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‘Relationships between relationships’ in forest stands: intercepts and exponents analyses

Vladimir L. Gavrikov

Abstract Relationships between diameter at breast height (dbh) versus stand density, and tree height versus dbh (height curve) were explored with the aim to find if there were functional links between correspondent parameters of the relationships, exponents and intercepts of their power functions. A geometric model of a forest stand using a conic approximation suggested that there should be interrelations between correspondent exponents and intercepts of the relationships. It is equivalent to a type of ‘relationship between relationships’ that might exist in a forest stand undergoing self-thinning, and means that parameters of one relationship may be predicted from parameters of another. The predictions of the model were tested with data on forest stand structure from published databases that involved a number of trees species and site quality levels. It was found that the correspondent exponents and intercepts may be directly recalculated from one another for the simplest case when the total stem surface area was independent of stand density. For cases where total stem surface area changes with the drop of density, it is possible to develop a generalization of the model in which the interrelationships between correspondent parameters (exponents and intercepts) may be still established.

Keywords Total stem surface area · Self-thinning · Conic approximation · Power function · Exponent · Intercept · Scots pine

Introduction

In forest science, a large proportion of studies represent the establishment of relationships—how one measure of a forest stand relates to another, the measures being either directly assessed or computed from basic values. Basic measures that can be obtained in the field include stem diameter (frequently as diameter at breast height), stem height and number of trees per unit area (stand density). For some time, forest mensuration practitioners have found that all three measures relate to each other, producing—as forest stand growth progresses—curvilinear interrelations (e.g., Chapman 1921).

The relationship between diameter at breast height (dbh) and stem height is known as a height curve. Typically, stem height increases in a curvilinear way with an increase in dbh and levels off closer to maximum diameter values. A number of mathematical functions have been proposed to fit height curves; they are often enumerated in forestry textbooks (Van Laar and Akça 2007) and include various polynomials, logarithmic, as well as simple power functions.

The development of stand density with time has been a frequent topic of forestry research but even greater attention has been given to relationships of various measures of tree size and number of trees because stand density has a profound effect on tree growth, and determination of stem
growth, form and crown development. Most famous relationsh
ships are self-thinning rules by Reinke (1933) and
Yoda et al. (1963) which link number of trees per unit area
and mean tree size. Analyses of the intrinsic mechanics of
the rules and their importance for contemporary forest
science may be found in a number of studies (Sterba 1987,
Pretzsch and Biber 2005; Pretzsch 2006; Vanclay and
Sands 2009; Larjavaara 2010; Gavrikov 2015).

It can be noted from the literature that a relationship
between stand variables is often studied separately from
other relationships between variables in the same stand.
Meanwhile, because of intense interactions between trees
in dense forest stands, the interactions may influence all
observable relationships leading to parameters of one
relationship beginning to depend on parameters from
another relationship. For example, a number of researchers
explored covariations between exponents in relationships
of biomass, tree height and dbh (Niklas and Spatz 2004;
Zhang et al. 2016).

These ‘relationships between relationships’ present a
rather profound interest because they may provide a deeper
understanding of self-thinning in forest stands. Inoue
(2009) developed an allometric model of maximum size–
density that related stem surface area to stand density. To
derive the model, Inoue (2009) considered allometric
relationships between mean tree height H and mean surface
area S, i.e., \(H \propto S^\alpha\), on the one hand, and the relationship
between biomass density \(B\) and mean surface area \(S\), i.e.,
\(B \propto S^\beta\), \(\alpha\) and \(\beta\) being allometric exponents. When
\(\alpha + \beta \approx 1/2\), the total stem surface area becomes con-
stant, independent of stand density. In other words, in the
case of a constant total stem surface area, the allometric
exponents can be predicted from one another and the study
by Inoue (2009) gives an example of finding ‘relationships
between relationships’.

Gavrikov (2014) considered a geometrical model of a
forest stand in which dependence of stem length \(l\) on dbh
\(D\) (height curve) as well as dependence of \(D\) on stem
density \(N\) (thinning curve) was analyzed. The relationships
were presented as simple power functions in a generalized
form such as \(l(D) \propto D^a\) and \(D(N) \propto N^b\), \(a\) and \(b\) being
allometric exponents. When the total stem surface area
remains constant and independent of stand density

decrease, the exponents are tightly interrelated to each
other and therefore one exponent may be predicted from
the other. When the total stem surface area grows or falls
with stand density decrease, the exponents predictably
relate, more or less, to each other. It has been therefore
shown how different relationships may be interconnected
through power exponents.

Because of convenience of the mathematical form of the
simple power function, the analysis of its exponents may be
rather easy. History of self-thinning rule studies indicates

Materials and methods

Method

The method applied uses two approaches. The first consists
in using total stem surface area \(\hat{S}\) development as the basis
of analysis. To get estimations of \(\hat{S}\), a conic approximation
of tree stem was used which is reflected in the product of
dbh \(D\), height \(H\) as suggested by Inoue (2004). For con-
venience, mean dbh is represented by mean stem radius \(r\)
and mean stem height is substituted through cone gener-
atrix \(l\). The latter implies that because trees are narrow, long
shapes, the genuine stem height is approximately equal to
the generatrix, \(l \approx H\), though a small loss of accuracy may
take place. Thus total stem surface area is given through:
\[
\hat{S} = \delta \pi rl \cdot N,
\] (1)

where \(\delta\) is a normalization constant that will be discussed
under Results and Discussion. The second indicates that
height curve \(l(r)\), thinning curve \(r(N)\) and \(\hat{S}(N)\) may be
analyzed through fitting by simple power functions. The
supposition meets no difficulties with \(l(r)\) and \(r(N)\) since
they are mostly monotonic curves. The total stem surface
area develops, however, in such a way that the curve often
appears to be non-monotonic; it may grow and it may fall.
It is supposed, nevertheless, that monotonic sections of the
non-monotonic curves may be fitted by power functions
and parameters of the functions rightly reflect properties of
the curve sections. It is use of power functions that enables
a transparent analytical modeling of relationships between
forest stand measures in this study. Though use of power
functions does not imply that they are the best functions for
fitting, it is expected that power functions do provide
valuable information on the relationships studied.
The monotonic sections of $\hat{S}(N)$ are referred to here as ‘tendencies’. It is supposed that stand density $N$ can only decrease (thinning or self-thinning). A growing tendency is observed when $\hat{S}$ increases during a decrease of $N$. If $\hat{S}$ stays constant independent of $N$, this is called a flat tendency. Consequently, if $\hat{S}$ decreases with decreasing $N$ this is called a falling tendency.

Data used

To evaluate the results of modeling, a number of datasets was extracted from a database published by Usoltsev (2010). The database contains about 10,000 descriptions of sample plots in various forest stands over the whole of Eurasia. As a rule, each description includes data on species, bonitet (Russian system of site quality estimation), mean dbh, mean height, stand density per ha and other information. The descriptions are combined in groups by name of author and geographic location where the data were gathered. From these groups, the data on individual sample plots were collected to provide datasets for the study.

One of the problems with most of the published data is that they present static descriptions of different stands while modeling implies a dynamic situation. For the purposes of this study, descriptions within a group were collected in such a way that they resembled the development of one forest stand with time. In other words, to get datasets the descriptions had to be sub-sampled. Within datasets, the data may be differentiated by bonitet (site index). It is important to note that some datasets had to be divided into sections in which a monotonic development of $\hat{S}(N)$ is observed as explained above. Such sections are denoted as having either flat, growing or a falling tendency of the total stem surface area development in the course of thinning.

All the datasets were denoted by the names of the authors as cited by Usoltsev (2010). Table 1 gives an overview of the datasets used. The development of the total stem surface area with thinning in all the datasets is given graphically in Electronic Supplement (fig. S1 through fig. S19).

Estimations of regression parameters in the relationships studied were performed with STATISTICA 6 software. The software has the module of non-linear estimation that provides the tools to perform various regressions based on different loss functions. In this study, ordinary least squares were used as the loss function that was minimized by the software through the Levenberg–Marquardt algorithm. The user-specified regression model was a two-parameter power function of the form $Y = c \cdot X^a$ where $Y$ and $X$ are dependent and independent variables, respectively; $c$ and $a$ are intercept and exponent, respectively.

Results and discussion

Model and its analysis

The first part of the model is based on Eq. 1 that allows the generating of hypotheses on how total stem surface area may depend on stand density. As a reference point, consider the case where total stem surface area is equal to a constant $C$ and therefore independent of $N$. To find this in a real forest stand is not improbable, and has been reported in a number of publications (Gavrikov 2014; Inoue and Nishizono 2015). In other words, there is a flat tendency in the development of $\hat{S}(N)$. Through generalization, other tendencies may be further studied. From Eq. 1 one can therefore get an expression for $l(r)$:

$$l = \frac{C}{\delta \pi r N}. \tag{2}$$

By contrast to the analysis of exponents only, a model including intercepts as well requires a thorough consideration of dimensions. In the data used here, stand density $N$ is given in number of trees per hectare (ha$^{-1}$). Because $C$ is implied to be in square meters m$^2$ and $l$ and $r$ are naturally in meters, $\delta$ has to be in ha or m$^2$; for consistency, ha units are converted into m$^2$ in all further calculations. According to Eq. 1, $\delta$ gives an idea of proportion between ‘genuine’ stem surface area and the area for the conic approximation of stem.

The second part of the model comes from the consideration of tree radius $r$ dependence on stand density $N$. It is admitted here that the relationship $r(N)$ may be represented as in a geometric model of forest stand (Gavrikov 2014):

$$r = \varepsilon \sqrt{\frac{1}{N}}. \tag{3}$$

where $\varepsilon$ is a normalization constant. Resolving of $N$ given in ha$^{-1}$ from the square root gives $\sqrt{\frac{ha^2}{N^2}} = \frac{ha^2}{N^2} = 10000 m^2 \cdot N^{-2} = 100^2 \cdot N^{-2}$ and therefore Eq. 3 may be rewritten as

$$r = \varepsilon \cdot \left(100 m^2\right)^{1/2} \cdot N^{-2}, \tag{4}$$

where $N$ is dimensionless and $\varepsilon$ has to be in m$^{1-\gamma}$ since $r$ is naturally expressed in m.

To ensure that $l$ in Eq. 2 depends only on $r$, $N$ may be resolved from Eq. 4 as $N = \frac{1}{r^2 \varepsilon^2 \cdot 100^{-2}}$ and substituted to Eq. 2 to get the final form of $l(r)$ relationship:

$$l = \frac{C}{\delta} \cdot \frac{1}{100^2 \cdot \pi \cdot \varepsilon^2} \cdot r^{-2-1}. \tag{5}$$
In Eq. 4, there is only one unknown multiplier in the intercept ($c$) and only one unknown term in the exponent ($\gamma$).

In Eq. 5, the expression $C/\delta$ is written as a separate ratio for the following reason. It follows from Eq. 5 that one does not have to know $C$ and $\delta$ separately but only their ratio. This ratio may be determined from Eq. 2 as $C/\delta = \pi r LN$. In the right-hand term, the multipliers are either known or may be found from data and therefore the ratio $C/\delta$ may also be known. Hence, there is only one unknown term in the exponent of relation Eq. 5 ($\gamma$). After the term $\gamma$ is estimated from data then only one term remains unknown in the intercept $K = \frac{C}{\delta} \cdot \frac{1}{100^{2} \pi c}$ of Eq. 5; the term is $c$.

As a result of the derivation of Eqs. 4 and 5, both relationships contain the same parameter $c$ in their

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\(^a\) The dataset names are given according citations in Usoltsev (2010), the page number is also provided; a dataset may be sub-divided into bonitets

\(^b\) Tendency of total stem surface area development in the course of thinning (flat or growing or falling)

\(^c\) Reference to figure number in Electronic Supplement

\(^d\) Russian system of bonitation, 1st bonitet being the best and Vth bonitet being the worst conditions; bonitets are given as in Usoltsev (2010)

\(^e\) Ages in years, stand densities in trees per hectare
intercepts and the same parameter $\gamma$ in their exponents. Under the above supposition of constancy of $\dot{S}(N)$, this means that if the values of intercept and exponent in Eq. 4, for example, are known, then the corresponding values of intercept and exponent in Eq. 5 should be also computable.

To avoid confusion because $\gamma$ and $\varepsilon$ are estimated by separate fitting operations, relationships Eqs. 4 and 5 should be rewritten as follows:

$$l = \frac{C}{\delta} \cdot \frac{1}{100^2 \cdot \pi \cdot \varepsilon_i} \cdot \frac{1}{r^2 - 1}$$  \hspace{1cm} (6)

and $r = e_2 \cdot 100^{\frac{2}{3}} \cdot \frac{1}{N^{\frac{2}{3}}}$.  \hspace{1cm} (7)

The introduction of inferior indices at $\gamma$ and $\varepsilon$ allows for the formulating of a clear hypothesis that should be verified. If total stem surface area $\dot{S}$ is constant and independent of stand density, the values $\gamma$ and $\varepsilon$ should follow $\gamma_1 = \gamma_2$ and $\varepsilon_1 = \varepsilon_2$; if not constant, then $\gamma_1 \neq \gamma_2$ and $\varepsilon_1 \neq \varepsilon_2$.

Estimations of intercept and exponent components $\varepsilon$ and $\gamma$

Equation 7 was used for fitting against the data. Equation 6, however, had to be fitted first as $l = K \cdot r^2 - 1$ and then, having known values of $\gamma_1$ and $K$, value $\varepsilon_1$ was found. To compute the value $\varepsilon_1$ for a dataset, the value of ratio $C/\delta$ was taken as the mean product $\pi r l N$ for this particular dataset.

Results of the fittings are given in Table 2. Coefficient of determination ($R^2$) of relations in the fitted data is usually rather high, with a single exclusion. Figures 1 and 2 depict graphically the data from Table 2. Datasets that have a flat tendency is prone to the line denoting $\gamma_1 = \gamma_2$. Datasets with growing tendencies are located consistently in the area above the line where $\gamma_1 < \gamma_2$. Datasets with falling tendencies are located consistently below the line, i.e., where $\gamma_1 > \gamma_2$. Because datasets with growing tendencies are mostly from younger, dense stands and datasets with falling tendencies are from older, sparse ones, it is quite plausible that when tendencies change from growth to decline, the values of $\gamma_1$ and $\gamma_2$ satisfy $\gamma_1 = \gamma_2$.

Moeller-46 dataset presents a noticeable deviation from the $\gamma_1 = \gamma_2$ condition (Fig. 1, rightmost closed circle). The cause of this deviation is not known but the dataset was the only that showed low confirmation of the relation $l(r)$ (height curve) (Table 2). As noted previously, each dataset resembles the development of an individual forest stand. Perhaps the Moeller-46 dataset does not quite satisfy this assumption (see also fig. S10 in the Electronic Supplement).

Figure 2 plots $\varepsilon_1$ against $\varepsilon_2$. As with the $\gamma$ parameter, values of $\varepsilon_1$ and $\varepsilon_2$ for datasets with a flat tendency of $\dot{S}(N)$ development are very close to the straight line in Fig. 2. Again, datasets with a growing tendency are located consistently below the line denoting the condition $\varepsilon_1 > \varepsilon_2$ and datasets with a falling tendency are located consistently above the line that means $\varepsilon_1 < \varepsilon_2$. It may be therefore quite plausible that $\varepsilon_1 = \varepsilon_2$ when a growing tendency turns into a falling one through a flat tendency.

Among the datasets, more than half are Scots pine data. Fourteen of the total 32 datasets belong to other species. The computations showed no definite patterns relating to species, which may mean that the application of the approach depends not on species but solely on how total stem surface area develops with stand density decrease. The question of species influence requires, however, larger studies involving more data. From the data here, it might be inferred that, in terms of $\varepsilon$ values, Scots pine tends to occupy a middle position among other species involved.

Generalization of model

It has been shown previously that qualitative information of tendencies in $\dot{S}(N)$ development allows predicting of interrelations between correspondent intercepts of $l(r)$ and $r(N)$ relationships and between correspondent exponents of these relationships. If the tendency of $\dot{S}(N)$ is flat, i.e., $\dot{S}(N)$ is a constant, then $\varepsilon_1 = \varepsilon_2$ and $\gamma_1 = \gamma_2$. But if it is known that tendencies are growing or falling, then only predictions $\varepsilon_1 > \varepsilon_2$, $\gamma_1 < \gamma_2$ or $\varepsilon_1 < \varepsilon_2$, $\gamma_1 > \gamma_2$, respectively, are possible.

Let us consider a generalization of the model when a quantitative description of tendencies is available. In compliance with the approach used here, dependence of $\dot{S}(N)$ within monotonic sections may be given as a power function. Use of a power function form provides consistency throughout the model and a possibility to derive an analytical solution.

Thus, $\dot{S}(N)$ is presented as:

$$\dot{S} = \delta \pi r l N = A \cdot N^\lambda,$$  \hspace{1cm} (8)

where $A$ is a normalization constant and $\lambda$ is an exponent. It is $\lambda$ that quantitatively describes monotonic segments of $\dot{S}(N)$ (tendencies). $\lambda$ may be received through independent measurements. By analogy with derivations made above, $l = \frac{A}{\delta} \cdot \frac{1}{\pi \varepsilon_i^2} \cdot \frac{1}{100^{\frac{2}{3}} \cdot N^{\frac{2}{3}}} \cdot \frac{1}{r^2 - 1}$ and because (after resolving from Eq. 4 and raising to the power of $1 - \lambda$) $N^{1-\lambda} = \frac{1}{\varepsilon_i^{2(1-\lambda)} \cdot 100^{2(1-\lambda)}}$, the new expression for $l(r)$ will look as follows:

$$l = \frac{A}{\pi \varepsilon_i^2} \cdot \frac{1}{100^{\frac{2}{3}} \cdot N^{\frac{2}{3}}} \cdot \frac{1}{r^2 - 1}$$  \hspace{1cm} (9)
The ratio A/\hat{S} may be derived from Eq. 8 as \pi rlN^{1-\lambda} where all the terms are supposed to be known. By analogy with Eq. 6, there is one unknown term \gamma_1 in the exponent and one unknown term \epsilon_1 in the intercept of Eq. 9. Equa-

tion 9 obviously generalizes the model because the case of \lambda = 0, which means a flat tendency in \hat{S}(N), reduces Eq. 9.
to old form of Eq. 6. Note that Eq. 8 has an impact only on
l(r) relationship while r(N) remains in the old form of
Eq. 7.

Hypothetically, as it follows from Eqs. 9 and 7, pro-
vided λ is known, relations may be established between
correspondent exponents in l(r) and r(N) as well as
between intercepts in them. In other words, knowing λ and
an exponent in l(r), the exponent in r(N) may be computed
since γ1 in Eq. 9 is hypothetically equal to γ2 in Eq. 7. The
same is hypothetically true for the intercepts, i.e., e1 in
Eq. 9 is equal to e2 in Eq. 7. To verify the hypothesis,
computations for dataset may be carried out, for example,
the Mironenko-98 dataset, Ia bonitet, that shows a slightly
growing tendency (fig. S1 in Electronic Supplement). Since
Eq. 8 does not have an impact on Eq. 7, the values of
γ2 = 1.267 and e2 = 0.023 (Table 2, Mironenko-98, Ia)
are ready for comparison and γ1 and e1 have to be com-
puted. Exponent λ of Eq. 8 for this dataset is λ = −0.1192
(SE = 0.0523, significant at p < 0.1). Next, fitting of the
dataset with \( l = P \cdot r^{1-\frac{1}{2} (1-\delta)^{-1}} \) (see Eq. 9) gives γ1 = 1.265
(SE = 0.0524, significant at p < 0.05), \( P = 124.07 \)
(SE = 18.8, significant at p < 0.05), \( R^2 = 0.9565 \).

Already at this point one can note that independently
estimated γ1 (1.265) and γ2 (1.267) are close to each other.
The value of \( e_2 \) has to be extracted from \( P \). As noted
previously, the value of \( A/\delta \) ratio was taken as mean value
of \( n/rN^{1-\delta} \), for the dataset; the value was \( A/\delta = 14,962.2 \).

Then, resolving \( e_1 \) from \( P = \frac{A}{\delta} \cdot \frac{1}{e^1 \cdot 100^{-2(1-\delta)}} \) \( e_1 = \)
\[
\frac{1496.2 \cdot 1.15 \cdot 1.00^{-2(1-\delta)}}{100^{-2(1-\delta)}} \approx 0.0232, \ SE \ was \ estimated \ as \ 0.0023.
\]

Again, it is clear that independently estimated \( e_1 \)
(0.0232) and \( e_2 \) (0.0231) are close to each other.

To summarize, if \( \hat{S}(N) = \) constant, then exponents in
\( l(r) \) (Eq. 6) and \( r(N) \) (Eq. 7) are tightly related to each other.
so that information on one exponent may help to compute
the other one. This is done through a common term \( \gamma \) in the
exponents. Also, intercepts in \( l(r) \) (Eq. 6) and \( r(N) \) (Eq. 7)
can be computed from another through a common term \( \epsilon \).

If \( \hat{S}(N) \) is constant but only a tendency in \( \hat{S}(N) \) is
known, then relations between the exponents and intercepts
may be estimated in terms of ‘more/less’.

If however, \( \hat{S}(N) \) may be represented as a power func-
tion of \( N \), i.e., \( \hat{S}(N) = A \cdot N^\lambda \) and \( \lambda \) may be quantita-
tively estimated, then exponents in \( l(r) \) (Eq. 9) and \( r(N) \) (Eq. 7)
can be readily computed from one another with the help of \( \hat{S}(N) \) value. The same is true for the intercepts; they can be
computed from one another as well.

Conclusion

Numerous relationships have been established in forest
science that served to describe structure and growth of
forest stands. Some, like the ‘−3/2 self-thinning rule’, were
derived from other relations linking sizes of trees to stand
density.

In this study, the ‘relationships between relationships’
was considered; the \( H \) versus \( D \) relationship (height curve)
was sought to quantitatively relate to the \( D \) versus \( N \) rela-
tionship (thinning curve). In order to provide mathematical
consistency, all analyzed relations were presented in the
form of simple power functions that included an exponent
and an intercept. It has been shown that putting hypotheses

\[ \text{Fig. 1 Values of } \gamma_1 \text{ plotted against } \gamma_2 \text{ for all datasets. Key: filled circle datasets with a flat tendency of } \hat{S}(N) \text{ development, open triangle datasets with a growing tendency and diamond datasets with a falling tendency. Straight solid line denotes the position when } \gamma_1 = \gamma_2. \]

\[ \text{Fig. 2 Values of } \epsilon_1 \text{ plotted against } \epsilon_2 \text{ for all datasets. Legends are same as in Fig. 1. Straight solid line denotes the position when } \epsilon_1 = \epsilon_2. \]
on how total stem surface area develops during self-thinning or thinning helps to find analytical links between exponents/intercepts of the height curve and exponents/intercepts of the thinning curve. If it is known that total stem surface area does not change in the course of thinning or an exponent is known of the area dependent on stand density, the exponents/intercepts in the relationships may be directly computed from one another. This implies an existence of profound processes that govern the development of a forest stand and this deepens our knowledge on this development. Why such ‘relationships between relationships’ may appear is a topic of special research, but it may be hypothesized that the source of the phenomenon lies in interactions of trees in the course of growth, competition and dying-off.

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