Journal of Siberian Federal University. Mathematics & Physics 2018, 2, 191–193

УДК 517.98

Regularization of the Cauchy Problem for Elliptic Operators

Anastasiya S. Peicheva*
Institute of Mathematics and Computer Science
Siberian Federal University
Svobodny 79, Krasnoyarsk, 660041
Russia

Received 10.05.2017, received in revised form 10.12.2017, accepted 20.03.2018

We regularize the ill-posed Cauchy problem for a first order elliptic matrix differential operator A with the use of a mixed problem for its Laplacian A^*A, depending on small parameter in boundary conditions.

Keywords: elliptic operators, the Cauchy problem, small parameter method
DOI: 10.17516/1997-1397-2018-18-2-191-193.

The Cauchy problem for elliptic linear differential operators is a long standing problem connected with numerous applications in physics, electrodynamics, fluid mechanics etc. (see [1,4] or elsewhere). It appears that the regularization methods (see [5]) are most effective for studying the problem. Recently, a new approach was developed, cf. [2] based on the simple observation that the calculus of the solutions to the Cauchy problems foran elliptic equations just amounts to the calculus of a (possibly non-coercive) mixed boundary value problems for an elliptic equations with a parameter.

Let D be a bounded domain with Lipschitz boundary ∂D in Euclidean space $\mathbb{R}^n$, $n \geqslant 2$, with coordinates $x = (x_1, \dots, x_n)$. For some multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ we will write ∂^α for the partial derivative $\dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$. We consider the complex-valued functions defined over the domain D and its closure $\overline{D}$. We also fix a relatively open connected set S with piecewise smooth boundary ∂S on the hypersurface ∂D. Let $C^s(\overline{D}, \overline{S})$, $s \in \mathbb{Z}_+$, be the set of s-times continuously differentiable functions in $\overline{D}$, which are disappearing in some (one-sided) neighborhood of $\overline{S}$ in D. Let $L^q(D)$, $1 \leqslant q \leqslant +\infty$, stand for the standard normed Lebesgue spaces of functions over D. We also write $H^s(D)$, $s \in \mathbb{N}$, for the Sobolev space of functions whose weak derivatives up to the order s belong to $L^2(D)$. Let the space $H^s_0(D)$ stand for the closure of the space $C^\infty_0(D)$ in $H^s(D)$. For positive non-integer s we denote by $H^s(D)$ the standard Sobolev-Slobodetskii space The closure of $C^s(\overline{D}, \overline{S})$ in the space $H^s(D)$ is denoted by $H^s(D, S)$. Also, we will need Sobolev spaces $H^{-s}(D)$ with negative smoothness which we define in the usual way as the dual to $H^s(D)$, with respect to the pairing $\langle \cdot, \cdot \rangle$, induced from $L^2(D)$ see, for instance, [3], [4, Sec. 1.1].

Let $A(x, \partial)$ be a first order matrix differential operator in a domain $X \subset \mathbb{R}^n$, i.e. $A = \sum\limits_{j=1}^{n} A_j(x)\partial_j + A_0(x)$. Here $A_j(x)$ are $(k \times k)$-matrices, whose components are complex-valued real-analytic functions. The operator A is called elliptic on X if $\det \left(\sum\limits_{j=1}^{n} A_j(x)\zeta_j \right) \neq 0$ for all $x \in X, \zeta \in \mathbb{R}^n \backslash \{0\}$. Let $A^*_j(x)$ be the adjoint matrix for the matrix $A_j(x)$ and $A^* = -\sum\limits_{j=1}^{n} \partial_j (A^*_j(x) \cdot) +$

*peichevaas@mail.ru

A_0^* be the formal adjoint for A. If A is elliptic, then the second order differential operator A^*A is strongly elliptic in X.

Problem 1. *Consider the ill-posed Cauchy problem for the operator A in the domain D with boundary data on the set S: given distributions u_0 on S and f over D, find a distribution u satisfying in a proper sense*

$$\begin{cases} Au &= f \quad in \quad D, \\ u &= 0 \quad on \quad S. \end{cases} \tag{1}$$

In order to control the behaviour of solutions to problem (1), it is natural to introduce the following function spaces. For $\varepsilon \geqslant 0$ we consider the Hermitian form $\varepsilon \geqslant 0$ on the space $[C^1(\overline{D}, \overline{S})]^k$: $(u, v)_{+,\varepsilon} = \varepsilon\, (u, v)_{[L^2(\partial D)]^k} + (Au, Av)_{[L^2(D)]^k}$. If $(u, v)_{+,1}$ is an inner product on $[C^1(\overline{D}, \overline{S})]^k$, then we write $H^+(D, S)$ for the completion of $[C^1(\overline{D}, \overline{S})]^k$ with respect to the norm $\|\cdot\|_{+,1}$ induced by the scalar product $(\cdot,\cdot)_{+,1}$. Obviously, in this case the norms $\|u\|_{+,\varepsilon}$ and $\|u\|_{+,\delta}$ are equivalent for any positive ε and δ. Everywhere below we assume that $H^+(D, S)$ is embedded continuously to $[L^2(D)]^k$; then let ι be the natural (continuous) embedding: $\iota : H^+(D, S) \to [L^2(D)]^k$. Clearly problem (1) can be treated as the investigation of the bounded linear operator

$$A : H^+(D; S) \to [L^2(D)]^k. \tag{2}$$

Lemma 1. *Let $\partial D \in C^\infty$. If the interior of S on ∂D is non empty then the null-space of the operator (2) is trivial. If the interior of $\partial D \setminus \overline{S}$ on ∂D is non empty then the range of the operator (2) is dense in $[L^2(D)]^k$.*

Proof. Follows from the Uniqueness theorem for the Cauchy problem for elliptic systems A and A^* [4, Theorem 10.3.5]. □

Thus we have described the closure of the image of the map (2). Description of the image of the map (2) itself is a more difficult task. However, we note that a function $u \in H^+(D, S)$ is a solution to problem (1) if and only if for all $v \in H^+(D, S)$

$$(Au, Av)_{[L^2(D)]^k} = (f, Av)_{[L^2(D)]^k}. \tag{3}$$

Taking into account this observation, perturbed Cauchy problem:

Problem 2. *Fix $\varepsilon \in (0, 1]$. Given any $f \in [L^2(D)]^k$, find an element $u_\varepsilon \in H^+(D, S)$, which for all $v \in H^+(D, S)$ will be satisfying*

$$(Au_\varepsilon, Av)_{[L^2(D)]^k} + \varepsilon\, (u_\varepsilon, v)_{[L^2(\partial D \setminus \overline{S})]^k)} = (f, Av)_{[L^2(D)]^k}. \tag{4}$$

The difference between Problems 1 and 2 is that the last one is well-posed in $H^+(D, S)$.

Lemma 2. *For every $\varepsilon > 0$ and $f \in [L^2(D)]^k$ there exists an unique solution $u_\varepsilon(f) \in H^+(D, S)$ to Problem 2. Moreover, it satisfies $\|u_\varepsilon(f)\|_{+,\varepsilon} \leqslant \|f\|_{[L^2(D)]^k}$.*

Proof. The proof follows from Schwarz inequality and Riesz theorem. □

Lemma 3. *For every $\varepsilon \in (0, 1]$ there are positive numbers $\{\lambda_k^{(\varepsilon)}\}_{k\in\mathbb{N}}$ and functions $\{b_k^{(\varepsilon)}\}_{k\in\mathbb{N}} \subset H^+(D, S)$ such that*

$$\left(Ab_k^{(\varepsilon)}, Av\right)_{[L^2(D)]^k} + \varepsilon \left(b_k^{(\varepsilon)}, v\right)_{[L^2(\partial D \setminus \overline{S})]^k} = \lambda_k^{(\varepsilon)} \left(b_k^{(\varepsilon)}, v\right)_{[L^2(D)]^k} \tag{5}$$

for all $v \in H^+(D, S)$. The system $\{b_k^{(\varepsilon)}\}_{k\in\mathbb{N}}$ яis an orthonormal basis $H^+(D, S)$ (with respect to $(\cdot,\cdot)_{+,\varepsilon}$), it is also an orthogonal basis in $[L^2(D)]^k$.

Proof. See [3, Lemma 3.1]. □

The behaviour of the family $\{u_\varepsilon(f)\}_{\varepsilon>0}$ reflects on the solvability of problem (1).

Theorem 1. *The family $\{\|u_\varepsilon(f)\|_{+,1}\}_{\varepsilon\in(0,1]}$ is bounded if and only if there exists $u \in H^+(D,S)$ satisfying* (3)*. Under this conditions $\lim_{\varepsilon\to+0}\|Au_\varepsilon(f)-f\|_{[L^2(D)]^k}=0$ and even $\{u_\varepsilon(f)\}_{\varepsilon\in(0,1]}$ converges weakly in $H^+(D,S)$, when $\varepsilon\to+0$, to the solution $u\in H^+(D,S)$ of problem* (1)*. Moreover, it converges to u in $[H^s(D)]^k$ for every $s<1/2$ and also in the space $[H^1_{\mathrm{loc}}(D\cup S)]^k$.*

Proof. Follows from Lemma 2, cf. [2, Theorem 3.1] for the Cauchy-Riemann system. □

Finally, we obtain a formula for solutions to Problem 1.

Corollary 1. *For any function $u\in H^+(D,S)$ we have:*

$$(u,v)_{+,1} = \lim_{\varepsilon\to+0}\lim_{N\to+\infty}\Big(\big(Au, A\mathcal{G}^{(N)}_\varepsilon(z,\cdot)\big)_{[L^2(D)]^k}, v(z)\Big)_{[L^2(D)]^k},$$

for all $v\in H^+(D,S)$, where $\mathcal{G}^{(N)}_\varepsilon(z,\zeta)=\sum\limits_{k=1}^{N}\dfrac{b^{(\varepsilon)}_k(z)\overline{b^{(\varepsilon)}_k(\zeta)}}{\|b^{(\varepsilon)}_k\|_{[L^2(D)]^k}}$.

The research for this work was carried out in Siberian Federal University. The author was supported by grant of the Ministry of Education and Science of the Russian Federation no. 1.2604.2017/PCh.

References

[1] M.M.Lavrent'ev, On the Cauchy problem for linear elliptic equations of the second order, *Dokl. AN SSSR*, **112**(1957), no. 2, 195–197 (in Russian).

[2] A.N.Polkovnikov, A.A.Shlapunov, Construction of Carleman formulas by using mixed problems with parameter-dependent boundary conditions, *Siberian Math. J.*, **58**(2017), no. 4, 676–686.

[3] A.A.Shlapunov, N.N.Tarkhanov, On completeness of root functions of Sturm-Liouville problems with discontinuous boundary operators, *Journal of Differential Equations*, **255**(2013), 3305–3337.

[4] N.N.Tarkhanov, The Cauchy problem for solutions of elliptic equations, Berlin, Acad. Verl., Vol. 7, 1995.

[5] A.N.Tikhonov, V.Y.Arsenin, Solution of Ill-posed Problems, Washington, Winston & Sons, 1977.

Регуляризация задачи Коши для эллиптических операторов

Анастасия С. Пейчева
Институт математики и фундаментальной информатики
Сибирский федеральный университет
Свободный, 79, Красноярск, 660041
Россия

Некорректная задача Коши для матричного эллиптического дифференциального оператора A регуляризована с помощью решений смешанных задач для его Лапласиана A^*A, зависящих от малого параметра $\varepsilon>0$ в граничных условиях.

Ключевые слова: эллиптические операторы, задача Коши, метод малого параметра.