.

1. Ideals and enveloping algebras of Lie algebra $N\Phi(K)$

The definition of an enveloping algebra R of an arbitrary finite dimensional Lie algebra L (see Introduction) shows that both algebras can be constructed on the same vector space. Also, every ideal of the enveloping ring R is an ideal of the Lie ring L.

In this section we study certain enveloping algebras of a Lie algebra $N\Phi(K)$ and their ideals. According to [1, Proposition 4.2.2], we have

$$[e_r, e_s] = N_{rs}e_{r+s} = -[e_s, e_r] \ (r + s \in \Phi), \quad [e_r, e_s] = 0 \ (r + s \notin \Phi \setminus \{0\}),$$

where $N_{rs} = \pm 1$ or $|r| = |s| < |r + s|$ and $N_{rs} = \pm 2$ or Φ is of type G_2 and $N_{rs} = \pm 2$ or ± 3.

Proposition 1.1 ([3, Proposition 1]). *A K-algebra with the basis $\{e_r \mid r \in \Phi^+\}$ is an enveloping algebra of $N\Phi(K)$ if the product is defined as follows: $e_r e_s = 0$ when $r+s \notin \Phi$, and if $r+s \in \Phi^+$ and $N_{rs} \geqslant 1$, then $e_r e_s = e_{r+s}$ and $e_s e_r = (1 - N_{rs})e_{r+s}$.*

We distinguish the following ideals in a Lie algebra $N\Phi(K)$ putting on $r \leqslant s$ $(r, s \in \Phi^+)$ if $s - r$ is a linear combination of simple roots with nonnegative coefficients:

$$T(r) := \sum_{r \leqslant s} Ke_s, \qquad Q(r) := \sum_{r < s} Ke_s.$$

Roots r and s are called *incident* ones if $T(r) \subseteq T(s)$ or $T(s) \subseteq T(r)$ (i.e., $s \leqslant r$ or $r \leqslant s$). Any set $\mathcal{L}$ of pairwise non-incident roots in Φ^+ is called *a set of corners in* Φ^+.

If $H \subseteq \sum_{r \in \mathcal{L}} T(r)$ and the inclusion fails under every substitution of $T(r)$ by $Q(r)$, then $\mathcal{L} = \mathcal{L}(H)$ is said to be *a set of corners in* H. By [7], a set $\mathcal{F}(H)$ is said to be a *frame* of H if

$$\mathcal{F}(H) \subseteq \sum_{r \in \mathcal{L}} Ke_r, \quad \mathcal{F}(H) = H \mod Q(\mathcal{L}) \quad \big(Q(\mathcal{L}) = \sum_{r \in \mathcal{L}} Q(r)\big).$$

An ideal H of a Lie ring $N\Phi(K)$ is said to be *standard* if $H = \mathcal{F}(H) + Q(\mathcal{L})$. Evidently, all standard ideals of Lie ring $N\Phi(K)$ are ideals of any enveloping ring R from Proposition 1.1

The representation [6] of Lie algebras $N\Phi(K)$ of classical Lie types determines uniquely their enveloping algebra R. All ideals of such enveloping ring R for $\Phi \neq D_n$ are exactly standard ideals of Lie ring $N\Phi(K)$. By [3], it is not true for Lie type D_n $(n \geqslant 4)$ and also, as a corollary, for Lie types E_n $(n = 6, 7, 8)$. We now show that both cases are possible for Lie type F_4.

Theorem 1.1. *For Lie type F_4 Proposition 1.1 allows to construct enveloping algebras R_1 having nonstandard ideals, and R_2 in which all ideals are standard.*

Proof. Note that the enveloping algebra R from Proposition 1.1 depends on choice of signs of structural constants N_{rs}.

Similarly to [1, Lemma 5.3.1], we use an ordering $\prec$ on the space containing roots Φ such that $r \prec s$ implies $h(r) \leqslant h(s)$. An ordered pair (r, s) of roots is called a *special pair* if $r + s \in \Phi$ and $0 \prec r \prec s$. An ordered pair (r, s) is called *extraspecial* if (r, s) is a special pair and if for all special pairs (r', s') with $r + s = r' + s'$ we have $r \preccurlyeq r'$.

Proposition 1.2. *The signs of the structure constants N_{rs} may be chosen arbitrarily for extraspecial pairs (r, s), and then the structure constants for all pairs are uniquely determined.*

Proof. See [1, Proposition 4.2.2]. □

For the root system Φ of type F_4, we need notation from [8]. The positive root systems of types B_n and C_n [9, Tables I-IV] can be written, respectively, as

$$C_n^+ = \{p_{iv} \mid 0 < |v| \leqslant i \leqslant n, v \neq i\}, \qquad p_{i,mj} = \epsilon_i - m\epsilon_j, \quad 1 \leqslant j \leqslant i \leqslant n, \ m = 0, 1, -1;$$

$$B_n^+ = \{q_{ij} \mid 0 \leqslant |j| < i \leqslant n\}, \qquad q_{i,mj} = \epsilon_i - m\epsilon_j.$$

Then the positive system F_4^+ is represented as the union $C_4^+ \cup B_4^+$ with the given intersection

$$B_4^+ \cap C_4^+ = \{q_{i0}, p_{i,-i} \ (1 \leqslant i \leqslant 4)\}.$$

Also, we use the following diagram from [8]. (The roots are accompanied by the notation (abcd) from [9, Table VIII].)

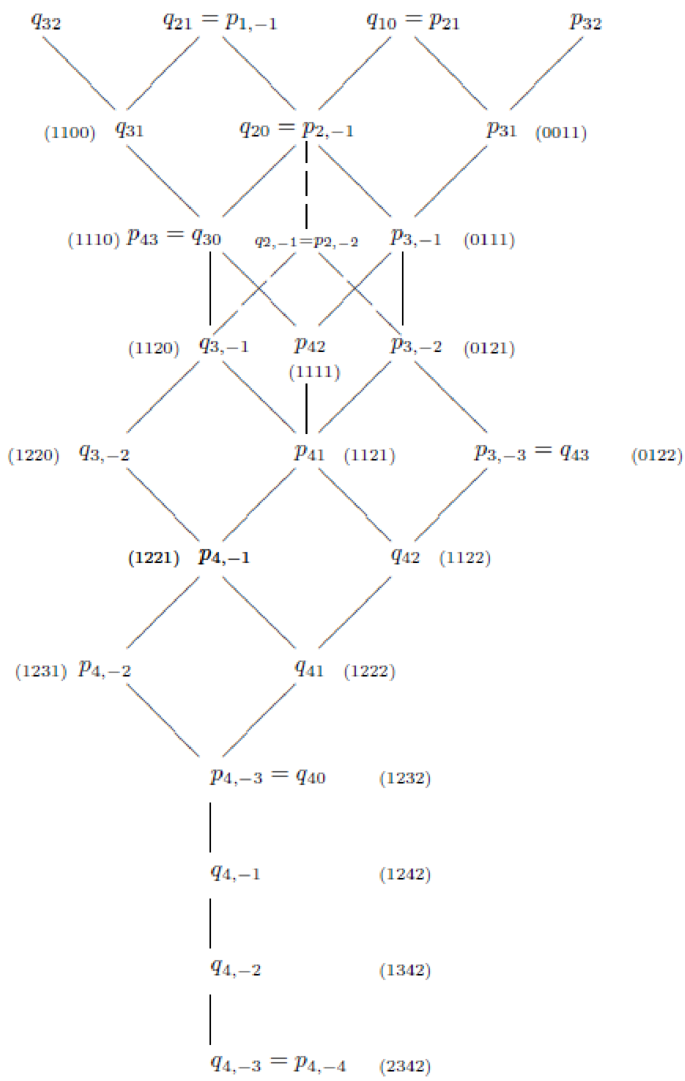

Fig. 1. The positive roots of the system F_4

The relation $q_{32} \prec q_{21} \prec q_{10} \prec p_{32}$ of simple roots determines uniquely the ordering $\prec$ (Fig. 1). Using Proposition 1.1 choose an arbitrary enveloping algebra R for Lie algebra $N\Phi(K)$ of type F_4. Recall that an ideal H of R is standard iff $Q(r) \subseteq H$ for all $r \in \mathcal{L}(H)$.

It is clear that if M is a subset in an ideal H of R and $\mathcal{F}(M) = Ke_r$, then $T(r) \subseteq H$. Let $L_i = \sum\limits_{i \geqslant h(r)} T(r),\ 1 \leqslant i < h$. It is not difficult to prove the following lemma.

Lemma 1.1. *Every ideal $H \subseteq L_4$ in the enveloping ring R is standard.*

Now we construct algebras R_i from Theorem 1.1. Assume $N(r,s) := N_{rs}$, and also,

$$\begin{gathered} N(q_{21}, q_{3,-1}) = 1, \quad N(q_{21}, p_{31}) = -1, \quad N(q_{32}, p_{3,-1}) = 1, \quad N(q_{10}, p_{3,-1}) = -1, \\ N(q_{32}, q_{2,-1}) = -1, \quad N(q_{32}, q_{21}) = -1, \quad N(q_{10}, p_{32}) = -1, \quad N(q_{21}, q_{10}) = -1. \end{gathered} \tag{1}$$

For algebra R_1 we additionally set $N(q_{32}, q_{20}) = -1,\ N(q_{10}, q_{20}) = 2$, and

$$N(p_{32}, q_{2,-1}) = -1, \quad N(p_{32}, q_{30}) = -1, \quad N(q_{10}, q_{31}) = 1.$$

By choosing arbitrarily the remaining structural constants N_{rs} we obtain the algebra R_1. One can see that ideals in the algebra R_1 of the form

$$K(e_{q_{30}} + ce_{q_{2,-1}}) + K(e_{p_{42}} + ce_{p_{3,-2}}) + T(q_{3,-1}) + T(q_{43}) \quad (c \in K^*) \tag{2}$$

are nonstandard. Moreover, It can easily be checked that all other ideals in the algebra R_1 are standard.

Lemma 1.2. *The algebra R_1 has nonstandrd ideals and they are exhausted by ideals* (2).

Further, we use the following lemma to construct algebra R_2 which is not isomorphic to R_1.

Lemma 1.3. *All ideals in the ring R are standard if the following equalities are satisfied:*

$$\begin{aligned} N(q_{21}, q_{3,-1}) &= -N(q_{21}, p_{31}), \quad & N(q_{32}, p_{3,-1}) &= -N(q_{32}, q_{2,-1}), \\ N(q_{10}, p_{3,-1}) &= -N(q_{10}, q_{31}), \quad & N(p_{32}, q_{30}) &= -N(p_{32}, q_{2,-1}), \\ N(q_{21}, p_{31}) &= N(q_{32}, q_{21}), \quad & N(q_{10}, q_{31}) &= -N(q_{10}, p_{32}), \\ N(q_{32}, q_{21}) &= N(q_{21}, q_{10}), \quad & N(q_{21}, q_{10}) &= N(q_{10}, p_{32}). \end{aligned}$$

Proof. The proof is by direct calculation. □

To construct algebra R_2, as before, assume (1). Also set $N(q_{32}, q_{20}) = -1$ and $N(q_{10}, q_{20}) = -2$. Then, by the Jacobi identity, $N(p_{32}, q_{20}) = 1$ and

$$N(p_{32}, q_{2,-1}) = 1, \quad N(p_{32}, q_{30}) = -1, \quad N(q_{10}, q_{31}) = 1.$$

By choosing arbitrarily the remaining structural constants N_{rs} we obtain the algebra R_2.

Lemma 1.4. *All ideals in the algebra R_2 are standard.*

Finally, by combining Lemmas 1.2 and 1.4 we prove Theorem 1.1. □

2. The completion of problem's (A) solution

Denote by $N\Phi(q)$ the algebra $N\Phi(K)$ over finite field $K = GF(q)$. Problem (A) of enumeration of standard ideals in Lie algebras $N\Phi(q)$ had been recently solved for classical Lie types (as Problem 1 in [2]) by G. P. Egorychev, V. M. Levchuk, and the author. In these section we complete the solution of Problem (A).

The following theorem gives the solution of Problem (A) for exceptional Lie types.

Theorem 2.1. *The number of standard ideals of a Lie algebra $N\Phi(q)$ of exceptional Lie type is equal to*

$$\begin{aligned}
&G_2: \ q+7;\\
&F_4: \ q^4+3\,q^3+44\,q^2+32\,q+25;\\
&E_6: \ q^9+3\,q^8+4\,q^7+67\,q^6+69\,q^5+230\,q^4+306\,q^3+94\,q^2+22\,q+37;\\
&E_7: \ 2\,(q^{12}+q^{11}+3\,q^{10}+32\,q^9+90\,q^8+118\,q^7+394\,q^6+449\,q^5+\\
&\qquad +708\,q^4+300\,q^3-79\,q^2+31\,q+32);\\
&E_8: \ q^{16}+3\,q^{15}+4\,q^{14}+7\,q^{13}+237\,q^{12}+239\,q^{11}+693\,q^{10}+1647\,q^9+3554\,q^8+\\
&\qquad +4283\,q^7+5829\,q^6+7055\,q^5+3773\,q^4-2361\,q^3-244\,q^2+239\,q+121.
\end{aligned}$$

Proof. We need the following definition. A subspace S of the space K^m is called *m-proper* if for all i, $1 \leqslant i \leqslant m$, there exists an element $(a_1, \dots, a_m) \in S$ such that $a_i \neq 0$.

Similarly to Section 1, every standard ideal H of Lie algebra $N\Phi(q)$ is characterized by a set of corners $\mathcal{L}(H) = \{r_1, r_2, \dots, r_m\}$ and a frame $\mathcal{F}(H)$. So, to each standard ideal H there corresponds a unique pair $(\mathcal{L}, S)$ such that H is equal to the ideal

$$H(\mathcal{L}, S) = Q(\mathcal{L}) + \{a_1e_{r_1} + a_2e_{r_2} + \dots + a_me_{r_m} \mid (a_1, a_2, \dots, a_m) \in S\}. \tag{3}$$

The second term in (3) is a frame of the ideal $H(\mathcal{L}, S)$. This yields that the enumeration of standard ideals coincides with the enumeration of ideals of the form (3). Denote by $\tilde{V}_{m,t}$ the number of all m-proper t-dimensional subspaces in K^m and by $B(\Phi, m)$ denote the number of sets of corners $\mathcal{L}$ in Φ^+ with $|\mathcal{L}| = m$. From the established one-to-one correspondence between standard ideals and pairs $(\mathcal{L}, S)$, we obtain the following

Lemma 2.1. *The number of standard ideals in the algebra $N\Phi(q)$ of Lie rank n is*

$$\Omega(\Phi, q) = 1 + \sum_{m=1}^{n} B(\Phi, m) \ \sum_{t=1}^{m} \tilde{V}_{m,t}. \tag{4}$$

Besides the solution of Problem 1 for type A_n, [10] provides the formula

$$\widetilde{V}_{m,t} = \sum_{1=j_1<j_2<\ldots<j_t\leqslant m} \frac{(q^t-1)^{m-j_t}}{(q-1)^{t-j_t}} \cdot \prod_{k=2}^{t-1} (\frac{q^k-1}{q-1})^{j_{k+1}-j_k-1} \quad (1 \leqslant t \leqslant m).$$

In his paper [11], G. P. Egorychev has found a simpler form of this formula.

Lemma 2.2 ([11, Lemma 4]). *The number of m-proper t-dimensional subspaces of the space K^m over the finite field $K = GF(q)$ is*

$$\widetilde{V}_{m,t} = \sum_{k=0}^{m-t} (-1)^{m-t-k} q^k \binom{m-1}{t+k-1} \begin{bmatrix} t+k-1 \\ k \end{bmatrix}_q. \tag{5}$$

By using Lemma 2.1 we immediately obtain $\Omega(\Phi, q) = q + 7$ for type G_2. In the remaining cases, we obtain the numbers $B(\Phi, m)$ by using the representations of Φ^+ of type F_4 in [8] and of types E_n ($n = 6, 7, 8$) in [12]. Tab. 1 represents the results of computations. (See also [13, Remark 5.2].)

Substituting the corresponding values of Tab. 1 and (5) for $B(\Phi, m)$ and $\widetilde{V}_{m,t}$ in (4), we prove Theorem 2.1. □

Table 1. The values of $B(\Phi, m)$ for types F_4 and E_n

Φ/m	0	1	2	3	4	5	6	7	8
F_4	1	24	55	24	1				
E_6	1	36	204	351	204	36	1		
E_7	1	63	546	1470	1470	546	63	1	
E_8	1	120	1540	6120	9518	6120	1540	120	1

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Перечисления идеалов в нильтреугольной подалгебре алгебры Шевалле

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В работе Г. П. Егорычева и В. М. Левчука 2001 г. была записана проблема 1, заключающаяся в перечислении стандартных идеалов нильтреугольных подалгебр $N\Phi(GF(q))$ алгебр Шевалле классических типов. Мы решаем аналог проблемы 1 для исключительных типов. С помощью недавно введенной конструкции В. М. Левчука обертывающих алгебр для $N\Phi(K)$ исключительного типа F_4 найдены обертывающие алгебры как с нестандартными иделами, так и без них.

Ключевые слова: алгебра Шевалле, нильтреугольная подалгебра, обертывающая алгебра, идеал.