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On a Sufficient Condition when an Infinite Group Is not Simple

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We describe the conditions of existing periodic part in Shunkov group.

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V. P. Shunkov in [14] proved his famous theorem on the local finiteness and almost solvability of a periodic group G containing an involution with a finite centralizer. V. V. Belyaev in [2] on the basis of ideas from the work of V.P. Shunkov proved that any group G containing a finite involution z with a finite centralizer is locally finite. The finiteness of the involution z means that the group $\langle z, z^g \rangle$ is finite for any $g \in G$. A. I. Sozutov in [11] showed, in particular, that any group G, containing an almost perfect involution z with a finite centralizer, is not simple. An involution z of a group G is said to be almost perfect if from the condition $|zz^g| = \infty$, where $g \in G$, implies the equality $z^g = z^x$ for some involution x from G. In all the above papers, [2, 11, 14] it was shown that the group G is not simple (under the assumption that G is an infinite group). It was natural to consider the situation when the group G contains an involution z such that $C_G(z)$ contains a finite number of elements of finite order, but $C_G(z)$ does not have to be finite, unlike the groups from the papers [2, 11, 14].

Hypothesis. *Let G be an infinite group, z be an involution from G such that $C_G(z)$ contains a finite number of elements of finite order. Then G is not a simple group.*

Obviously, for the groups of Shunkov, Belyaev, and Sozutov, the above hypothesis is correct. We note that the results of V. P. Shunkov on T_0-groups and groups with finitely embedded involution [10, 15] are close to the formulated hypothesis. In the present paper this hypothesis was confirmed for Shunkov groups saturated with groups from the set of finite simple nonabelian groups. A group G is called a Shunkov group if for any of its finite subgroups H in the factor group $N_G(H)/H$ any two conjugate elements of prime order generate a finite subgroup. Initially, such a group was called the conjugately birimitive finite group [9, 10]. The class of Shunkov groups is very extensive and includes some mixed groups. Therefore, for each given Shunkov group G, the following question is topical: does the group G have a periodic part, e.g. do elements of finite orders in G belong to a subgroup? The nontriviality of the answer to this question is emphasized by examples of solvable Shunkov groups that do not have a periodic part (see for example [4]).

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Theorem. *Suppose that the Shunkov group G is saturated by groups from the set of finite simple nonabelian groups, and in G there is an involution z such that $C_G(z)$ contains a finite number of elements of finite order. Then G has a periodic part isomorphic to a finite simple nonabelian group. In particular, if G is an infinite group, then G is not a simple group.*

1. Definitions, preliminary results

Definition 1. *The group G is saturated with groups from the set of groups $\mathfrak{X}$, if any finite subgroup K of G is embeddable in a subgroup M of the group G, that M is isomorphic to a group in $\mathfrak{X}$* [12].

Definition 2. *Let the group G be saturated with groups from the set of groups $\mathfrak{X}$. Then the set $\mathfrak{X}$ is called the saturating set for G* [7].

Definition 3. *Let G be a group, $\mathfrak{X}$ be the set of groups. Recording*

$$G \mathrel{\widetilde{\in}} \mathfrak{X}$$

means that the group G is isomorphic to some group in $\mathfrak{X}$. Accordingly, the record

$$G \mathrel{\widetilde{\notin}} \mathfrak{X}$$

means that the group G is not isomorphic to any group in the set $\mathfrak{X}$.

Definition 4. *Let G be a group, K be a subgroup of G, $\mathfrak{X}$ be the set of groups. Across*

$$\mathfrak{X}_G(K) = \{H \mid K \leqslant H \leqslant G, H \mathrel{\widetilde{\in}} \mathfrak{X}\}$$

we denote the set of all subgroups H of the group G containing the subgroup K and isomorphic to groups in the set $\mathfrak{X}$. If 1 is the identity subgroup of G, then

$$\mathfrak{X}_G(1) = \{H \mid H \leqslant G, H \mathrel{\widetilde{\in}} \mathfrak{X}\}$$

will denote the set of all subgroups H of the group G, isomorphic to groups in the set $\mathfrak{X}$. If the context is clear about which group G we are talking about, then instead of $\mathfrak{X}_G(K)$ we write $\mathfrak{X}(K)$, and accordingly $\mathfrak{X}_G(1)$ we write $\mathfrak{X}(1)$.

Definition 5. *Let G be a group. If all elements of finite orders in G are contained in a periodic subgroup of G, then it is called the periodic part of the group G and is denoted by $T(G)$* ([6, pp. 90, 150],).

From the theorem of V. D. Mazurov ([8]) follows

Preposition 1. *For any finite set of prime numbers π, there exists only a finite set of finite simple groups (up to isomorphism) $\mathfrak{M}_\pi$ with the property that if a prime number p divides $|K|$, where $K \in \mathfrak{M}_\pi$, then $p \in \pi$* [8].

Preposition 2 (Dicman's lemma)**.** *A finite invariant set of elements of finite order in any group generates a finite normal subgroup* [5].

Preposition 3 (Theorem of Brauer)**.** *There exists a finite number of finite simple nonabelian groups (up to isomorphism) with a given centralizer of involution* [3].

Preposition 4. *The Shunkov group with an infinite number of elements of finite order has an infinite locally finite subgroup* [13].

Preposition 5. *Let G be a Shunkov group, a be an element of prime order in G, x an involution in G. Then $\langle x, a\rangle$ is a finite group.*

Proof. It follows from the definition of the Shunkov group that $\langle a, a^x\rangle$ is a finite group. It is easy to see, that $x \in N_G(\langle a, a^x\rangle)$. Consequently, $\langle a, a^x\rangle\langle x\rangle$ is a finite group. Since the group $\langle x, a\rangle$ coincides with the group $\langle a, a^x\rangle\langle x\rangle$, then $\langle x, a\rangle$ is also a finite group. □

2. Proof of the theorem

Let G be a counterexample to the statement of the theorem, and let $\mathfrak{M}$ be the saturating set for the group G consisting of finite simple non-abelian groups. Fix an involution z from the condition of the theorem.

Lemma 1. *$C_G(z)$ has a finite periodic part $T(C_G(z))$.*

Proof. Let P be the set of all elements of finite order from $C_G(z)$. By the condition of the theorem P is a finite set. Since P is an invariant set. Then by Dicman's lemma (Preposition 2) $C_G(z)$ possesses finite periodic part of $T(C_G(z))$. □

Lemma 2. *The group G contains infinitely many elements of finite order.*

Proof. Suppose the converse. By Dicman's lemma (Preposition 2), G possesses finite periodic part of $T(G)$. A contradiction with the fact that G is a counterexample. □

Lemma 3. *The group G contains an infinite locally finite subgroup.*

Proof. The statement of the lemma is a consequence of Lemma 2 and Preposition 4. □

Lemma 4. *The set $\mathfrak{M}(1)$ contains groups of arbitrarily large order.*

Proof. By Lemma 3, for any natural m in the group G there is a finite subgroup K_m such that $|K_m| > m$. By the saturation condition, $K_m \leqslant L_m$ and $L_m \in \mathfrak{M}(1)$. By the arbitrariness of the choice of m, the set $\mathfrak{M}(1)$ contains groups of arbitrarily large order. □

Lemma 5. *Let $P_{\mathfrak{M}(1)}$ be the set of prime divisors of the orders of groups in $\mathfrak{M}(1)$. Then $P_{\mathfrak{M}(1)}$ is an infinite set.*

Proof. Suppose the converse. Then, by Proposition 1, the orders of groups in the set $\mathfrak{M}(1)$ are bounded in the collection. A contradiction with the assertion of Lemma 4. □

By the condition of the theorem, in the group G there exists an involution z such that $C_G(z)$ has a finite periodic part $T(C_G(z))$ ($C_G(z)$ contains a finite number of elements of finite order). By Definition 4

$$\mathfrak{M}(\langle z\rangle) = \{M_z \mid M_z \in \mathfrak{M}(1), z \in M_z\}$$

is the set of all finite simple nonabelian subgroups of G, containing the involution z.

Lemma 6. *The set $\mathfrak{M}(\langle z\rangle)$ contains groups whose order is greater than any of a given natural m.*

Proof. Let $\{a_1, a_2, \cdots, a_k, \cdots\}$ be an infinite set of elements of groups from the set $\mathfrak{M}(1)$ such that $|a_k| = p_k$ is a prime number and all p_k are distinct (Lemma 5). By Proposition 5, the group $\langle z, a_k\rangle$ is finite for any k. In view of the saturation condition

$$\langle z, a_k\rangle \leqslant M_z \in \mathfrak{M}(\langle z\rangle).$$

It follows from the definition of primes p_k that for any natural m there is a p_k such that $p_k > m$. Hence, $m < p_k < |\langle z, a_k\rangle| \leqslant |M_z|$. □

We now complete the proof of the theorem. Let $\{M_z^{(k)} | k = 1, 2, \dots\}$ be an infinite subset of the set $\mathfrak{M}(\langle z\rangle)$ such that

$$|M_z^{(1)}| < |M_z^{(2)}| < \dots < |M_z^{(k)}| < \cdots$$

(Lemma 6). By Proposition 3, there exists an infinite strictly increasing sequence of natural numbers

$$k_1 < k_2 < \dots < k_m < \cdots$$

such that

$$|C_{M_z^{(k_1)}}(z)| < |C_{M_z^{(k_2)}}(z)| < \dots < |C_{M_z^{(k_m)}}(z)| < \cdots$$

is an infinite strictly increasing sequence of natural numbers. This contradicts the fact that for any k_m, $|C_{M_z^{(k_m)}}(z)| \leqslant |T(C_G(z)|$ (Lemma 1). The contradiction completes the proof of the theorem.

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Об одном достаточном условии, при котором бесконечная группа не будет простой

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В работе рассмотрены условия существования периодической части группы Шункова.

Ключевые слова: группа Шункова, группы насыщенные заданным множеством групп.