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Multidimensional Boundary Analog of the Hartogs Theorem in Circular Domains

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This paper presents some results related to the holomorphic extension of functions, defined on the boundary of a domain $D \subset \mathbb{C}^n$, $n > 1$, into this domain. We study a functions with the one-dimensional holomorphic extension property along the complex lines.

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Introduction

This paper presents some results related to the holomorphic extension of functions, defined on the boundary of a domain $D \subset \mathbb{C}^n$, $n > 1$, into this domain. We consider a functions with the one-dimensional holomorphic extension property along the complex lines.

The first result related to our subject was obtained M.L.,Agranovsky and R.E.Valsky in [1], who studied functions with the one-dimensional holomorphic continuation property into a ball. The proof was based on the properties of the automorphism group of a sphere.

E. L. Stout in [2] used the complex Radon transformation to generalize the Agranovsky and Valsky theorem for an arbitrary bounded domain with a smooth boundary. An alternative proof of the Stout theorem was obtained by A. M .Kytmanov in [3] by using the Bochner–Martinelli integral. The idea of using the integral representations (Bochner–Martinelli, Cauchy–Fantappiè, logarithmic residue) has been useful in the study of functions with the one-dimensional holomorphic continuation property (see review [4]).

The question of finding different families of complex lines sufficient for holomorphic extension was put in [5]. As shown in [6], a family of complex lines passing through a finite number of points, generally speaking, is not sufficient. Thus, a simple analog of the Hartogs theorem should be not expected.

Various other families are given in [7–11]. In [12–16] it is shown that for holomorphic extension of continuous functions defined on the boundary of ball,there are enough $n + 1$ points inside the bal, not lying on a complex hyperplane. This result was generalized by the authors n-circular domains.

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1. Main results

Let D be a bounded domain in $\mathbb{C}^n$ with a smooth boundary. Consider the complex line of the form

$$l_{z,b} = \{\zeta \in \mathbb{C}^n : \zeta = z + bt, t \in \mathbb{C}\} = \{(\zeta_1, \dots \zeta_n) : \zeta_j = z_j + b_j t, j = 1, 2, \dots, n, t \in \mathbb{C}\}, \qquad (1)$$

where $z \in \mathbb{C}^n$, $b \in \mathbb{CP}^{n-1}$.

We will say that a function $f \in \mathcal{C}(\partial D)$ has the *one-dimensional holomorphic extension property along the complex line* $l_{z,b}$, if $\partial D \cap l_{z,b} \neq \varnothing$ and there exists a function $F_{l_{z,b}}$ with the following properties:

1) $F_{l_{z,b}} \in \mathcal{C}(\overline{B} \cap l_{z,b})$,

2) $F_{l_{z,b}} = f$ on the set $\partial D \cap l_{z,b}$,

3) function $F_{l_{z,b}}$ is holomorphic at the interior (with respect to the topology of $l_{z,b}$) points of set $\overline{D} \cap l_{z,b}$.

Let Γ be a set in $\mathbb{C}^n$. Denote by $\mathfrak{L}_\Gamma$ the set of all complex lines $l_{z,b}$ such that $z \in \Gamma$, and $b \in \mathbb{CP}^{n-1}$, i.e., the set of all complex lines passing through $z \in \Gamma$.

We will say that a function $f \in \mathcal{C}(\partial D)$ has the *one-dimensional holomorphic extension property along the family* $\mathfrak{L}_\Gamma$, if it has the one-dimensional holomorphic extension property along any complex line $l_{z,b} \in \mathfrak{L}_\Gamma$.

We will call the *set* $\mathfrak{L}_\Gamma$ *sufficient for holomorphic extension*, if the function $f \in \mathcal{C}(\partial D)$ has the one-dimensional holomorphic extension property along all complex lines of the family $\mathfrak{L}_\Gamma$, and then the function f extends holomorphically into D (i.e., f is a CR-function on ∂D).

Theorem A. *Let $n = 2$ and D be a bounded strictly convex circular domain with twice smooth boundary and a function $f(\zeta) \in \mathcal{C}(\partial D)$ have the one-dimensional holomorphic extension property along the family $\mathfrak{L}_{\{a,c,d\}}$, and the points $a, c, d \in D$ do not lie on one complex line in $\mathbb{C}^2$, then the function $f(\zeta)$ extends holomorphically into D.*

We denote by $\mathfrak{A}$ the set of points $a_k \in D \subset \mathbb{C}^n$, $k = 1, \dots, n + 1$, which do not lie on a complex hyperplane in $\mathbb{C}^n$.

Theorem B. *Let D be a bounded strictly convex circular domain with twice smooth boundary in $\mathbb{C}^n$ and the function $f(\zeta) \in \mathcal{C}(\partial D)$ have the one-dimensional holomorphic extension property along the family $\mathfrak{L}_{\mathfrak{A}}$, then the function $f(\zeta)$ extends holomorphically into D.*

2. Construction of the Szegö kernel

Let $\mathcal{H}(D)$ be the space of holomorphic functions in D with the topology of uniform convergence on compact subsets of D, and $\mathcal{H}(\overline{D})$ be the space of holomorphic functions in a neighborhood of $\overline{D}$ with the corresponding topology. Consider the measure $d\mu = g(\zeta)d\sigma$, where $g(\zeta) \in \mathcal{C}^1(\partial D)$, $g(\zeta) > 0$, and $d\sigma$ is the Lebesgue measure on ∂D. The space $\mathcal{H}(\overline{D})$ is the subspace in $\mathcal{L}^2(\partial D)$ with the measure $d\mu$ on ∂D. By the Maximum Modulus Theorem the mapping $\mathcal{H}(\overline{D}) \longrightarrow \mathcal{L}^2(\partial D)$ is injective. By $\mathcal{H}^2 = \mathcal{H}^2(\partial D)$ we denote the closure of $\mathcal{H}(\overline{D})$ in $\mathcal{L}^2$.

Consider a restriction mapping $r : \mathcal{H}(\overline{D}) \longrightarrow \mathcal{H}(D)$. The mapping r extends by continuity from $\mathcal{H}^2$ in $\mathcal{H}(D)$.

Lemma 1 (Lemma 4.1. [17])**.** *The restriction mapping $r : \mathcal{H}(\overline{D}) \longrightarrow \mathcal{H}(D)$ is continuous, if $\mathcal{H}(\overline{D})$ is considered in the topology induced by the space $\mathcal{L}^2$.*

Therefore, the mapping r extends by continuity to the map $i : \mathcal{H}^2 \longrightarrow \mathcal{H}(D)$. In this case, we say that for functions $f \in \mathcal{H}^2$ there is a holomorphic continuation $\tilde{f} = i(f)$ in D. Further on, this continuation will be denoted by the same symbol f.

In [17] as the measure considered by the Lebesgue measure $d\sigma$ on the boundary of the domain, in our case, for the measure $d\mu = g(\zeta)d\sigma$ the proof is similar.

Since the space $\mathcal{H}^2$ is a Hilbert separable space, then there exists an orthonormal basis

$$\{\varphi_k\}_{k=1}^{\infty} \tag{2}$$

in the metric $\mathcal{L}^2$. Therefore, any function $f \in \mathcal{H}^2$ extens in a Fourier series:

$$f(\zeta) = \sum_{k=1}^{\infty} c_k \varphi_k(\zeta) \tag{3}$$

with respect to the basis (2), which converges in the topology of $\mathcal{L}^2$, where $c_k = (f, \varphi_k) = \int_{\partial D} f(u)\bar{\varphi}_k(u)\, d\mu(u)$. Then

$$f(\zeta) = \sum_{k=1}^{\infty} \left(\int_{\partial D} f(u)\bar{\varphi}_k(u)\, d\mu(u) \varphi_k(\zeta) \right) = \int_{\partial D} f(u) \sum_{k=1}^{\infty} \bar{\varphi}_k(u)\varphi_k(\zeta)\, d\mu(u).$$

Denote $K(\zeta, \bar{u}) = \sum_{k=1}^{\infty} \varphi_k(\zeta)\bar{\varphi}_k(u)$ and $K(\zeta, \bar{u}) \in \mathcal{H}(\overline{D})$ on $\zeta \in \overline{D}$ for a fixed $u \in D$.

Lemma 2. *We can choose an orthonormal basis* $\{\varphi_k\}_{k=1}^{\infty}$ *in* $\mathcal{H}^2$ *which consists of functions* φ_k *in* $\mathcal{H}(\overline{D})$*.*

Proof. Since the space $\mathcal{H}(\overline{D})$ is separable, then there exists a countable everywhere dense set. It will be the same in $\mathcal{H}^2$, since $\mathcal{H}^2$ is the closure of $\mathcal{H}(\overline{D})$. Using the process of Gram-Schmidt orthogonalization for the functions from this set, we get orthonormal basis in $\mathcal{H}^2$ consisting of functions $\varphi_k \in \mathcal{H}(\overline{D})$. □

Lemma 3. *If D is a bounded strictly convex domain with a smooth boundary, then we can choose a polynomials basis* $\{\varphi_k\}_{k=1}^{\infty}$*.*

Proof. Since the domain D is strictly convex, the set $\overline{D}$ is polynomially convex and compact. On such sets functions, holomorphic in its neighborhood, are uniformly approximated by the polynomials [18]. Consequently, the polynomials are dense in the class of functions from $\mathcal{H}(\overline{D})$ and therefore from $\mathcal{H}^2$. Applying the Gram-Schmidt orthogonalization to this set we get an orthonormal basis in $\mathcal{H}^2$ consisting of polynomials. □

Let us call the function $g(\zeta)$ invariant under rotations, if $g(\zeta_1, \dots, \zeta_n) = g(e^{i\varphi}\zeta_1, \dots, e^{i\varphi}\zeta_n)$ for all $\varphi \in [0, 2\pi)$.

Lemma 4. *If D is a bounded strictly convex circular domain with a smooth boundary and a function* $g(\zeta)$ *is invariant under rotations, we can choose a basis* $\{\varphi_k\}_{k=1}^{\infty}$ *of homogeneous polynomials.*

Proof. Indeed, in this case, the measure $d\mu$ is also invariant under rotations, so the homogeneous polynomials of different degrees of homogeneity are orthogonal in $\mathcal{H}^2$. □

Further on, we assume that the basis is chosen in accordance with Theorem 5.1 [17]. According to this theorem the continuation of the kernel $K(\zeta, \bar{u})$ has the property:

$$i(f)(z) = \int_{\partial D} f(\zeta)K(z, \bar{\zeta})\, d\mu(\zeta), \quad z \in D,$$

where $K(z,\bar\zeta)=\sum\limits_{k=1}^{\infty} i(\varphi_k)(z)i(\bar\varphi_k)(\zeta)$ and the series converges uniformly on compact subsets of $D\times D$. This kernel we call *the Szegö kernel.* Then

$$f(z)=\int_{\partial D} f(\zeta)K(z,\bar\zeta)\,d\mu(\zeta), \tag{4}$$

where $f(z)$ is identified with $\tilde f(z)=i(f)(z)$ and $f\in\mathcal{H}^2$.

We define the Poisson kernel

$$P(z,\zeta)=\frac{K(z,\bar\zeta)\cdot K(\zeta,\bar z)}{K(z,\bar z)}=\frac{K(z,\bar\zeta)\cdot \overline{K}(z,\bar\zeta)}{K(z,\bar z)}=\frac{|K(z,\bar\zeta)|^2}{K(z,\bar z)},$$

and $K(z,\bar z)=\sum\limits_{k=1}^{\infty}\varphi_k(z)\bar\varphi_k(z)=\sum\limits_{k=1}^{\infty}|\varphi_k(z)|^2\geqslant 0$.

Lemma 5. *The kernel $K(z,\bar z)>0$ for any $z\in D$.*

Proof. Let $k(z,\bar z)=0$ for some $z\in D$. Then $\varphi_k(z)=0$ for all $k=1,2,\ldots$, so

$$\varphi_k(z)=\int_{\partial D}\varphi_k(\zeta)K(z,\bar\zeta)\,d\mu(\zeta)=0. \tag{5}$$

Since any function $f\in\mathcal{H}^2$ decomposes into the Fourier series (3), $f(\zeta)=\sum\limits_{k=1}^{\infty}c_k\varphi_k(\zeta)$. Applying the mapping i, we get that $f(z)=\sum\limits_{k=1}^{\infty}c_k i(\varphi_k)(z)=0$ in virtue of (5), i.e. $f(z)=0$ in D for all functions $f\in\mathcal{H}^2$, which is impossible. □

Lemma 6. *A function $f\in\mathcal{H}(\overline{D})$ admits the integral representation*

$$f(z)=\int_{\partial D} f(\zeta)P(z,\zeta)\,d\mu(\zeta), \tag{6}$$

for $z\in D$.

Proof. By definition of the kernel $P(z,\zeta)$ and from the integral representation (4) we have

$$\int\limits_{\partial D} f(\zeta)P(z,\zeta)\,d\mu(\zeta)=\int\limits_{\partial D} f(\zeta)\frac{K(z,\bar\zeta)\cdot K(\zeta,\bar z)}{K(z,\bar z)}\,d\mu(\zeta)=$$
$$=\frac{1}{K(z,\bar z)}\int\limits_{\partial D}\big(f(\zeta)K(\zeta,\bar z)\big)K(z,\bar\zeta)\,d\mu(\zeta)=\frac{f(z)K(z,\bar z)}{K(z,\bar z)}=f(z).$$

□

Corollary 1. *If the space $\mathcal{H}(\overline{D})$ is dense in the space $\mathcal{H}(D)\cap\mathcal{C}(\partial D)=\mathcal{A}(D)$, then a function $f\in\mathcal{A}(D)$ admits the integral representation* (6).

Suppose that the domain D satisfies the condition

(A)*: for any point $\zeta\in\partial D$ and any neighborhood $U(\zeta)$ the Szegö kernel $K(z,\bar\zeta)$ is uniformly bounded by $z\in D$ and $z\notin U(\zeta)$.*

Further, we assume that the domain D satisfies the condition (A).

Theorem 1. *Let D be a strictly convex domain in $\mathbb{C}^n$ and the kernel $K(z,\bar\zeta)$ satisfies the Hölder condition with exponent $\frac{1}{2} < \alpha \leqslant 1$ for $\zeta \in \partial D$ and a fixed $z \in D$. Then the domain D and the kernel $K(z,\bar\zeta)$ satisfy the condition (A).*

Proof. Let

$$D = \{z \in \mathbb{C}^n : \ \rho(z) < 0\}, \tag{7}$$

where $\rho \in \mathcal{C}^2(\overline{D})$ and $\operatorname{grad}\rho\big|_{\partial D} \neq 0$. For the proof we use Corollary 26.13 [3] for the Leray integral representations for holomorphic functions $f \in \mathcal{A}(D)$ in strictly convex domains:

$$f(z) = \frac{(n-1)!}{(2\pi i)^n}\int\limits_{\partial D}\frac{f(\zeta)\sum_{k=1}^{\infty}\delta_k\, d\bar\zeta[k]\wedge d\zeta}{\left[\rho'_{\zeta_1}(\zeta_1 - z_1) + \ldots + \rho'_{\zeta_n}(\zeta_n - z_n)\right]^n},$$

where

$$\delta_k = \begin{vmatrix} \rho'_{\zeta_1} & \ldots & \rho'_{\zeta_n} \\ \rho''_{\zeta_1\bar\zeta_1} & \ldots & \rho''_{\zeta_n\bar\zeta_1} \\ & [k] & \\ \rho''_{\zeta_1\bar\zeta_n} & \ldots & \rho''_{\zeta_n\bar\zeta_n} \end{vmatrix}, \quad k = 1,\ldots,n,$$

$d\zeta = d\zeta_1 \wedge \ldots \wedge d\zeta_n$, $d\bar\zeta[k] = d\bar\zeta_1 \wedge \ldots \wedge d\bar\zeta_{k-1} \wedge d\bar\zeta_{k+1} \wedge \ldots \wedge d\bar\zeta_n$.

The denominator of the kernel $\rho'_{\zeta_1}(\zeta_1 - z_1) + \ldots + \rho'_{\zeta_n}(\zeta_n - z_n) \neq 0$ for $\zeta \in \partial D$, $z \in \overline{D}$ and $\zeta \neq z$. Indeed, the equality $\rho'_{\zeta_1}(\zeta_1 - z_1) + \ldots + \rho'_{\zeta_n}(\zeta_n - z_n) = 0$ defines a complex tangent plane to ∂D at the point ζ. If the domain D is strictly convex, then the tangent plane intersects the boundary of D only at a point ζ.

For the domain D the Szegö kernel $K(z,\bar\zeta)$ is the (generalized) Cauchy-Fantappiè (Leray) kernel by Corollary 26.13 [3], so the same domain satisfy the condition (A).

□

Consider the restriction of the form

$$L(z,\zeta,\bar\zeta) = \frac{\sum_{k=1}^{\infty}\delta_k\, d\bar\zeta[k]\wedge d\zeta}{\left[\rho'_{\zeta_1}(\zeta_1 - z_1) + \ldots + \rho'_{\zeta_n}(\zeta_n - z_n)\right]^n}$$

to ∂D, then it would be

$$L(z,\zeta,\bar\zeta) =$$

$$= \frac{\psi(\zeta,\bar\zeta)\, d\sigma(\zeta)}{\left[\rho'_{\zeta_1}(\zeta_1 - z_1) + \ldots + \rho'_{\zeta_n}(\zeta_n - z_n)\right]^n} = \frac{\psi(\zeta,\bar\zeta)\, d\mu(\zeta)}{g(\zeta)\left[\rho'_{\zeta_1}(\zeta_1 - z_1) + \ldots + \rho'_{\zeta_n}(\zeta_n - z_n)\right]^n} =$$

$$= \frac{\psi_1(\zeta,\bar\zeta)\, d\mu(\zeta)}{\left[\rho'_{\zeta_1}(\zeta_1 - z_1) + \ldots + \rho'_{\zeta_n}(\zeta_n - z_n)\right]^n} = \widetilde{L}(z,\zeta,\bar\zeta)\, d\mu(\zeta).$$

The proof of Theorem 1 shows that

$$K(z,\bar\zeta) = \widetilde{L}(z,\zeta,\bar\zeta) \tag{8}$$

for $\zeta \in \partial D$.

Lemma 7. *The function $K(z,\zeta)$ is unbounded as $z \to \zeta$ and $\zeta \in \partial D$, $z \in D$.*

Proof. Consider the point $z^0 \in D$, then the domain D is a strongly star-shaped with respect to z^0, i.e. for any point $\zeta^0 \in \partial D$ the segment $[z^0, \zeta^0] \in \overline{D}$. Let this segment have the form $\{z \in D : \ z = \zeta^0 + t(z^0 - \zeta^0),\ 0 \leqslant t \leqslant 1\}$. Then

$$\rho'_{\zeta_1}(\zeta_1^0 - z_1) + \ldots + \rho'_{\zeta_n}(\zeta_n^0 - z_n) = t\big(\rho'_{\zeta_1}(\zeta_1^0 - z_1^0) + \ldots + \rho'_{\zeta_n}(\zeta_n^0 - z_n^0)\big).$$

If $z \to \zeta^0$, then $t \to 0$ and $\big(\rho'_{\zeta_1}(\zeta_1^0 - z_1^0) + \ldots + \rho'_{\zeta_n}(\zeta_n^0 - z_n^0)\big) \to 0$. Then $K(z,\zeta) \to \infty$ for $z \to \zeta$, $\zeta \in \partial D$. □

3. Poisson kernel and its properties

For a function $f \in \mathcal{C}(\partial D)$ we define the Poisson integral:

$$P[f](z) = F(z) = \int_{\partial D} f(\zeta) P(z,\zeta)\, d\mu(\zeta).$$

In strictly convex domain that satisfy the condition (A), from Equality (8) and the form of the kernel $P(z,\zeta)$, it follows that this kernel is a continuous function for $z \in D$ and then the function $F(z)$ is continuous in D.

Theorem 2. *Let D be a bounded strictly convex domain in $\mathbb{C}^n$ satisfying the condition (A), and $f \in \mathcal{C}(\partial D)$, then the function $F(z)$ continuously extend onto $\overline{D}$ and $F(z)\big|_{\partial D} = f(z)$.*

Proof. Theorem 1 and Lemma 7 show that the kernel $P\big(\zeta, t(z^0 - z)\big)$ tends uniformly to zero outside any neighborhood of the point ζ for $\zeta, z \in \partial D$, $z^0 \in D$, $\zeta \neq z$ and $t \to 1$. Moreover $P(z,\zeta) > 0$ and $P[1](\zeta) = 1$. Consequently, the Poisson kernel $P(z,\zeta)$ is an approximative unit [19, Theorem 1.9]. □

Consider the differential form

$$\omega = c \sum_{k=1}^{n} (-1)^{k-1} \bar{\zeta}_k\, d\bar{\zeta}[k] \wedge d\zeta,$$

where $c = \dfrac{(n-1)!}{(2\pi i)^n}$. Find the restriction of this form to ∂D for the domain D of the form (7). Then by Lemma 3.5 [20], we get

$$d\bar{\zeta}[k] \wedge d\zeta = (-1)^{k-1} 2^{n-1} i^n \frac{\partial \rho}{\partial \bar{\zeta}_k} \cdot \frac{d\sigma}{|\operatorname{grad} \rho|}.$$

Therefore, the restriction of ω to ∂D is equal to

$$d\mu = \omega\big|_{\partial D} = \frac{(n-1)!}{2\pi^n} \sum_{k=1}^{n} \bar{\zeta}_k \frac{\partial \rho}{\partial \bar{\zeta}_k} \cdot \frac{d\sigma}{|\operatorname{grad} \rho|}.$$

We denote

$$g(\zeta) = \frac{(n-1)!}{2\pi^n} \sum_{k=1}^{n} \bar{\zeta}_k \frac{\partial \rho}{\partial \bar{\zeta}_k} \cdot \frac{1}{|\operatorname{grad} \rho|}.$$

Lemma 8. *If D is a strictly convex circular domain, then $g(\zeta)$ is a real-valued function that does not vanish on ∂D.*

Proof. For circular domain $\rho(\zeta_1, \ldots, \zeta_n) = \rho(\zeta_1 e^{i\theta}, \ldots, \zeta_n e^{i\theta})$, $0 \leqslant \theta \leqslant 2\pi$, differentiating this equality with respect θ, we get

$$0 = \sum_{k=1}^{n} i\zeta_k e^{i\theta} \frac{\partial \rho}{\partial \zeta_k} - \sum_{k=1}^{n} i\bar{\zeta}_k e^{-i\theta} \frac{\partial \rho}{\partial \bar{\zeta}_k}.$$

Then we get $\displaystyle\sum_{k=1}^{n} \zeta_k \frac{\partial \rho}{\partial \zeta_k} = \sum_{k=1}^{n} \bar{\zeta}_k \frac{\partial \rho}{\partial \bar{\zeta}_k}$ for $\theta = 0$. The function $g(\zeta)$ means being real that

$$\sum_{k=1}^{n} \bar{\zeta}_k \frac{\partial \rho}{\partial \bar{\zeta}_k} = \overline{\sum_{k=1}^{n} \bar{\zeta}_k \frac{\partial \rho}{\partial \bar{\zeta}_k}} = \sum_{k=1}^{n} \zeta_k \frac{\partial \rho}{\partial \zeta_k}.$$

The function $g(\zeta) \neq 0$ on ∂D, since the complex tangent plane does not pass through zero at the point ζ. Therefore, the function $g(\zeta)$ preserves sign on ∂D. $\square$

Therefore, we can assume that $g(\zeta) > 0$ on ∂D. Therefore, $d\mu = g d\sigma$ is a measure and for it all previous constructions are true.

Lemma 9. *Let D be a strictly convex $(p_1, \ldots, p_n)$-circular domain, i.e.*

$$\rho(\zeta_1, \ldots, \zeta_n) = \rho(\zeta_1 e^{ip_1\theta}, \ldots, \zeta_n e^{ip_n\theta}), \quad 0 \leqslant \theta \leqslant 2\pi,$$

where $p_1, \ldots, p_n$ are positive rational numbers. Then the function

$$\sum_{k=1}^{\infty} \bar{\zeta}_k p_k \frac{\partial \rho}{\partial \bar{\zeta}_k}$$

is real-valued and not zero.

Proof repeats the proof of the previous Lemma 8. $\square$

The function ρ can be chosen so that $|\operatorname{grad} \rho|\big|_{\partial D} = 1$, then

$$d\mu = c_1 \sum_{k=1}^{n} \bar{\zeta}_k \frac{\partial \rho}{\partial \bar{\zeta}_k}\, d\sigma,$$

where $c_1 = \dfrac{(n-1)!}{2\pi^n}$.

Consider the family of complex lines $l_{z^0,b}$ of the form (1) passing through the point $z^0 \in D$, where $b \in \mathbb{CP}^{n-1}$. Calculate the form ω in the variables b and t, we get

$$\begin{aligned} d\zeta = d\zeta_1 \wedge \ldots \wedge d\zeta_n = d(z_1^0 + b_1 t) \wedge \ldots \wedge d(z_n^0 + b_n t) = \\ = d(b_1 t) \wedge \ldots \wedge d(b_n t) = t^{n-1} dt \wedge (b_1 db[1] - b_2 db[2] + \ldots + (-1)^{n-1} b_n db[n] = \\ = t^{n-1} dt \wedge \sum_{k=1}^{n} (-1)^{k-1} b_k\, db[k] = t^{n-1} dt \wedge \nu(b), \end{aligned}$$

where $\nu(b) = \sum\limits_{k=1}^{n} (-1)^{k-1} b_k\, db[k]$. Here we use the fact that $b \in \mathbb{CP}^{n-1}$.

Now we calculate

$$\begin{aligned} &\sum_{k=1}^{n} (-1)^{k-1} \zeta_k\, d\zeta[k] = \\ &= \sum_{k=1}^{n} (z_k^0 + b_k t) d(z_1^0 + b_1 t) \wedge \ldots \wedge d(z_{k-1}^0 + b_{k-1} t) \wedge d(z_{k+1}^0 + b_{k+1} t) \wedge \ldots \wedge d(z_n^o + b_n t) = \\ &= \sum_{k=1}^{n} (-1)^{k-1} z_k^0\, d\zeta[k] + \sum_{k=1}^{n} (-1)^{k-1} b_k t\, d\zeta[k] = \\ &= \sum_{k=1}^{n} (-1)^{k-1} z_k^0 t^{n-2}\, dt \wedge \chi(b) + \sum_{k=1}^{n} (-1)^{k-1} z_k^0 t^{n-1}\, db[k] + \sum_{k=1}^{n} b_k t^n\, db[k], \end{aligned}$$

where $\chi(b)$ is a differential form of degree $(n-2)$. From here we get that

$$\omega\big|_{\partial D} = c\sum_{k=1}^{n}(-1)^{k-1}\bar\zeta_k\, d\bar\zeta[k]\wedge d\zeta\big|_{\partial D} =$$
$$= c\sum_{k=1}^{n}(-1)^{k-1}\bar z_k^0\bar t^{n-1}t^{n-1}\, d\bar b[k]\wedge dt\wedge\nu(b) + c\sum_{k=1}^{n}(-1)^{k-1}\bar b_k\bar t^{n}t^{n-1}\, d\bar b[k]\wedge dt\wedge\nu(b) =$$
$$= (-1)^n c\, dt\wedge\left(\sum_{k=1}^{n}(-1)^{k-1}\bar z_k^0|t|^{2n-2}\, d\bar b[k]\wedge\nu(b) + \bar t|t|^{2n-2}\nu(\bar b)\wedge\nu(b)\right) =$$
$$= (-1)^{n-1}c|t|^{2n-2}\, dt\wedge\left(\sum_{k=1}^{n}(-1)^{k-1}\bar z_k^0\, d\bar b[k] + \bar t\nu(\bar b)\right)\wedge\nu(b).$$

Thus, we have Lemma:

Lemma 10. *The form $\omega\big|_{\partial D}$ in the variables b and t has the form*

$$\omega\big|_{\partial D} = (-1)^{n-1}c|t|^{2n-2}\, dt\wedge\left(\sum_{k=1}^{n}(-1)^{k-1}\bar z_k^0\, d\bar b[k] + \bar t\nu(\bar b)\right)\wedge\nu(b).$$

Consider the modified Poisson kernel

$$Q(z,w,\zeta) = \frac{K(z,\bar\zeta)\cdot K(\zeta,w)}{K(z,w)}.$$

For $w=\bar z$ we obtain $Q(z,\bar z,\zeta) = P(z,\zeta)$ and $K(z,\bar z) > 0$. Therefore, there exists a neighborhood U of the diagonal $w=\bar z$ in $D_z\times D_w$ in which $K(z,w)\neq 0$.

Consider the function

$$\Phi(z,w) = \int_{\partial D} f(\zeta)Q(z,w,\zeta)\, d\mu(\zeta),$$

which is defined for $(z,w)\in U$. It is holomorphic in $(z,w)\in U$, and for $w=\bar z$ we have $\Phi(z,w) = F(z)$ and

$$\left.\frac{\partial^{\delta+\gamma}\Phi(z,w)}{\partial z^\delta\partial w^\gamma}\right|_{w=\bar z} = \frac{\partial^{\delta+\gamma}F(z)}{\partial z^\delta\partial\bar z^\gamma}, \tag{9}$$

where

$$\frac{\partial^{\delta+\gamma}\Phi(z,w)}{\partial z^\delta\partial w^\gamma} = \frac{\partial^{\delta_1+\ldots+\delta_n+\gamma_1+\ldots+\gamma_n}\Phi(z,w)}{\partial z_1^{\delta_1}\cdots\partial z_n^{\delta_n}\partial w_1^{\gamma_1}\cdots\partial w_n^{\gamma_n}},$$
$$\frac{\partial^{\delta+\gamma}F(z)}{\partial z^\delta\partial\bar z^\gamma} = \frac{\partial^{\delta_1+\ldots+\delta_n+\gamma_1+\ldots+\gamma_n}F(z)}{\partial z_1^{\delta_1}\cdots\partial z_n^{\delta_n}\partial\bar z_1^{\gamma_1}\cdots\partial\bar z_n^{\gamma_n}},$$

and $\delta = (\delta_1,\ldots,\delta_n)$, $\gamma = (\gamma_1,\ldots,\gamma_n)$.

4. Additional construction

Consider a mapping $\zeta = \chi(\eta) : \overline{B}\longrightarrow\overline{D}$, where B is the unit ball in $\mathbb{C}^n$ centered at zero taking zero to a $a\in D$. The mapping χ is be constructed as follows: Consider the complex lines $\lambda_b = \{\eta\in\mathbb{C}^n:\ \eta = b\tau,\ b\in\mathbb{CP}^{n-1},\ \tau\in\mathbb{C}\}$ and $l_{a,b} = \{\zeta\in\mathbb{C}^n:\ \zeta = a+bt,\ b\in\mathbb{CP}^{n-1},\ t\in\mathbb{C}\}$. The intersection $D_{a,b} = D\cap l_{a,b}$ is a strictly convex domain in $\mathbb{C}$; therefore, there exists a conformal mapping $t = \chi_b(\tau)$ of the unit disk $B\cap\lambda_b$ into $D_{a,b}$ taking $\tau = 0$ to $t = 0$. By the

Carathéodory Theorem [21], this mapping extends to a homeomorphism of the closed domains. Then to a point $\eta = b\tau \in B \cap \lambda_b$ there is assigned the point $\chi(\eta) = a + b\chi_b(\tau) \in D_{a,b}$.

Lemmas 11-14 are formulated and proved in the same way as in the paper [22].

Lemma 11. *Let D be a bounded strictly convex circular domain with twice smooth boundary in $\mathbb{C}^n$. Then $\chi(\eta)$ is well defined and is a $\mathcal{C}^1$-diffeomorphism from $\overline{B}$ onto $\overline{D}$.*

Henceforth, we assume that D is a bounded strictly convex circular domain with twice smooth boundary.

Lemma 12. *The derivatives of $\chi(\eta)$ are holomorphic functions in τ for b fixed and where $\eta = b\tau$.*

Lemma 13. *Let the function $f \in \mathcal{C}(\partial D)$ have the one-dimensional holomorphic extension property along complex lines passing through $a \in D$. Then the function $f^\star(\eta) = f(\chi(\eta))$ is continuous on ∂B and has the one-dimensional holomorphic extension property along complex lines passing through zero.*

Performing a change of variables in integral for Φ, we obtain

$$\Phi(z,w) = \int_{\partial D} f(\zeta) Q(z,w,\zeta)\, d\mu(\zeta) =$$

$$= \int_{\partial B} f(\chi(\eta)) Q(z,w,\chi(\eta))\, d\mu(\chi(\eta)) = \int_{\partial B} f^\star(\eta) Q^\star(z,w,\eta)\, d\mu^\star(\eta).$$

Consider the form

$$\omega^\star(\eta) = \omega(\chi(\eta)) = \sum_{k=1}^{n} (-1)^{k-1} \bar{\chi}_k(\eta)\, d\bar{\chi}(\eta)[k] \wedge d\chi(\eta).$$

By Lemma 12, the form $d\chi(b\tau)$ is holomorphic in τ for b fixed, while the form $d\bar{\chi}(b\tau)[k]$ is antiholomorphic in τ for b fixed.

Lemma 14. *The forms $d\bar{\chi}(b\tau)\big|_{|\tau|=1}$, $k = 1, \ldots, n$, are forms with holomorphic coefficients with respect to τ.*

Theorem 3. *Let D be a bounded strictly convex circular domain with twice smooth boundary and the function $f \in \mathcal{C}(\partial D)$ have the one-dimensional holomorphic extension property along complex lines passing through $a \in D$. Then*

$$\frac{\partial^\gamma \Phi(z,w)}{\partial w^\gamma}\Big|_{\substack{z=a\\ w=\bar{a}}} = 0$$

for $\|\gamma\| > 0$, where $\gamma = (\gamma_1, \ldots, \gamma_n)$ and $\|\gamma\| = \gamma_1 + \ldots + \gamma_n$.

The proof of this Theorem is essentially as in the proof of Theorem 3 of [22].

Corollary 2. $\Phi(a,w) = \mathrm{const}$ *under the conditions of Theorem 3.*

the same way as the previous theorem we prove the statement:

Theorem 4. *Let D be a bounded strictly convex circular domain with twice smooth boundary and the function $f \in \mathcal{C}(\partial D)$ have the one-dimensional holomorphic extension property along complex lines passing through $a \in D$. Then the derivatives $\dfrac{\partial^\delta \Phi(z,w)}{\partial z^\delta}\Big|_{\substack{z=a,\\ w=\bar{a}}}$ are polynomials in w of degree at most $\|\delta\|$.*

Theorem 5. *Let D be a bounded strictly convex circular domain with twice smooth boundary and the function $f(\zeta) \in \mathcal{C}(\partial D)$, and $a, c \in D$. Assume that $\Phi(z, w)$ satisfies the conditions $\Phi(a, w) = \mathrm{const}$, $\Phi(c, w) = \mathrm{const}$ and $\dfrac{\partial^\alpha \Phi(a, w)}{\partial z^\alpha}$, $\dfrac{\partial^\alpha \Phi(c, w)}{\partial z^\alpha}$ are polynomials in w of degree at most $\|\alpha\|$. Then, for every fixed z on the complex line*

$$l_{a,c} = \{(z, w) : \ z = at + c(1 - t), \, w = \bar{a}t + \bar{c}(1 - t), \, t \in \mathbb{C}\}$$

we have $\Phi(z, w) = \mathrm{const}$ with respect to w; i.e., $\dfrac{\partial^\gamma \Phi(z, w)}{\partial w^\gamma} = 0$ for $\|\gamma\| > 0$.

The proof of this Theorem is essentially the same as the proof of Theorem 5 of [22].

Corollary 3. *Under the conditions of Theorem 5, $\dfrac{\partial^\gamma F(z)}{\partial \bar{z}^\gamma}\Big|_{z=at+(1-t)c} = 0$ for $\|\gamma\| > 0$.*

5. Proof of the main assertions

Theorem 6. *Let $n = 2$ and D be a bounded strictly convex circular domain with twice smooth boundary and the function $f \in \mathcal{C}(\partial D)$ have the one-dimensional holomorphic extension property along the family $\mathfrak{L}_{\{a,c,d\}}$ and the points $a, c, d \in D$ do not lie on one complex line in $\mathbb{C}^2$. Then $\dfrac{\partial^\gamma \Phi(z, w)}{\partial w^\gamma} = 0$ for any $z \in D$ and $\|\gamma\| > 0$, and $f(\zeta)$ extends holomorphically into D.*

Proof. Let $\tilde{z}$ be an arbitrary point on $l_{a,c}$. Then by Theorem 5, we have

$$\frac{\partial^\gamma \Phi(\tilde{z}, w)}{\partial w^\gamma} = 0 \tag{10}$$

for$\|\gamma\| > 0$. Joining $\tilde{z}$ with d by the line $l_{\tilde{z},d}$ and again applying Theorem 5 with $\tilde{\tilde{z}} \in l_{\tilde{z},d}$, we conclude that $\dfrac{\partial^\gamma \Phi(\tilde{\tilde{z}}, w)}{\partial w^\gamma} = 0$ for $\|\gamma\| > 0$. Therefore, (10) is fulfilled for all $\tilde{z}$ in some open set.

Inserting $w = \bar{z}$ in (10), we have $\dfrac{\partial^\gamma F(z)}{\partial \bar{z}^\gamma} = 0$ in some open set in D. The real analiticity of $F(z)$ implies that $\dfrac{\partial F(z)}{\partial \bar{z}_j} = 0$ for any $z \in D$ and $j = 1, \ldots, n$. Since by Theorem 2 we have $F(\zeta)\big|_{\partial D} = f(\zeta)$, the function $f(\zeta)$ extends holomorphically into D. □

Denote by $\mathfrak{A}$ the set of noncomplanar points $a_k \in D \subset \mathbb{C}^n$, $k = 1, \ldots, n + 1$.

Theorem 7. *Let D be a bounded strictly convex circular domain with twice smooth boundary in $\mathbb{C}^n$ and the function $f \in \mathcal{C}(\partial D)$ have the one-dimensional holomorphic extension property along the family $\mathfrak{L}_{\mathfrak{A}}$. Then $\dfrac{\partial^\gamma \Phi(z, w)}{\partial w^\gamma} = 0$ for any $z \in D$ and $\|\gamma\| > 0$, and $f(\zeta)$ extends holomorphically into D.*

Proof. Proceed by induction on n. The induction base is Theorem 6 ($n = 2$). Suppose that the theorem holds for all $k < n$. Consider the complex plane Γ passing through $a_1, \ldots, a_n$, the dimension of Γ is by hypothesis equal to $n - 1$ and $a_{n+1} \notin \Gamma$. The intersection $\Gamma \cap D$ is a strictly convex domain in $\mathbb{C}^{n-1}$.

The function $f\big|_{\Gamma \cap \partial D}$ is continuous and has the property of holomorphic extension along the family $\mathfrak{L}_{\mathfrak{A}_1}$, where $\mathfrak{A}_1 = \{a_1, \ldots, a_n\}$. By the induction assomption, $\dfrac{\partial^\gamma \Phi(z', w)}{\partial w^\gamma} = 0$ for $\|\gamma\| > 0$ for all $z' \in \Gamma \cap D$.

Joining $z' \in \Gamma$ with a_{n+1}, we find by Theorem 6 that $\dfrac{\partial^\gamma \Phi(z,w)}{\partial w^\gamma} = 0$ for$\|\gamma\| > 0$ for some open set in $D \times D$. In much the way as Theorem 6, this implies that $F(z)$ is holomorphic in D, and so $f(\zeta)$ extends holomorphically into D. □

Theorems 6 and 7 obviously imply Theorems A and B.

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Многомерные граничные аналоги теоремы Гартогса в круговых областях

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В статье представлены некоторые результаты, связанные с голоморфным продолжением функций, определенных на границе области $D \subset \mathbb{C}^n$, $n > 1$, в эту область. Речь идет о функциях с одномерным свойством голоморфного продолжения вдоль комплексных прямых.

Ключевые слова: функции с одномерным свойством голоморфного продолжения, круговые области.