On intersection of primary subgroups of odd order in finite almost simple groups

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Abstract

The question of the determination of subgroups A and B such that $A \cap B^g \neq 1$ for any $g \in G$ for a finite almost simple group G and its primary subgroups A and B of odd order set up. We prove that there exist four possibilities for ordered pair (A, B) only.

Introduction

Let G be a finite group and A, B be its subgroups. By the definition, M is the set of subgroups which are minimal by inclusion among all subgroups of type $A \cap B^g$, $g \in G$, and m consists of those elements of the set M the order of which is minimal. Set $\mathrm{Min}_G(A, B) = \langle M \rangle$, and $\mathrm{min}_G(A, B) = \langle m \rangle$. Evidently, $\mathrm{Min}_G(A, B) \geq \mathrm{min}_G(A, B)$ and the following three conditions are equivalent:

а) $A \cap B^g \neq 1$ for any $g \in G$;
б) $\mathrm{Min}_G(A, B) \neq 1$;
в) $\mathrm{min}_G(A, B) \neq 1$.

The second author showed that $\mathrm{Min}_G(A, B) \leq F(G)$ for any pair of abelian subgroups A and B of G [4, Theorem 1], where $F(G)$ is the Fitting subgroup of G (the greatest normal nilpotent subgroup of G). On the other hand, V. I. Zenkov and V. D. Mazurov proved that $\mathrm{Min}_G(A, B) = 1$ for any pair of primary subgroups A and B of a simple nonabelian group G [3, Theorem 1]. But also in almost simple group $G \simeq \mathrm{Aut}(L_2(7))$ we have $\mathrm{Min}_G(S, S) = \mathrm{min}_G(S, S) = S$ for a Sylow 2-subgroup S of G. Moreover it is proved in [5] that for a group G with socle $L_2(q)$, $q > 3$, if subgroups A and B are primary, then the inequality $\mathrm{Min}_G(A, B) \neq 1$ is valid only for $q = 9$ and for the Mersenne prime $q = 2^n - 1$; in these cases subgroups A and B are 2-groups.

In the present article we consider the case when A and B are primary subgroups of odd order in a finite almost simple group G. Our main result is the following theorem.

Theorem 1. *Let G be a finite almost simple group and A, B be its primary subgroups of odd order. Then the following are equivalent.*
(1) $\mathrm{Min}_G(A, B) \neq 1$.
(2) G contains a normal subgroup of index 1 or 2, which is isomorphic to $D_4(3) \rtimes Z_3$, and $(A, B) \in \{S, S_0\}^2$. Where S is a Sylow 3-subgroup of G, $S_0 = O_3(N_G(P))$ and P is a minimal proper parabolic subgroup of the group $D_4(3)$ corresponding to the central node of the Coxeter graph of type D_4. Moreover $S_0 = \mathrm{min}_G(S, S)$.

In section 2 we establish some necessary for the proof of theorem 1 properties of subgroups of Chevalley groups, which in the opinion of the authors and are of independent interest.

1. Notations and preliminary results

A finite group G is called *almost simple*, if $\mathrm{Inn}(K) \le G \le \mathrm{Aut}(K)$, where K is a finite simple non-abelian group, and $\mathrm{Inn}(K)$ and $\mathrm{Aut}(K)$ are groups of inner and respectively all automorphisms of a group K. In this article the following shortcuts and notations are used:

$g^h = h^{-1}gh$ for elements g and h of a group G;

$A^B = \{a^b \mid a \in A, \ b \in B\}$ for subsets A and B of a group G;

$A \le G$ denotes that A is a subgroup of a group G;

$\langle M \rangle$ is the group generated by a set M;

$N_G(A)$ is the normalizer of a subgroup A in a group G;

$O_p(G)$ is the maximal normal p-subgroup of a finite group G;

$A \rtimes B$ is the semidirect product of groups A and B with a normal subgroup A.

Actually, the proof of the implication $(1) \Rightarrow (2)$ of Theorem 1 is reduced to the analysis of situation in the group $D_4(3) \rtimes Z_3$ by the usage of the following result of the second author.

Lemma 1. [5, Theorem B(2a)] *Let G be a finite almost simple group, p be an odd prime and S be a Sylow p-subgroup of G. If $S \cap S^g \ne 1$ for any element g of G, then $p = 3$ and G contains a normal subgroup of index 1 or 2, which is isomorphic to $D_4(3) \rtimes Z_3$.*

We will also need the following two technical lemmas, which will be use in the sequel.

Lemma 2. [5, Lemma 3.1] *Let G be a finite group and M_1 be a subgroup of G. Let P_1 be a Sylow p-subgroup of M_1 such that $P_1 \cap P_1^k = O_p(M_1)$ for some $k \in M$, and M_2 be a conjugate with M_1 subgroup of G. Then there exists Sylow p-subgroup P_2 of M_2 such that $P_1 \cap P_2 \le O_p(M_1) \cap O_p(M_2)$.*

Lemma 3. *Let A, B, S be subgroups of a finite group G such that $\mathrm{Min}_G(A, B) \ne 1$, $A \le S$ and $A \cap B^h = S \cap S^h = T$ for some element $h \in G$ and some cyclic subgroup T of prime order. Then $T^S \le A$.*

Proof. By the lemma conditions we have $T^s = S \cap S^{hs} \ge A \cap B^{hs} \ne 1$ for any $s \in S$. Therefore taking into account that T is a cyclic subgroup of prime order, we deduce the inclusion $T^s \le A$ for any $s \in S$, that is $T^S \le A$. The lemma is proved.

2. Some properties of intersections of Sylow p-subgroups of Chevalley groups over a finite field of characteristic p

Further Φ is a reduced indecomposable root system, $\Pi = \{r_1, ..., r_l\}$ is its set of fundamental roots, Φ^+ is a positive root system respect to Π, and also $\Phi^- = -\Phi^+$. We always suggest, that r_1 is a short root and sum $r_i + r_j$, $i \le j$, is a root if and only if:

$(i, j) = (l - 3, l)$ or $(i, i + 1), 1 \le i \le l - 2$, if $\Phi = E_l$;

$(i, j) = (1, 3)$ or $(i, i + 1), 2 \le i \le l - 1$, if $\Phi = D_l$;

$(i, j) = (i, i + 1)$ in all other cases.

We will need the following strengthening the Lemma 3.6.2 from the book R. Carter [2, p. 50].

Lemma 4. *Let a fundamental root r_{i_1} be a part with nonzero coefficient of the expression of a root $r \in \Phi^+$ as an integral combination of fundamental roots with non-negative coefficients. Then r can be expressed as a sum of fundamental roots*

$$r = r_{i_1} + r_{i_2} + \cdots + r_{i_k}$$

in such a way that $r_{i_1} + r_{i_2} + \cdots + r_{i_s}$ is a root for all $s \le k$.

Proof. Let

$$r = c_1 r_1 + \cdots + c_l r_l.$$

be the expression of a root $r \in \Phi^+$ as an integral combination of fundamental roots with non-negative coefficients. Obviously, that we have two following cases only: 1) $c_i \leq 1$; 2) at least one of numbers c_i is more than one.

For any subgraph of the Coxeter graph of type Φ, whose vertices are labeled by fundamental roots $r_{j_1}, \ldots, r_{j_m}$, the sum $r_{j_1} + \cdots + r_{j_m}$ if and only if is a root when this subgraph is connected. Therefore in the first case the lemma is true. The second case is reduced by induction to the first one by virtue of the following assertion.

A) If $c_i > 1$ for some i, then there exists j such that $c_j > 1$ and the difference $r - r_j$ is a root.

Assertion A) can be directly verified for every root system.

For the root system of type A_l the first case is possible only.

For the root system of type B_l in the case 2) we have $r = 2r_1 + \cdots + 2r_s + r_{s+1} + \cdots + r_t$, where $1 \leq s < t \leq l$. Here the one variant $j = s$ is possible only, in order that the difference $r - r_j$ is a root.

For the root system of type C_l in the case 2) we have $r = r_t + \cdots + r_{s-1} + 2r_s + \cdots + 2r_{l-1} + r_l$, where $1 \leq t \leq s \leq l - 1$ (for $s = 1$ we set $r = 2r_1 + \cdots + 2r_{l-1} + r_l$). Here also one variant $j = s$ is possible only.

For the root system of type D_l in the case 2) we have $r = r_1 + r_2 + 2r_3 + \cdots + 2r_s + r_{s+1} + \cdots + r_t$, where $3 \leq s < t \leq l$. Here also one variant $j = s$ is possible only.

In the Tables V–VIII from [1] for the exceptional types E_l and F_4 all positive roots, which have even one of numbers c_j more than one are listed. Using these tables, it is not difficult to check the validity of the assertion A) for the types E_l and F_4. Note that even here for some roots the parameter j is not uniquely defined.

For type G_2 the conclusion A) correctness is easily checked and in this case the parameter j is defined uniquely.

Hence, the assertion A), along with lemma is proved.

Further $\Phi(q)$ is an adjoint Chevalley group of type Φ of rank l over the finite field $\mathbb{F}_q$ of the order $q = p^n$, where p is a prime. The group $\Phi(q)$ is generated by the root subgroups

$$X_r = \langle x_r(t) \mid t \in K \rangle, \ r \in \Phi,$$

where $x_r(t)$ is the corresponding root element in the group $\Phi(q)$. We will need the following natural subgroups of the group $\Phi(q)$:

unipotent subgroups

$$U = \langle X_r \mid r \in \Phi^+ \rangle,$$

$$V = \langle X_r \mid r \in \Phi^- \rangle,$$

monomial subgroup

$$N = \langle n_r(t) \mid r \in \Phi, \ t \in \mathbb{F}_q^* \rangle,$$

diagonal subgroup

$$H = \langle h_r(t) \mid r \in \Phi, \ t \in \mathbb{F}_q^* \rangle$$

and Borel subgroup

$$B = UH.$$

Here, $\mathbb{F}_q^*$ is the multiplicative subgroup of the field $\mathbb{F}_q$ and

$$n_r(t) = x_r(t)x_{-r}(-t^{-1})x_r(t),$$

$$h_r(t) = n_r(t)n_r(-1).$$

We set also

$$I = \{1, 2, \ldots, l\}.$$

Overgroups of the Borel subgroup B and conjugate with them are called *parabolic*. Due to familiar result of J.Tits, parabolic subgroups containing subgroup B are

$$P_J = \langle B, n_{r_j}(1) \mid j \in J \rangle,$$

3

where $J \subseteq I$.

Lemma 5. *Fix a monomial element n_0 with condition $U^{n_0} = V$ and a positive integer $i \in I$. Set $n = n_0 n_{r_i}(1)$. Then $U \cap U^n = X_{r_i}$.*

Proof. The root subgroups X_{r_i} and X_{-r_i} normalize the subgroup

$$V_{r_i} = \langle X_r \mid r \in \Phi^- \backslash \{-r_i\} \rangle$$

and $V = V_{r_i} X_{-r_i}$ [2, Lemma 8.1.1]. Therefore $U^n = V^{n_{r_i}(1)} = V_{r_i} X_{r_i}$. Clearly, that $U \cap V_{r_i} X_{r_i} = X_{r_i}$. The lemma is proved.

For $l = 1$ the root subgroup X_{r_i} coincides with a Sylow p-subgroup of the group $\Phi(q)$ and in this case the element n from Lemma 5 is diagonal.

Lemma 6. *Let $P = P_{I \backslash \{i\}}$ be the parabolic maximal subgroup of the group $\Phi(q)$ of type A_l, D_l or E_l of rank $l \geq 2$ and the monomial element n as in Lemma 5. Then $U \cap U^n = X_{r_i}$ and $\langle X_{r_i}^U \rangle = O_p(P)$.*

Proof. For Chevalley groups $\Phi(q)$ of any type, the equality

$$O_p(P) = \langle X_r \mid r = c_k r_k + \cdots + c_i r_i + \cdots + c_m r_m, \ 1 \leq k \leq i \leq m \leq l, \ c_j \geq 1 \rangle.$$

holds [2, Theorem 8.5.2]. For types A_l, D_l and E_l, all structure constants of Chevalley's commutator formula are equal to 1. Hence, using Lemma 4, we can obtain the equality $\langle X_{r_i}^U \rangle = O_p(P)$. Really, let $X_r \in O_p(P)$. Then by Lemma 4 as $i = i_1$ for the root r we have the following representation

$$r = r_{i_1} + r_{i_2} + \cdots + r_{i_k},$$

where the sum $r_{i_1} + r_{i_2} + \cdots + r_{i_s}$ is the root for all $s \leq k$. Therefore, we obtain inclusions

$$[X_{r_{i_1}}, X_{r_{i_2}}] = X_{r_{i_1}+r_{i_2}} \subset \langle X_{r_i}^U \rangle,$$

$$[X_{r_{i_1}+r_{i_2}}, X_{r_{i_3}}] = X_{r_{i_1}+r_{i_2}+r_{i_3}} \subset \langle X_{r_i}^U \rangle,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$[X_{r_{i_1}+\cdots+r_{i_{k-1}}}, X_{r_k}] = X_{r_{i_1}+\cdots+r_{i_k}} = X_r \subset \langle X_{r_i}^U \rangle.$$

Hence, $\langle X_{r_i}^U \rangle = O_p(P)$. Lemma 5 gives the equality $U \cap U^n = X_{r_i}$. The lemma is proved.

Lemma 6 conclusion can not be adapted in general for each types B_l, C_l, F_4 and G_2. For example, the following result holds.

Lemma 7. *Let $P = P_{I \backslash \{1\}}$ be the parabolic maximal subgroup of the group $\Phi(2)$ of type B_l, $l \geq 2$, over the field of two elements, where r_1 is short root. There is no a root subgroup X_r such that $\langle X_r^U \rangle = O_p(P)$.*

Proof. Due to the parabolic maximal subgroup P draw, the following equality is valid

$$O_p(P) = \langle X_s \mid s = c_1 r_1 + \cdots + c_k r_k, \ 1 \leq k \leq l, \ c_j \geq 1 \rangle.$$

It is trivial, that the equality $\langle X_r^U \rangle = O_p(P)$ is admissible only for $r = r_1$. Note that subgroup $\langle X_{r_1}^U \rangle$ contains the product $x_{r_1+r_2}(1) x_{2r_1+r_2}(1)$, but individually the elements $x_{r_1+r_2}(1)$ and $x_{2r_1+r_2}(1)$ do not belong to $\langle X_{r_1}^U \rangle$. The lemma is proved.

The Picture 1 depicts correspondence between nodes of the Coxeter graph and roots from the fundamental root system, which are associated with the Chevalley group of type D_4, is recognized.
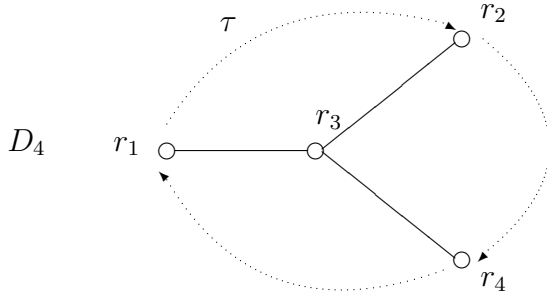
Pic. 1

Lemma 8. *Let $G = D_4(3) \rtimes \langle \tau \rangle$ and $S = U \rtimes \langle \tau \rangle$, where τ is the graph automorphism of order 3 of the group $D_4(3)$ as on the picture 1, and let the monomial element $n = n_0 n_{r_1}(1) \in D_4(3)$ be as in Lemma 5. Then $S \cap S^n = X_{r_1}$, in particular, $U \cap U^n = X_{r_1}$.*

Proof. In the Weyl group of type D_4 there exists an element w_0 such that $w_0(r) = -r$ for any root r. Moreover, w_0 coincides with cube c^3 of the Coxeter element

$$c = w_{r_3} w_{r_1} w_{r_2} w_{r_4}.$$

Hence in our case the element n_0 is the preimage of the element w_0 under the homomorphism of the monomial subgroup of the group $D_4(3)$ on the Weyl group of type D_4. Moreover, we can suggest, that

$$n_0 = (n_{r_3}(1)n_{r_1}(1)n_{r_2}(1)n_{r_4}(1))^3.$$

Since the graph automorphism τ centralizes the monomial element

$$n_{r_3}(1)n_{r_1}(1)n_{r_2}(1)n_{r_4}(1)$$

(see Picture 1 and [2, Proposition 12.2.3]), then

$$\tau^n = \tau^{n_{r_1}(1)} = \tau n_{r_2}(-1)n_{r_1}(1).$$

It is clear that $\tau n_{r_2}(-1)n_{r_1}(1) \notin \langle \tau \rangle$. Therefore, and from the equality $U \cap U^n = X_{r_1}$, which is given by Lemma 5, we obtain the equality $S \cap S^n = X_{r_1}$. The lemma is proved.

Lemma 9. *Let G, S and τ as Lemma 8. Set $S_0 = U_{r_3} \rtimes \langle \tau \rangle$, where $U_{r_3} = \langle X_r \mid r \in \Phi^+ \backslash \{r_3\} \rangle$. Then $S_0 = \min_G(S, S)$.*

Proof. The subgroup S is a Sylow 3-subgroup of G. Therefore, by Lemma 1 $S \cap S^x \neq 1$ for any $x \in G$, hence $\min_G(A, B) = \langle m \rangle \neq 1$ and by Lemma 5 the set m consists of subgroups of order 3. Further, in the Coxeter graph of type D_4 the roots r_1, r_2 and r_4 coincide with symmetric nodes (see Picture 1), hence Lemma 8 is valid, if the root r_1 is exchanged with the root r_2 or r_4. Therefore by Lemma 6 $O_3(P_{\{i\}}) \leq \min_G(S, S)$ for any $i = 1, 2, 4$. Thus, Lemmas 1, 5, 6 and 8 along with the equality

$$\langle O_3(P_{I \backslash \{1\}}),\ O_3(P_{I \backslash \{2\}}),\ O_3(P_{I \backslash \{4\}}) \rangle = O_3(P_{\{3\}})$$

give the inclusion $S_0 \leq \min_G(S, S)$.

Suppose $S_0 < \min_G(S, S)$. Then there is exists an element (subgroup) D of the set m such that $D = S \cap S^g \nleq S_0$ for some $g \in G$. Because of $|S \cap S^g| = 3$ by Lemma 8, then $S_0 \cap S_0^g = 1$. Subgroups S and S_0 satisfy the conditions of Lemma 2 as

$$G = D_4(3) \rtimes \langle \tau \rangle,$$

$$M_1 = N_G(P_{\{3\}}),$$

$$P_1 = S,$$

$$O_3(M_1) = S_0,$$
$$k = n_{r_3}(1)$$
$$O_3(M_2) = S_0.$$

Thus by Lemma 2

$$S \cap S^x \leq S_0 \cap S_0^g = 1$$

for suitable $x \in G$. Hence, $S \cap S^x = 1$. Contradiction with Lemma 1. The lemma is proved.

3. The proof of the Theorem 1

Let a group G satisfies the conditions of Theorem 1. Further, we use the notations of the preceding paragraph for subgroups and elements of the Chevalley group $D_4(3)$.

(1) $\Rightarrow$ (2) Let $\mathrm{Min}_G(A, B) \neq 1$. Since A and B are primary subgroups, then the condition $\mathrm{Min}_G(A, B) \neq 1$ implies that subgroups A and B are p-groups for an odd prime p . Hence, without loosing generality, we can suggest, that subgroups A and B lie in one fixed Sylow p-subgroup S of the group G. Now again by the condition $\mathrm{Min}_G(A, B) \neq 1$ we get the inequality $S \cap S^g \neq 1$ for any element g of G. Therefore by Lemma 1 we can suggest, that $p = 3$ and the group G contains the normal subgroup

$$G_0 = D_4(3) \rtimes \langle \tau \rangle$$

of index 1 or 2, where τ is a graph automorphism of order 3 of the group $D_4(3)$. We can suggest, that

$$S = U \rtimes \langle \tau \rangle.$$

Let

$$n_0 = (n_{r_3}(1)n_{r_1}(1)n_{r_2}(1)n_{r_4}(1))^3.$$

Then (see the proof of Lemma 8)

$$S \cap S^{n_0} = \langle \tau \rangle.$$

Since

$$\langle \tau \rangle = S \cap S^{n_0} \geq A \cap B^{n_0} \neq 1,$$

then we obtain

$$A \cap B^{n_0} = \langle \tau \rangle.$$

By Lemma 8 there exists a monomial element $n \in D_4(3)$ such that

$$S \cap S^n = X_{r_1}.$$

Since

$$X_{r_1} = S \cap S^n \geq A \cap B^n \neq 1,$$

then

$$A \cap B^{n_0} = X_{r_1}.$$

Now by Lemma 3, we have

$$X_{r_1}^S \leq A.$$

By Lemma 6,

$$\langle X_{r_1}^U \rangle = O_3(P_{I \setminus \{1\}}).$$

As $\tau \in A$,

$$\langle O_3(P_{I \setminus \{1\}}), \ O_3(P_{I \setminus \{1\}}^\tau), \ O_3(P_{I \setminus \{1\}}^{\tau^2}) \rangle = O_3(P_{\{3\}}) \leq A.$$

Hence

$$O_3(N_G(P_{\{3\}})) \leq A.$$

Suppose that

$$S_0 = O_3(N_G(P_{\{3\}})).$$

Note, that $|S : S_0| = 3$. Therefore, either $A = S_0$ or $A = S$. The condition $A \cap B^g \neq 1$ for any $g \in G$ is equivalent to the condition $B \cap A^{g^{-1}} \neq 1$ for any $g \in G$. Thus

$$(A, B) \in \{S, S_0\}^2.$$

It remains only to prove equality $S_0 = \min_G(S, S)$. We have two cases only: a) $G = G_0$; b) $|G : G_0| = 2$.

In case a), the equality $S_0 = \min_G(S, S)$ is valid by Lemma 9.

The case b) follows from a) and invariance of subgroup S with respect to outer (graph) automorphisms of order 2 of the group $D_4(3)$.

$(1) \Leftarrow (2)$. It is clear that for pair (S, S) the inequality $S \cap S^g \neq 1$ for any element $g \in G$ follows from recently found equality $S_0 = \min_G(S, S)$.

As it is already known, that $S \cap S^g \neq 1$ for any $g \in G$ and $S \cap S^{n_{r_3}(1)} = S_0$, then by Lemma 2 (see the proof of Lemma 9) we have $S_0 \cap S_0^g \neq 1$ and, moreover, $S_0 \cap S^g \neq 1$ and $S \cap S_0^g \neq 1$ for any $g \in G$. This concludes the proof of our theorem.

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