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The Normal Structure of the Unipotent Subgroup of a Chevalley Group of Type $\mathbf{E_6}$, $\mathbf{E_7}$, $\mathbf{E_8}$

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The normal structure of the unipotent subgroup of a Chevalley group of Lie type E_6, E_7, E_8 over an arbitrary field is found.

Keywords: normal structure, unipotent subgroup, Chevalley group, associated Lie ring, ideal.

Introduction

In any Chevalley group over a field K, associated with the root system Φ, the unipotent subgroup $U\Phi(K)$ is generated by the root subgroups corresponding to the positive roots. The group $U\Phi(K)$ of Lie type A_{n-1} is isomorphic to the unitriangle group $UT(n, K)$; its normal subgroups are described in [1] on the basis of the correspondence with the ideals of the associated Lie ring. The approach from [1] was applied to investigate the normal structure of the unipotent subgroups of some certain types for the case $K = 2K$ in [2] –[5]. However, some particular features of the descriptions have shown an inadequacy of the method.

A new approach was developed and applied in [6] for the classical types. In the present work this approach made it possible to investigate the normal structure of the groups $U\Phi(K)$ for the exceptional types E_6, E_7, E_8.

Let $\Phi(K)$ be a Chevalley group over a field K, associated with the root system Φ. For the case of $2K = K$ the normal structure of $UE_m(K)$ was studied by L.A.Martynova [3]. We revise the cases when the known for the type A_n correspondence of the normal subgroups of $UE_m(K)$ and the ideals of the associated Lie ring is realized. In this paper the normal structure of its unipotent subgroup $U\Phi(K) = \langle X_r \mid r \in \Phi^+\rangle$ for the type $\Phi = E_m$ $(m = 6, 7, 8)$ over a field of characteristic 2 is investigated.

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1. The Representation of the Unipotent Subgroups

The unipotent subgroup $U\Phi(K)$ is generated by the root subgroups $x_r(K) = X_r$, corresponding to the roots $r \in \Phi^+$. Each element A of $U\Phi(K)$ is uniquely represented by the product of the root elements $x_r(t_r)$, $r \in \Phi^+$, disposed corresponding to the fixed ordering of roots [7, 5.3.3], [8, Lemma 18]. We'll use the representation π of the group $U\Phi(K)$, which was found in [9]. Choose the subalgebra $N\Phi(K)$ with the base e_r ($r \in \Phi^+$) in the Chevalley algebra of type Φ over K with the base e_r ($r \in \Phi$), ... (cf. [7, § 4.4]) and let

$$\pi(A) = \sum_{r\in\Phi^+} t_r e_r, \qquad \alpha \circ \beta = \pi(\pi^{-1}(\alpha)\pi^{-1}(\beta)) \ \ (\alpha, \beta \in N\Phi(K)).$$

The adjoint multiplication $\circ$ is a group operation on $N\Phi(K)$ and the mapping $\pi : U\Phi(K) \to (N\Phi(K), \circ)$ is a group isomorphism. Instead of $\circ$ in the product we'll usually write $+$, when the cofactors don't depend on the choice of π.

Further we'll use the concepts of a corner and a frame from [6].

Let $\{r\}^+$ for $r \in \Phi$ is a set of all $s \in \Phi^+$ with non-negative coefficients in the linear expression of $s - r$ through the base $\Pi(\Phi)$. Let

$$T(r) = \langle X_s \mid s \in \{r\}^+\rangle, \quad Q(L) = \langle X_s \mid s \in \cup_{r\in L}\{r\}^+ \setminus L\rangle, \ L \subset \Phi^+.$$

Definition 1. *If $H \subseteq T(r_1)T(r_2)\dots T(r_m)$ and the inclusion is not fulfilled for any replacement of $T(r_i)$ by $Q(r_i)$, then call $\{r_1, r_2, \cdots, r_m\} = \mathcal{L}(H)$ the set of corners for H. Call the frame for H the set $\mathcal{F}(H)$ such that*

$$\mathcal{F}(H) = H \mod \prod_{s\in\mathcal{L}(H)} Q(s), \quad \mathcal{F}(H) \subseteq \prod_{s\in\mathcal{L}(H)} X_s. \tag{1}$$

Call r, s from Φ *connected* in H, if s-projection of each element from H is equal to the product of its r-projection and a fixed scalar $\neq 0$; and call them *p-connected for $p \in \Phi^+$*, if also $r + p, s + p \in \Phi$.

This terminology for the $U\Phi(K)$ will be also used for $N\Phi(K)$. An element e_r of the Chevalley base we denote for brevity by r, first of all, in the notations $Ke_r = Kr$ of the root subgroups. As in [10, Tables V – VII], the root system of type E_m ($m = 6, 7, 8$) with the base

$$\alpha_1 = \varepsilon_1 + \varepsilon_8 - \frac{1}{2}\sum_{i=1}^{8}\varepsilon_i, \ \alpha_2 = \varepsilon_2 + \varepsilon_1, \ \alpha_j = \varepsilon_j - \varepsilon_{j-1} \ (1 < j < m)$$

we choose in 8-dimentional Euclidean space with the orthonormalized base $\{\varepsilon_1, \dots, \varepsilon_8\}$.

Consider the next conditions for the root r in $H \subseteq NE_m(K)$ and the fundamental root p:

(A) $\mathcal{F}([H, X_p]) + Q(r + p) \subseteq H$,

(B) there exists a corner s in H, p-connected with r, and there exist fundamental roots p_j and roots $r_j = r + p_1 + p_2 + \dots + p_j$, $s_j = s + p_1 + p_2 + \dots + p_j$ with $p_1 = p$, $1 \leqslant j \leqslant t$, $1 < t \leqslant m - 3$, such that (r, s)-projection and (r_j, s_j)-projections of H for $j < t - 1$ generate in

K-module (K, K) the submodule $K(a, b)$, (r_{t-1}, s_{t-1})-projection is equal to $K(a, b)$ or in H there is p_t-connected with r_{t-1} corner $\neq s_{t-1}$,

$$Q(r_2, \dots, r_t) + \mathcal{F}([H, X_{p_t}]) + \sum_{j=2}^{t} K(ae_{r_j} + be_{s_j}) \subseteq H,$$

and also if there exists a fundamental root $q \neq p_2$ such, that in $[H, X_p]$ the corner $r + p$ is not q-connected, that $T(r + p + q) \subseteq H$. Moreover, either

(B_1) $\mathcal{F}([H, X_p]) + T(r + s + p) \subseteq H$, or

(B_2) $|H_r| = 2$, there exists a corner $u \neq s$, p-connected with r, the set $\{r, s, u\}$ coincides with one of the sets of form

$$\{\alpha_2, k_1\alpha_1 + \alpha_3, \alpha_5 + k_2\alpha_6 + k_3\alpha_7 + k_4\alpha_8\}, \quad k_i = 0, 1,$$

and $K\{ae_{r+p} + be_{s+p} + abe_{r+s+p} + ce_{u+p} \mid a \in H_r^*, b \in H_s^*, c \in H_u^*\} \subseteq H$.

Theorem 1. *The subgroup H of the adjoint group $NE_m(K)$, over a field of characteristic 2, is normal if and only if for each its corner r and each fundamental root p with the root $r + p$ one of the conditions (A), (B) is satisfied.*

As the theorem shows, the normal subgroups are not the ideals of the Lie ring $N\Phi(K)$ if and only if they don't contain at least one frame $\mathcal{F}([H, X_p])$ (and such $p = \alpha_4$ is unique). Earlier L.A.Martynova [3] has proved that the class of all normal subgroups of the adjoint group $NE_m(K)$ coincides with the class of all ideals of the associated Lie ring for the case $2K = K$.

2. Proof of the Main Theorem

We now need the following lemmas.

Lemma 1. *Let $H \subseteq N\Phi(K)$, $p \in \Phi^+$ and $[H, X_p] \neq 0$. Then the corners in $[H, X_p]$ have the form $p + s_i$, where $s_i \in \cup_{r \in \mathcal{L}(H)} \{r\}^+$, $1 \leqslant i \leqslant k$, and $1 \leqslant k \leqslant 3$. When $k = 3$, then $\Phi = D_n$ or E_m, and $\{p, s_1, s_2, s_3\}$ is a base of the system of type D_4.*

Proof. It is obvious that $|\mathcal{L}(H)| \leqslant$ rank of Φ and

$$[H, X_p] \subseteq \langle T(s + p) \mid s \in \cup_{r \in \mathcal{L}(H)} \{r\}^+, \ s + p \in \Phi^+ \rangle,$$

so $\mathcal{L}([H, X_p]) = \{p + s_1, p + s_2, \cdots, p + s_k\}$ and the sets $\{p + s_i\}^+$ are pairwise not incidental. The least in Φ subsystem of roots, which contains $\mathcal{L}([H, X_p])$ and all roots p, s_i, have the connected Coxeter graph. When its rank $k + 1 > 3$, then from the known classification of the root systems, the subsystem has type D_4 and Φ is of type D_n or E_m. □

As we can observe from the Definition 1, elements of H in the Lemma 1 give the frame $\mathcal{F}(H)$, if in their canonical decompositions we throw out all cofactors ae_s with $s \notin \mathcal{L}(H)$. The addition and the multiplication in H coincide modulo $\sum_{r \in \mathcal{L}(H)} Q(r)$. Hence from the Chevalley commutator formula we see that for the subgroup H of the additive or adjoint group $N\Phi(K)$ the frame in $[H, X_p]$ is a K-module. So we have

Lemma 2. *If H is a subgroup of the additive or adjoint group $N\Phi(K)$, then under the conditions of lemma 3 the frame in $[H, X_p]$ is a K-submodule in $N\Phi(K)$ and equals to the frame of the Lie product of H and X_p in subalgebra $N\Phi(K)$.*

The next lemma is established by direct calculations.

Lemma 3. *Let Φ be a system roots of Lie type E_m. Let Φ^+ contain fundamental roots p, q and not incidental roots r, s with $r+p$, $s+p$, $r+q \in \Phi^+$. Then $s+q \notin \Phi^+$.*

It is clear that r-projection H_r of corner r in H does not depend on the root ordering. It is also clear, that $r+p$ is a corner in $[H, X_p]$, and we have

Lemma 4. *If $H \trianglelefteq U = UE_m(K)$, $s \in \cup_{r\in\mathcal{L}(\mathcal{H})} \{r\}^+ \setminus \mathcal{L}(H)$, then s is a corner of a subset in $[U, H]$.*

Lemma 5. *Let $H \trianglelefteq N\Phi(K)$, $\Phi = E_m$, $\mathcal{L}(H) = \{r\}$. Then $H = Q(r) + H_r e_r$.*

Proof. Let $h(\Phi)$ be the Coxeter number of the system Φ and $ht(r)$ be the height of r. The derived group $[H, X_p]$ for $p \in \Pi(\Phi)$ with $r+p \in \Phi^+$ by lemma 5 has a unique corner $r+p$. The induction on $h(\Phi) - ht(r)$ gives the inclusion $T(r+p) \subset H$. □

Lemma 6. *Let $A, B \subseteq K$, $\mu : B \to K$. The set $A\{(x, x^\mu) \mid x \in B\}$ additively generates (K, K), if either $A = K$ and there exists two K-linear independent elements in $\{(x, x^\mu) \mid x \in B\}$, or $B = K, x^\mu = cx^\theta, c \in K^*$ and θ is an automorphism of K, not identical on $A(A \cap K^*)^{-1}$.*

Proof. The case with $A = K$ is obvious. For all elements $s \in A \cap K^*$, $t \in As^{-1}$ in the case $B = K$ and $x^\mu = cx^\theta$ we obtain the equalities for $x \in K$:

$$[s(xt, (xt)^\mu) - st(x, x^\mu)] = (0, csx^\theta(t^\theta - t)),\ csK^\mu(t^\theta - t) = K(t^\theta - t).$$

When there exists $t \neq t^\theta$, we obtain the conclusion of the lemma. □

Consider the following conditions for the corner r in $H \subseteq NE_m(K)$ and the fundamental root p:

(C) there exist a corner s, p-connected r, and there exist fundamental roots p_j and roots $r_j = r+p_1+p_2+\cdots+p_j$, $s_j = s+p_1+p_2+\cdots+p_j$ with $p_1 = p$, $1 \leqslant j \leqslant t, 1 \leqslant t \leqslant m-3$, such that (r, s)-projection and (r_j, s_j)-projections in H for $j < t$ generate in K-module (K, K) the submodule $K(a, b)$, and $Q(r_1, \ldots, r_t) + \sum\limits_{j=1}^{t} K(ae_{r_j} + be_{s_j}) \subseteq H$.

Lemma 7. *Let a subgroup $H \trianglelefteq N\Phi(K)$, $\Phi = E_m$, have exactly two corners. Then for each its corner r and each fundamental root p with $r+p$ one of the conditions (A) and (C) is satisfied.*

Proof. Under the conditions of the theorem $r+p$ is a corner in $[H, X_p]$. When the corner is unique, the normal closure of the derived group $[H, X_p]$ by Lemma 5 contains $Q(r+p)$, and hence also contains $\mathcal{F}([H, X_p])$. The same inclusions are obtained by Lemmas 3–5, if $\mathcal{L}([H, X_p]) = \{r+p, s+p\}$ and corners in $[H, X_p]$ are not connected, in particular, when $s \notin \mathcal{L}(H)$.

Further assume $Q\{r+p\} \not\subseteq H$. Then the corners in $[H, X_p]$ are connected and there exist fundamental roots p_j and the roots $r_j = r+p_1+p_2+\cdots+p_j$, $s_j = s+p_1+p_2+\cdots+p_j$ with $p_1 = p$, $1 \leqslant j \leqslant t, 1 \leqslant t \leqslant m-3$, where t is the maximal index, such that $X_{r_t} \not\subseteq H$. The inclusion $Q(r_1, \ldots, r_t) \subseteq H$ we obtain by Lemmas 3 and 5. Let (r, s)-projection in H generates in K-module (K, K) the submodule $K(a, b)$. Using the relations $H \supseteq [[\ldots[[H, X_{p_1}], X_{p_2}]\ldots], X_{p_t}]$, $H \supseteq [[\ldots[[H, X_{p_{j+1}}], X_{p_2}]\ldots], X_{p_t}]$ $(1 \leqslant j < t)$ and Lemma 6, we obtain that (r_j, s_j)-projections in H for $j < t$ generate in K-module (K, K) the submodule $K(a, b)$, since otherwise $X_{r_t} \subseteq H$, against the choice of t. □

Proof of the theorem. It's sufficient to consider the case when the derived group $[H, X_p]$ has three connected corners, the other cases by analogy with the proof of Lemma 7 give (A) or (B) with case (B_1).

Assume that there exists a corner s in H, p-connected with r, and that there exist fundamental roots p_j and the roots $r_j = r + p_1 + p_2 + \cdots + p_j$, $s_j = s + p_1 + p_2 + \cdots + p_j$ with $p_1 = p$, $1 \leqslant j \leqslant t, 1 \leqslant t \leqslant m-3$, where t is the maximal index, such that $X_{r_t} \not\subseteq H$, and let $u \neq s$ is a corner, p-connected with r.

Since for $t = 1$, it is clear that the case (A) is satisfied, we may further assume $t > 1$. Then the frame $\mathcal{F}([H, X_{p_j}])$ for any $j > 1$ is situated in H. If $j < t-1$, then (r_j, s_j)-projections in H for $j < t$ generate in K-module (K, K) the submodule $K(a, b)$, otherwise $T(r_t) \subset H$.

If $\mathcal{F}([H, X_p]) \not\subseteq H$, then the subgroup $T(r+p+s)$ is not situated in H. Directly calculating all roots which in addition with $p = \alpha_1$ is again a root, we note, that among them there are no three pairwise not incidental roots, such that at least two of them have the height $\leqslant 4$ (otherwise $H \supseteq T(r+p+s)$).

Let Φ be a system roots of Lie type E_8. Consider the case $p = \alpha_2$. In the set $(\Phi^+ + \alpha_2) + p$ for $p \neq \alpha_4$ all roots are pairwise incidental, and for this case $[[H, X_{\alpha_2}], X_p]$ has the unique corner, hence H again contains the subgroup $T(r+p+s)$. The set $((\Phi^+ + \alpha_2) + \alpha_4) + \alpha_3$ contains the pair of not incidental roots $\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$ and $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$, but the height of both input roots > 4. The set $((\Phi^+ + \alpha_2) + \alpha_4) + \alpha_5$ contains the pairs of not incidental roots from the set $\{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8\}$, but again all input roots have the height > 4.

If we consider the other cases by analogy (the sets of form $(\Phi + p) + q$ for all fundamental roots p, q were calculated using a computer program in Turbo Pascal)), we obtain triples of corners of form $\{\alpha_2, k_1\alpha_1 + \alpha_3, \alpha_5 + k_2\alpha_6 + k_3\alpha_7 + k_4\alpha_8\}, k_i = 0, 1$.

The following equality is obtained modulo the sum of the subgroup $Q(r_2, \ldots, r_t) + Q(s_2, \ldots, s_t) + \sum_{j=2}^{t} K(ae_{r_j} + be_{s_j})$ and the subgroups of form $T(r+p+q)$ and $T(s+p+q)$

$$[X_p, H] = K\{e_{r+p} + cae_{s+p} + cae_{r+s+p} + dae_{u+p} \mid a \in H_r^*, c, d \in K^*\}.$$

Hence using the condition $T(r+s+p) \not\subseteq H$ we have $|H_r| = 2$, and (B), case (B_2). □

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