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Conditional Correctness and Approximate Solution of Boundary Value Problem for the System of Second Order Mixed-type Equations

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In this paper, we consider the system of second-order mixed type equations. Theorems of uniqueness and conditional stability in the set of correctness are proven. The approximate solution is constructed by the method of regularization and by the quasi-inverse method.

Keywords: system of equations, boundary value problem, ill-posed problem, a priori estimate, theorem of the uniqueness, conditional stability, set of correctness, approximate solution, regularization.
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Introduction

The general theory of boundary value problems for the mixed type equations with variable coefficients and with a manifold of type change have been the subject of research M. A. Lavryent'yev, A. V. Bitsadze, M. M. Smirnov, M. S. Salakhitdinov, T. D. Djuraev, V. N. Vragov, K. B. Sabitov, A. I. Kozhanov and many others [1, 2].

These type of the equations have many different applications, for example, the problems encountered in applications, in particular the problem of transonic flow of a compressible medium, and without torque shell theory. Many important practical applications, such as jet aircraft and astronautics, rocketry, gas-dynamic lasers, caused an avalanche growth of research in the field of boundary value problems for equations of mixed type (see. [3, 4]).

Here we consider the system of mixed type equations. Systematic study of such equations began from the work of F. Trikomi and S. Gellerstedt [1, 2]. The theory of the solvability of boundary value problems for linear models described by a such equations has been constructed in the papers S. A. Tersenov, I. E. Egorov, A. A. Kerefov, N. V. Kislov, S. G. Pyatkov and others [5–7].

The problem considered in this paper belongs to the class of ill-posed problems of mathematical physics. Namely, in this problem the solution does not continuously depend on the initial data. Ill-posed problems for such equations were considered in [8–11].

In this paper, we establishe the conditional correctness of this problem and construct the approximate solution of the problem by regularization and quasi-inverse methods.

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1. Problem statement

Let the pair of functions $(v(x,t), u(x,t))$ is a solution of equation

$$\begin{cases} \operatorname{sign}(x)\, v_{tt}(x,t) - Lv(x,t) = f(x,t), \\ \operatorname{sign}(x)\, u_{tt}(x,t) - Lu(x,t) = v(x,t), \end{cases} \tag{1}$$

in the region $\Omega = \{-1 < x < 1,\, x \neq 0,\, 0 < t < T\}$ and satisfy
the initial conditions

$$\left.\begin{aligned} v(x,0) = \varphi_1(x),\ \ v_t(x,0) = \varphi_2(x), \\ u(x,0) = \varphi_3(x),\ \ u_t(x,0) = \varphi_4(x), \end{aligned}\right\} -1 \leqslant x \leqslant 1, \tag{2}$$

the boundary conditions

$$v(-1,t) = v(1,t) = 0, \quad u(-1,t) = u(1,t) = 0, \quad 0 \leqslant t \leqslant T \tag{3}$$

and the gluing conditions

$$\left.\begin{aligned} v(-0,t) = v(+0,t),\ \ v_x(-0,t) = v_x(+0,t), \\ u(-0,t) = u(+0,t),\ \ u_x(-0,t) = u_x(+0,t), \end{aligned}\right\} 0 \leqslant t \leqslant T, \tag{4}$$

where $Lv \equiv -\dfrac{\partial}{\partial x}\big(p(x)v_x(x,t)\big) + q(x)v(x,t)$ and $p(x)$, $p'(x)$, $q(x)$ continuous functions on the segment $[-1,\,1]$, $q(x) \geqslant 0$, $p(x) > p_0$, p_0 some positive constant.

Definition 1. By solution of the problem (1)–(4) we understood the pair of functions (v,u) having corresponding continuous derivatives involved in the system of equations satisfying equation (1) and the conditions (2)–(4).

Consider the spectral problem

$$\begin{cases} \operatorname{sign}(x)\dfrac{d}{dx}\left(p(x)X'(x)\right) - \operatorname{sign}(x)q(x)X(x) + \lambda X(x) = 0, \\ X(-1) = X(1) = 0, \\ X(-0) = X(+0),\ X'(-0) = X'(+0). \end{cases} \tag{5}$$

Let $\left\{X_k^+\right\}_{k=1}^{\infty}$, $\left\{X_k^-\right\}_{k=1}^{\infty}$ be eigenfunctions of the problem (5), corresponding respectively to positive and negative λ_k^+, λ_k^- eigenvalues, and the number of λ_k^+, $-\lambda_k^-$ form a increasing sequence. We denote by $(u,v) = \int\limits_{-1}^{1} uv dx$ scalar product in $L_2(-1,1)$, $\|u\|^2 = (u,u)$,

$$\big(\operatorname{sign}(x)X_k^+, X_j^-\big) = 0, \forall k,j;$$

$$\big(\operatorname{sign}(x)X_k^\pm, X_j^\pm\big) = \delta_{kj},$$

where δ_{kj} is the Kronecker delta.

Let $P^\pm$ is spectral projections, defined by the equalities $P^\pm u = \sum\limits_{k=1}^{\infty}\big(\operatorname{sign}(x)u, X_k^\pm\big)X_k^\pm$.

Then, according to [12], we have

$$\left(P^+ - P^-\right)u = u, \quad \big(\operatorname{sign}(x)\left(P^+ - P^-\right)u, u\big) = \|u\|_0^2,$$

$$\big(\operatorname{sign}(x)P^\pm u,\, v\big) = \big(\operatorname{sign}(x)\, u,\, P^\pm v\big), \quad u,v \in H_0 = L_2(-1,1),$$

$$\|u(x,t)\|_0^2 = \sum_{k=1}^{\infty}\left\{\left|\left(\mathrm{sign}(x)u(x,t),X_k^+\right)\right|^2 + \left|\left(\mathrm{sign}(x)u(x,t),X_k^-\right)\right|^2\right\}. \quad (6)$$

According to the results of [12], the eigenfunctions of the problem (5) form a Riesz basis in the H_0 and the norm in the space of $L_2(-1,\,1)$, defined by equation (6), equivalent to the original.

By a generalized solution of problem (1)–(4) we understand the pair of functions $(v(x,t),\,u(x,t)) \in C\left([0,T];\,L_2(-1,1)\right)$ satisfies the following conditions

$$\int_0^T\int_{-1}^1 v(x,t)\left(\mathrm{sign}(x)V_{tt} - LV\right)dxdt = \int_0^T\int_{-1}^1 f(x,t)V\,dxdt + \\ + \int_{-1}^{1}\mathrm{sign}(x)V(x,0)\varphi_2(x)\,dx - \int_{-1}^{1}\mathrm{sign}(x)V_t(x,0)\varphi_1(x)\,dx,$$

$$\int_0^T\int_{-1}^1 u(x,t)\left(\mathrm{sign}(x)U_{tt} - LU\right)dxdt = \int_0^T\int_{-1}^1 V(x,t)U\,dxdt + \\ + \int_{-1}^{1}\mathrm{sign}(x)U(x,0)\varphi_4(x)\,dx - \int_{-1}^{1}\mathrm{sign}(x)U_t(x,0)\varphi_3(x)\,dx$$

for any pair of functions $\left(V(x,t),U(x,t)\right)$, $V,U \in W_2^2(\Omega)$, $V(x,T)=0$, $V_t(x,T)=0$, $V(-1,t)=0$, $V(1,t)=0$, $U(x,T)=0$, $U_t(x,T)=0$, $U(-1,t)=0$, $U(1,t)=0$.

Assume that the solution $(v(x,t),u(x,t))$ of problem (1)–(4) exists. Then for the system (1) using the properties of eigenfunctions of the problem (5) and the definition of a generalized solution, we have

$$\begin{cases} \left\{v_k^{\pm}(t)\right\}_{tt} - \lambda_k^{\pm}v_k^{\pm}(t) = f_k^{\pm}(t), \\ \left\{u_k^{\pm}(t)\right\}_{tt} - \lambda_k^{\pm}u_k^{\pm}(t) = v_k^{\pm}(t), \end{cases} \quad k=1,\,2,\,3,\,\ldots\,. \quad (7)$$

with the initial conditions

$$v_k^{\pm}(t)\big|_{t=0} = \varphi_{1_k}^{\pm},\ \left\{v_k^{\pm}(t)\right\}_t\big|_{t=0} = \varphi_{2_k}^{\pm},$$
$$u_k^{\pm}(t)\big|_{t=0} = \varphi_{3_k}^{\pm},\ \left\{u_k^{\pm}(t)\right\}_t\big|_{t=0} = \varphi_{4_k}^{\pm},$$

where $f_k^{\pm}(t) = \int\limits_{-1}^{1} f(x,t)X_k^{\pm}(x)dx$, $\varphi_{j_k}^{\pm} = \int\limits_{-1}^{1}\varphi_j(x)X_k^{\pm}(x)dx$, $j=1,\,2,\,3,\,4,\ ,k=1,2,3,\ldots$.

2. Main results

Lemma 1. *Let $\omega(t)$ be solution of the equation*

$$\omega''(t) - \lambda\omega(t) = g(t)$$

and satisfies conditions $\omega(0)=0$, $\omega'(0)=w_1$. Then solution of the equation satisfies the inequality

$$\int_0^t w^2(\tau)d\tau \leqslant c(t)\gamma^{1-r(t)}\left(\int_0^T w^2(\tau)d\tau + \gamma\right)^{r(t)}, \quad (8)$$

for $\forall\, t\in(0,T)$, where λ is some constant, $g(t)$ is given function, $\gamma = (2T^2+1)\int\limits_0^T g^2(t)dt + 2Tw_1^2$,

$$r(t) = \frac{1-e^{-2t}}{1-e^{-2T}},\ c(t) = \exp\left(\frac{2T+1}{2}\cdot\frac{\left(1-e^{-2t}\right)T - \left(1-e^{-2T}\right)t}{1-e^{-2T}}\right).$$

Proof. Let $\xi(t) = \int\limits_0^t w^2(\tau)d\tau + \gamma$, then it is easy to notice that $\xi'(t) = w^2(t)$, $\xi''(t) = 2w(t)w'(t)$. Hence

$$\xi'(t) = 2\int_0^t w(\tau)w'(\tau)d\tau,$$

$$\xi'''(t) = 2w'^2(t) + 2w(t)w''(t) = 2w'^2(t) + 2w(t)g(t) + 2w^2\lambda.$$

We calculate the derivative of the function $\lambda w^2(t)$.

$$\frac{d}{dt}\left(\lambda w^2\right) = 2\lambda w w' = 2w'w'' - 2w'g = \frac{d}{dt}\left(w'^2\right) - 2w'g.$$

Integrating the last expression we have

$$\lambda w^2 = w'^2 - 2\int_0^t w'(\tau)g(\tau)d\tau - w_1^2.$$

From here

$$\xi'''(t) = 4w'^2(t) + 2w(t)g(t) - 4\int_0^t w'(\tau)g(\tau)d\tau - 2w_1^2,$$

and integrating last equality from 0 till t we get

$$\xi''(t) = 4\int_0^t w'^2(\tau)d\tau + 2\int_0^t w(\tau)g(\tau)d\tau - 4\int_0^t\int_0^\tau w'(\tau_1)g(\tau_1)d\tau_1 d\tau - 2w_1^2 t,$$

or

$$4\int_0^t w'^2(\tau)d\tau = \xi''(t) - 2\int_0^t w(\tau)g(\tau)d\tau + 4\int_0^t\int_0^\tau w'(\tau_1)g(\tau_1)d\tau_1 d\tau + 2w_1^2 t. \tag{9}$$

Using $2|ab| \leqslant sa^2 + \frac{1}{s}b^2$ inequality, where $s > 0$, and some elementary transformations in (9), we obtain

$$4\int_0^t w'^2(\tau)d\tau \leqslant \xi''(t) + \int_0^t w^2(\tau)d\tau + \int_0^t g^2(\tau)d\tau + \frac{2T}{s}\int_0^t w'^2(\tau)d\tau + 2Ts\int_0^t g^2(\tau)d\tau + 2w_1^2 T.$$

If $s = T$, then we have

$$2\int_0^t w'^2(\tau)d\tau \leqslant \xi''(t) + \int_0^t w^2(\tau)d\tau + (2T^2+1)\int_0^t g^2(\tau)d\tau + 2w_1^2 T,$$

or

$$2\int_0^t (t-\tau)w'^2(\tau)d\tau \leqslant \xi'(t) + T\int_0^t w^2(\tau)d\tau + (2T^3+T)\int_0^t g^2(\tau)d\tau + 2w_1^2T^2 \leqslant \xi'(t) + T\xi(t).$$

Let $\eta(t) = \ln(\xi(t))$. Then it is easy to see

$$\eta''(t) = \frac{\xi''(t)\xi(t) - \xi'^2(t)}{\xi^2(t)} =$$

$$= \frac{\left(4\int\limits_0^t w'^2(\tau)d\tau + 2\int\limits_0^t w(\tau)g(\tau)d\tau - 4\int\limits_0^t\int\limits_0^\tau w'(\tau_1)g(\tau_1)d\tau_1 d\tau - 2w_1^2 t\right)\left(\int\limits_0^t w^2(\tau)d\tau + \gamma\right)}{\xi^2(t)} -$$

$$- \frac{\left(2\int\limits_0^t w(\tau)w'(\tau)d\tau\right)^2}{\xi^2(t)} = A_1 + A_2 + A_3.$$

We estimate the last expressions

$$A_1 = \frac{4\int\limits_0^t w'^2(\tau)d\tau \int\limits_0^t w^2(\tau)d\tau - \left(2\int\limits_0^t w(\tau)w'(\tau)d\tau\right)^2 + 4\gamma\int\limits_0^t w'^2(\tau)d\tau}{\xi^2(t)} \geqslant 0,$$

$$A_2 = \frac{\left(2\int\limits_0^t w(\tau)g(\tau)d\tau - 2w_1^2 t\right)\left(\int\limits_0^t w^2(\tau)d\tau + \gamma\right)}{\xi^2(t)} =$$

$$\frac{2\int\limits_0^t w(\tau)g(\tau)d\tau - 2w_1^2 t}{\xi(t)} \geqslant \frac{-\int\limits_0^t w^2(\tau)d\tau - \int\limits_0^t g^2(\tau)d\tau - 2w_1^2 T}{\xi(t)} \geqslant -1,$$

$$A_3 = \frac{-4\int\limits_0^t\int\limits_0^\tau w'(\tau_1)g(\tau_1)d\tau_1 d\tau\left(\int\limits_0^t w^2(\tau)d\tau + \gamma\right)}{\xi^2(t)} =$$

$$\frac{-4\int\limits_0^t (t-\tau)w'(\tau)g(\tau)d\tau}{\xi(t)} \geqslant \frac{-2\xi'(t) - 2T\xi(t)}{\xi(t)} = -2\eta'(t) - 2T.$$

By combining the above taken inequalities we obtain

$$\eta''(t) + 2\eta'(t) + 2T + 1 \geqslant 0$$

or

$$\eta(t) \leqslant \eta(0)(1 - r(t)) + \eta(T)r(t) + c(t).$$

From this follows the required inequality (8). □

Theorem 1. *Let $(v(t), u(t))$ solution of (1) satisfying the conditions (2)–(4), and $\varphi_1(x) = 0$, $\varphi_3(x) = 0$. Then the estimates*

$$\int_0^t \|v(x,\tau)\|_0^2\, d\tau \leqslant c(t)(\beta)^{1-r(t)}\left(\int_0^T \|v(x,t)\|_0^2\, dt + \beta\right)^{r(t)}, \tag{10}$$

$$\int_0^t \|u(x,\tau)\|_0^2\, d\tau \leqslant c(t)(\theta)^{1-r(t)}\left(\int_0^T \|u(x,t)\|_0^2\, dt + \theta\right)^{r(t)} \tag{11}$$

for $v(x,t)$ and $u(x,t)$ are valid, where
$\beta = (2T^2+1)\int_0^T \|f(x,t)\|_0^2\, dt + 2T\|\varphi_2(x)\|^2$, $\theta = (2T^2+1)\int_0^T \|v(x,t)\|_0^2\, dt + 2T\|\varphi_4(x)\|^2$.

Proof. Using equation (1), conditions (2)–(4) and the spectral problem (5), we have

$$v(x,t) = \sum_{k=1}^{\infty}\left(v_k^+(t)X_k^+(x) + v_k^-(t)X_k^-(x)\right),$$

$$u(x,t) = \sum_{k=1}^{\infty}\left(u_k^+(t)X_k^+(x) + u_k^-(t)X_k^-(x)\right),$$

where $\left(v_k^{\pm}(t), u_k^{\pm}(t)\right)$ $(k = 1, 2, 3, \dots)$ satisfies the equation (7), respectively. According to Lemma 1

$$\int_0^t \left(v_k^{\pm}(\tau)\right)^2 d\tau \leqslant c(t)\left(\gamma_k^{\pm}\right)^{1-r(t)} \left(\int_0^T \left(v_k^{\pm}(t)\right)^2 d\tau + \gamma_k^{\pm}\right)^{r(t)}, \tag{12}$$

where $\gamma_k^{\pm} = (2T^2+1)\int\limits_0^T \left(f_k^{\pm}(t)\right)^2 dt + 2T\left(\varphi_{2_k}^{\pm}\right)^2$, $k = 1, 2, 3, \dots$. Summing (12) over k and using Holder's inequality we obtain

$$\int_0^t \sum_{k=1}^{\infty} \left(v_k^{+}(\tau)\right)^2 + \left(v_k^{-}(\tau)\right)^2 d\tau \leqslant$$
$$\leqslant c(t)\left(\sum_{k=1}^{\infty} \gamma_k^{+} + \gamma_k^{-}\right)^{1-r(t)} \left(\int_0^T \sum_{k=1}^{\infty} \left(v_k^{+}(t)\right)^2 + \left(v_k^{-}(t)\right)^2 dt + \sum_{k=1}^{\infty} \gamma_k^{+} + \gamma_k^{-}\right)^{r(t)}.$$

Hence according to (6) we have (10). By similar way for $u(x,t)$ we have the inequality (11). □

We introduce the set of correctness M as follows

$$M = \left\{(u, v) : \|u(x,t)\|_0^2 + \|v(x,t)\|_0^2 \leqslant m^2\right\}. \tag{13}$$

Theorem 2. *Let the solution of the problem (1)–(4) exists and $(v(x,t), u(x,t)) \in M$. Then the solution of the problem is unique.*

Proof. Let two pair of functions $(v_1(x,t), u_1(x,t))$, $(v_2(x,t), u_2(x,t))$ are solutions of problem (1)–(4). We denote $v(x,t) = v_1(x,t) - v_2(x,t)$, $u(x,t) = u_1(x,t) - u_2(x,t)$. Then the pair of functions $(v(x,t), u(x,t))$ satisfies the homogeneous equations (1) and the corresponding conditions (2)–(4). Then as in the formula in (10) $\beta = 0$ we have $\int\limits_0^t \|v(x,\tau)\|_0^2 d\tau = 0$ or $v(x,t) = 0$. By the same way we find $\theta = 0$ and $\int\limits_0^t \|u(x,\tau)\|_0^2 d\tau = 0$. Hence $u(x,t) = 0$ for all $(x, t) \in \Omega$. From here we have $v_1(x,t) \equiv v_2(x,t)$, $u_1(x,t) \equiv u_2(x,t)$, that is solution of problem (1)–(4) is unique. The theorem is proved. □

Theorem 3. *Let the solution of the problem (1)–(4) exists, $(v(x,t), u(x,t)) \in M$, $\varphi_1(x) = 0$, $\varphi_3(x) = 0$, $\|f(x,t) - f_\varepsilon(x,t)\|_0 \leqslant \varepsilon$, $\|\varphi_2(x) - \varphi_{2_\varepsilon}(x)\|_0 \leqslant \varepsilon$, $\|\varphi_4(x) - \varphi_{4_\varepsilon}(x)\|_0 \leqslant \varepsilon$. Then the solution of problem (1)–(4) satisfies the inequalities*

$$\int_0^t \|v(x,\tau)\|_0^2 d\tau \leqslant c(t)(\beta_\varepsilon)^{1-r(t)}\left(m^2T + \beta_\varepsilon\right)^{r(t)},$$

$$\int_0^t \|u(x,\tau)\|_0^2 d\tau \leqslant c(t)(\theta_\varepsilon)^{1-r(t)}\left(m^2T + \theta_\varepsilon\right)^{r(t)},$$

where $\theta_\varepsilon = c(t)(2T^2+1)(\beta_\varepsilon)^{1-r(t)}\left(m^2T + \beta_\varepsilon\right)^{r(t)} + 2T\varepsilon^2$, $\beta_\varepsilon = (2T^3 + 3T)\varepsilon^2$.

Proof. Consider the problem (1)–(4) with $f(x,t)$ replaced by $f(x,t) - f_\varepsilon(x,t)$, and $\varphi_2(x)$, $\varphi_4(x)$ respectively $\varphi_2(x) - \varphi_{2_\varepsilon}(x)$, $\varphi_4(x) - \varphi_{4_\varepsilon}(x)$. From the corresponding formula (10) we have

$$\beta = (2T^2+1)\int_0^T \|f(x,t) - f_\varepsilon(x,t)\|_0^2 dt + 2T\|\varphi_2(x) - \varphi_{2_\varepsilon}(x)\|^2,$$

then we have $\beta \leqslant (2T^3+3T)\varepsilon^2$ and $\int\limits_0^T \|v(x,t)\|_0^2\,dt \leqslant m^2T$. We use the notation $\beta_\varepsilon = (2T^3+3T)\varepsilon^2$, then

$$\int_0^t \|v(x,\tau)\|_0^2\,d\tau \leqslant c(t)(\beta_\varepsilon)^{1-r(t)}\big(m^2T+\beta_\varepsilon\big)^{r(t)}.$$

In the same way from the formula (11) we have

$$\theta = (2T^2+1)\int_0^T \|v(x,t)\|_0^2\,dt + 2T\,\|\varphi_4(x)-\varphi_{4_\varepsilon}(x)\|_0^2,$$

then

$$\theta \leqslant c(t)(2T^2+1)(\beta_\varepsilon)^{1-r(t)}\big(m^2T+\beta_\varepsilon\big)^{r(t)} + 2T\varepsilon^2.$$

From whence

$$\int_0^t \|u(x,\tau)\|_0^2\,d\tau \leqslant c(t)(\theta_\varepsilon)^{1-r(t)}\big(m^2T+\theta_\varepsilon\big)^{r(t)},$$

where $\theta_\varepsilon = c(t)(2T^2+1)(\beta_\varepsilon)^{1-r(t)}\big(m^2T+\beta_\varepsilon\big)^{r(t)} + 2T\varepsilon^2$. The theorem is proved. □

3. Approximate solution

Let $f(x,t)=0$, $\varphi_1(x)=0$, $\varphi_3(x)=0$, $\varphi_4(x)=0$. The solution of problem (1)–(4) exists and $(v(x,t),u(x,t)\,)\in M$, $\|\varphi_2(x)-\varphi_{2_\varepsilon}(x)\|_0 \leqslant \varepsilon$. For an approximate solution of problem (1)–(4) we use two methods:

I. Tikhonov regularization. Approximate solution for accurate data we define as the following sequence of functions

$$v^N(x,t) = \sum_{k=1}^N \big(v_k^+(t)X_k^+(x) + v_k^-(t)X_k^-(x)\big),$$

$$u^N(x,t) = \sum_{k=1}^N \big(u_k^+(t)X_k^+(x) + u_k^-(t)X_k^-(x)\big),$$

where

$v_k^+(t) = \varphi_{2_k}^+ \sinh\sqrt{\lambda_k^+}t/\sqrt{\lambda_k^+}$, $\;v_k^-(t) = \varphi_{2_k}^- \sin\sqrt{-\lambda_k^-}t/\sqrt{-\lambda_k^-}$,

$u_k^+(t) = \int_0^t v_k^+(\tau)\sinh\sqrt{\lambda_k^+}\,(t-\tau)\,d\tau/\sqrt{\lambda_k^+}$, $\;u_k^-(t) = \int_0^t v_k^-(\tau)\sin\sqrt{-\lambda_k^-}\,(t-\tau)\,d\tau/\sqrt{-\lambda_k^-}$,

N is integer parameter (regularization parameter). As an approximate solution with an approximate data it is possible to consider the sequence of functions

$$v_\varepsilon^N(x,t) = \sum_{k=1}^N \big(v_{k\varepsilon}^+(t)X_k^+(x) + v_{k\varepsilon}^-(t)X_k^-(x)\big),$$

$$u_\varepsilon^N(x,t) = \sum_{k=1}^N \big(u_{k\varepsilon}^+(t)X_k^+(x) + u_{k\varepsilon}^-(t)X_k^-(x)\big),$$

where

$v_{k\varepsilon}^+(t) = \varphi_{2_{k\varepsilon}}^+ \sinh\sqrt{\lambda_k^+}t/\sqrt{\lambda_k^+}$, $\;v_{k\varepsilon}^-(t) = \varphi_{2_{k\varepsilon}}^- \sin\sqrt{-\lambda_k^-}t/\sqrt{-\lambda_k^-}$, $\;\varphi_{2_{k\varepsilon}}^\pm = \int_{-1}^1 \varphi_{2_\varepsilon}(x)X_k^\pm(x)dx$,

$u_{k\varepsilon}^+(t) = \int_0^t v_{k\varepsilon}^+(\tau)\sinh\sqrt{\lambda_k^+}\,(t-\tau)\,d\tau/\sqrt{\lambda_k^+}$, $\;u_{k\varepsilon}^-(t) = \int_0^t v_{k\varepsilon}^-(\tau)\sin\sqrt{-\lambda_k^-}\,(t-\tau)\,d\tau/\sqrt{-\lambda_k^-}$.

We estimate the norm of the difference between the exact and approximate solutions

$$\left\|v(x,t)-v_\varepsilon^N(x,t)\right\|_0 \leqslant \left\|v(x,t)-v^N(x,t)\right\|_0+\left\|v^N(x,t)-v_\varepsilon^N(x,t)\right\|_0, \tag{14}$$

$$\left\|u(x,t)-u_\varepsilon^N(x,t)\right\|_0 \leqslant \left\|u(x,t)-u^N(x,t)\right\|_0+\left\|u^N(x,t)-u_\varepsilon^N(x,t)\right\|_0. \tag{15}$$

The second term on the right side of (14) can be written as

$$\left\|v^N(x,t)-v_\varepsilon^N(x,t)\right\|_0^2=\sum_{k=1}^N\left(\left(\varphi_{2_k}^+-\varphi_{2_{k\varepsilon}}^+\right)^2\frac{\sinh^2\sqrt{\lambda_k^+}t}{\lambda_k^+}+\left(\varphi_{2_k}^--\varphi_{2_{k\varepsilon}}^-\right)^2\frac{\sin^2\sqrt{-\lambda_k^-}t}{-\lambda_k^-}\right).$$

If we denote $\mu^2=\max\left(1/\lambda_1^+,\ 1/\left|\lambda_1^-\right|\right)$, then

$$\left\|v^N(x,t)-v_\varepsilon^N(x,t)\right\|_0^2 \leqslant \mu^2\sinh^2\sqrt{\lambda_N^+}t\left(\sum_{k=1}^N\left(\varphi_{2_k}^+-\varphi_{2_{k\varepsilon}}^+\right)^2+\left(\varphi_{2_k}^--\varphi_{2_{k\varepsilon}}^-\right)^2\right)$$

or

$$\left\|v^N(x,t)-v_\varepsilon^N(x,t)\right\|_0^2 \leqslant \mu^2\sinh^2\sqrt{\lambda_N^+}t\,\varepsilon^2. \tag{16}$$

We turn to the evaluation of the first term on the right side of inequality (14), which is expressed in the form

$$\left\|v(x,t)-v^N(x,t)\right\|_0^2=\sum_{k=N+1}^\infty\left(\left(\varphi_{2_k}^+\right)^2\frac{\sinh^2\sqrt{\lambda_k^+}t}{\lambda_k^+}+\left(\varphi_{2_k}^-\right)^2\frac{\sin^2\sqrt{-\lambda_k^-}t}{-\lambda_k^-}\right). \tag{17}$$

From (13) we have

$$\sum_{k=1}^\infty\left(\varphi_{2_k}^+\right)^2\sinh^2\sqrt{\lambda_k^+}T/\lambda_k^+ \leqslant m^2. \tag{18}$$

It can be seen that under the conditions (18) the right side of (17) reaches its maximum value if the coefficients $\varphi_k^\pm$ satisfy the following conditions

$$\varphi_{2_k}^\pm=0,\ \ k\neq N+1,$$

$$\varphi_{2_{N+1}}^+=\frac{m\sqrt{\lambda_{N+1}^+}}{\sinh\sqrt{\lambda_{N+1}^+T}}.$$

Then

$$\left\|v(x,t)-v^N(x,t)\right\|_0^2 \leqslant \frac{m^2\sinh^2\sqrt{\lambda_{N+1}^+t}}{\sinh^2\sqrt{\lambda_{N+1}^+T}}+\delta(N), \tag{19}$$

where $\delta(N)=\sum\limits_{k=N+1}^\infty\left(\varphi_{2_k}^-\right)^2/\left|\lambda_k^-\right|$. Here, $\delta(N)\to 0$ at $N\to\infty$.

Next, we consider the inequality (15). Each term on the right side of inequality (15) we estimate separately

$$\left\|u^N(x,t)-u_\varepsilon^N(x,t)\right\|_0^2=\sum_{k=1}^N\frac{\left(\varphi_{2_k}^+-\varphi_{2_{k\varepsilon}}^+\right)^2}{\left(\lambda_k^+\right)^2}\left(\int_0^t\sinh\sqrt{\lambda_k^+}\,(t-\tau)\sinh\sqrt{\lambda_k^+}\tau\,d\tau\right)^2+$$

$$+\sum_{k=1}^N\frac{\left(\varphi_{2_k}^--\varphi_{2_{k\varepsilon}}^-\right)^2}{\left(\lambda_k^-\right)^2}\left(\int_0^t\sin\sqrt{-\lambda_k^-}\,(t-\tau)\sin\sqrt{-\lambda_k^-}\tau\,d\tau\right)^2 \leqslant \frac{\mu^4}{4}t^2\cosh^2\sqrt{\lambda_N^+}t\,\varepsilon^2. \tag{20}$$

As well as

$$\left\|u(x,t)-u^N(x,t)\right\|_0^2=\sum_{k=N+1}^{\infty}\frac{\left(\varphi_{2_k}^+\right)^2}{\left(\lambda_k^+\right)^2}\left(\int_0^t \sinh\sqrt{\lambda_k^+}\,(t-\tau)\,\sinh\sqrt{\lambda_k^+}\tau\,d\tau\right)^2+$$

$$+\sum_{k=N+1}^{\infty}\frac{\left(\varphi_{2_k}^-\right)^2}{\left(\lambda_k^-\right)^2}\left(\int_0^t \sin\sqrt{-\lambda_k^-}\,(t-\tau)\sin\sqrt{-\lambda_k^-}\tau\,d\tau\right)^2$$

or

$$\left\|u(x,t)-u^N(x,t)\right\|_0^2\leqslant\sum_{k=N+1}^{\infty}\frac{\left(\varphi_{2_k}^+\right)^2}{4\left(\lambda_k^+\right)^2}\left(t\,\cosh\sqrt{\lambda_k^+}t-\sinh\sqrt{\lambda_k^+}t\right)^2+\sigma(N), \tag{21}$$

where $\sigma(N)=\sum\limits_{k=N+1}^{\infty}\left(\varphi_{2_k}^-\right)^2t^2/\left(\lambda_k^-\right)^2$, $\sigma(N)\to 0$, at $N\to\infty$. From (13) follow $\|u(x,T)\|_0^2\leqslant m^2$. Using this inequality we get

$$\sum_{k=1}^{\infty}\frac{\left(\varphi_{2_k}^+\right)^2}{4\left(\lambda_k^+\right)^2}\left(T\,\cosh\sqrt{\lambda_k^+}T-\sinh\sqrt{\lambda_k^+}T\right)^2\leqslant m^2. \tag{22}$$

Choose the coefficients $\varphi_{2_k}^+$ so that, the right part of (21) reaches its maximum value

$$\varphi_{2_k}^+=0,\ k\neq N+1,$$

$$\varphi_{2_{N+1}}^+=2m\lambda_{N+1}^+/(T\,\cosh\sqrt{\lambda_{N+1}^+}T-\sinh\sqrt{\lambda_{N+1}^+}T).$$

Then

$$\left\|u(x,t)-u^N(x,t)\right\|_0^2\leqslant\frac{m^2\left(t\,\cosh\sqrt{\lambda_{N+1}^+}t-\sinh\sqrt{\lambda_{N+1}^+}t\right)^2}{\left(T\,\cosh\sqrt{\lambda_{N+1}^+}T-\sinh\sqrt{\lambda_{N+1}^+}T\right)^2}+\sigma(N). \tag{23}$$

As a result of (16), (19), (20), (23) we get estimate for the norm of difference between the exact and approximate solutions of problem (1)–(4) in the form

$$\left\|v(x,t)-v_\varepsilon^N(x,t)\right\|_0^2\leqslant 2\mu^2\sinh^2\sqrt{\lambda_N^+}t\,\varepsilon^2+\frac{2m^2\sinh^2\sqrt{\lambda_{N+1}^+}t}{\sinh^2\sqrt{\lambda_{N+1}^+}T}+2\delta(N),$$

$$\left\|u(x,t)-u_\varepsilon^N(x,t)\right\|_0^2\leqslant\frac{\mu^4}{2}t^2\cosh^2\sqrt{\lambda_N^+}t\,\varepsilon^2+\frac{2m^2\left(t\,\cosh\sqrt{\lambda_{N+1}^+}t-\sinh\sqrt{\lambda_{N+1}^+}t\right)^2}{\left(T\,\cosh\sqrt{\lambda_{N+1}^+}T-\sinh\sqrt{\lambda_{N+1}^+}T\right)^2}+2\sigma(N).$$

Minimizing the last relations with $\varepsilon>0$ with respect to the variable N we find a formula for calculation of the regularization parameter, which depends on ε, m and T.

II. Quasi-inverse method. As an approximate solution of the original problem we take the following sequence of functions that depends on a parameter α, $\alpha>0$. (α is called a regularization parameter)

$$v^\alpha(x,t)=\sum_{k=1}^{\infty}\left(v_k^+(t)e^{-\alpha\lambda_k^+t}X_k^+(x)+v_k^-(t)X_k^-(x)\right),$$

$$u^\alpha(x,t)=\sum_{k=1}^{\infty}\left(u_k^+(t)e^{-\alpha\lambda_k^+t}X_k^+(x)+u_k^-(t)X_k^-(x)\right).$$

Then an approximate solution under approximate data can be taken as

$$v_\varepsilon^\alpha(x,t) = \sum_{k=1}^{\infty}\Big(v_{k\varepsilon}^{+}(t)e^{-\alpha\lambda_k^{+}t}X_k^{+}(x) + v_{k\varepsilon}^{-}(t)X_k^{-}(x)\Big),$$
$$u_\varepsilon^\alpha(x,t) = \sum_{k=1}^{\infty}\Big(u_{k\varepsilon}^{+}(t)e^{-\alpha\lambda_k^{+}t}X_k^{+}(x) + u_{k\varepsilon}^{-}(t)X_k^{-}(x)\Big).$$

Since this solution has a certain accuracy, we estimate the norm of the difference between the exact and regularized solutions

$$\begin{aligned}\|v(x,t)-v_\varepsilon^\alpha(x,t)\|_0 &\leqslant \|v(x,t)-v^\alpha(x,t)\|_0 + \|v^\alpha(x,t)-v_\varepsilon^\alpha(x,t)\|_0,\\ \|u(x,t)-u_\varepsilon^\alpha(x,t)\|_0 &\leqslant \|u(x,t)-u^\alpha(x,t)\|_0 + \|u^\alpha(x,t)-u_\varepsilon^\alpha(x,t)\|_0.\end{aligned} \quad (24)$$

Using Lagrange's method to find the constrained optimization under the condition $(v(x,t),u(x,t)) \in M$. After some elementary transformations we get

$$\|v(x,t)-v^\alpha(x,t)\|_0 \leqslant \frac{4mt\alpha}{\left(1-e^{-2\sqrt{\lambda_1^{+}}T}\right)(T-t)^2e^2}.$$

Similarly for the second term on the right side (24) is true the following estimate

$$\|v^\alpha(x,t)-v_\varepsilon^\alpha(x,t)\|_0 \leqslant \frac{\mu}{2}e^{\frac{t}{4\alpha}}\varepsilon.$$

Besides that we have

$$\|u^\alpha(x,t)-u_\varepsilon^\alpha(x,t)\|_0 \leqslant \frac{\mu^2}{2}te^{\frac{t}{4\alpha}}\varepsilon,$$

and

$$\|u(x,t)-u^\alpha(x,t)\|_0 \leqslant \frac{8mt^2\alpha}{|T-1|e^2(T-t)^2}.$$

So

$$\|v(x,t)-v_\varepsilon^\alpha(x,t)\|_0 \leqslant \frac{4mt\alpha}{\left(1-e^{-\sqrt{\lambda_1^{+}}T}\right)(T-t)^2e^2} + \frac{\mu}{2}e^{\frac{t}{4\alpha}}\varepsilon, \quad (25)$$

$$\|u(x,t)-u_\varepsilon^\alpha(x,t)\|_0 \leqslant \frac{8mt^2\alpha}{|T-1|e^2(T-t)^2} + \frac{\mu^2}{2}te^{\frac{t}{4\alpha}}\varepsilon. \quad (26)$$

Minimizing the right side of (25) and (26) for any $\varepsilon > 0$, respectively, we find the formula for α regularization parameter, where $t \neq T$, $T \neq 1$.

Conclusion

In this paper was considered ill-posed problem for system of mixed-type partial differential equations. On the base Tikhonov definitions we have show this problem has unique and conditional stable solution on the set of correctness M. Here was constructed approximated solutions (by two way) and value of this solutions were calculated numerically. Calculations show this approximate solutions are close to exact solution if we choose parameter of regularization from the minimization of the estimate norm of the difference between exact and approximated solutions.

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Условная корректность и приближенное решение краевой задачи для системы уравнений смешанного типа второго порядка

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В данной работе рассматривается система уравнений смешанного типа второго порядка. Доказаны теоремы о единственности решения и его условной устойчивости на множестве корректности. Построено приближенное решение методом регуляризации и методом квазиобращения.

Ключевые слова: система уравнений, начально-краевая задача, некорректная задача, априорная оценка, теорема о единственности, условная устойчивость, множества корректности, приближенное решение, регуляризация.