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On a Question about Generalized Congruence Subgroups

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Elementary net (carpet) $\sigma = (\sigma_{ij})$ is called admissible (closed) if the elementary net (carpet) group $E(\sigma)$ does not contain a new elementary transvections. This work is related to the problem proposed by Y.N.Nuzhin in connection with the problem 15.46 from the Kourovka notebook proposed by V.M.Levchuk (admissibility (closure) of the elementary net (carpet) $\sigma = (\sigma_{ij})$ over a field K). An example of field K and the net $\sigma = (\sigma_{ij})$ of order n over the field K are presented so that subgroup $\langle t_{ij}(\sigma_{ij}), t_{ji}(\sigma_{ji})\rangle$ is not coincident with group $E(\sigma) \cap \langle t_{ij}(K),\ t_{ji}(K)\rangle$.

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1. Notations and problem statement

Let us consider problem 15.46 from the Kourovka notebook [1] on the admissibility (closure) of carpets (elementary nets) proposed by V. M. Levchuk. This problem (or rather, its SL-version) is as follows. Let $\sigma = (\sigma_{ij})$ be an elementary net (carpet) of order $n \geqslant 3$ over a field K. Is it true that for the admissibility of the carpet (elementary net) $\sigma = (\sigma_{ij}),\ 1 \leqslant i \neq j \leqslant n$ is necessary and sufficient the admissibility of subcarpets (subnets) $\begin{pmatrix} * & \sigma_{ji} \\ \sigma_{ij} & * \end{pmatrix}$ of second order (for any $i \neq j$)? We note that solution of this problem for locally finite fields results from [2]. In connection with the theorem on the decomposition of the elementary transvection in the elementary net group $E(\sigma)$ [3], a sufficient condition to solve the problem was proposed by Y. N. Nuzhin. It is linked with the validity of the equality

$$E(\sigma) \cap \langle t_{ij}(K),\ \ t_{ji}(K)\rangle = \langle t_{ij}(\sigma_{ij}),\ \ t_{ji}(\sigma_{ji})\rangle \tag{1}$$

for all $i \neq j$, where $\sigma = (\sigma_{ij})$ is the closed elementary net of degree $n \geqslant 3$ over a field $K,\ \ E(\sigma)$ is the elementary net subgroup. Inclusion ($\supseteq$) is obvious. To test the validity of equality (1) one need to test the validity of inclusion ($\subseteq$) in (1).

In this paper we present an example of field K and elementary closed (admissible) irreducible net $\sigma = (\sigma_{ij})$ of order $n \geqslant 3$ over the field K for which the subgroup $E(\sigma) \cap \langle t_{ij}(K), t_{ji}(K)\rangle$ is not contained in the group $\langle t_{ij}(\sigma_{ij}), t_{ji}(\sigma_{ji})\rangle$. Without the loss in generality we assume that $i = 1,\ j = 2$. The proposed example results from [4, 5]. We should note that solution of problem 15.46 from the Kourovka notebook is not presented in the paper.

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In the paper the following standard notations are adopted: δ_{ij} is the Kronecker delta; $t_{ij}(\alpha) = e + \alpha e_{ij}$ is the elementary transvection, where e is the identity matrix of order n, e_{ij} is the matrix, its entries at (i, j) are equal to 1, and all other entries are equal to zero, $\alpha \in K$; $t_{ij}(A) = \{t_{ij}(\alpha) : \alpha \in A\}$;

$$E(\sigma) = \langle t_{ij}(\sigma_{ij}) : 1 \leqslant i \neq j \leqslant n\rangle$$

is the elementary net group defined for an elementary net $\sigma = (\sigma_{ij})$;

$$E_{ij}(\sigma) = \langle t_{ij}(\sigma_{ij}), t_{ji}(\sigma_{ji})\rangle, \quad i \neq j;$$

F is arbitrary commutative ring with 1; $F[x]$ is the ring of polynomials with respest to one variable x with coefficients from F; $K = F(x)$ is the the field of all rational functions $\dfrac{f}{g}, f, g \in F[x], \ g \neq 0$.

2. Deriviation of the example

To begin with, recall well-known definitions that we use in this paper. A set of additive subgroups $\sigma = (\sigma_{ij})$, $1 \leqslant i, j \leqslant n$, of a field (or ring) K is called a *net* of order n over K if $\sigma_{ir}\sigma_{rj} \subseteq \sigma_{ij}$ for all values of i, r, j [6]. The term *carpet* is also used instead of the term net [7, 8]. The same system but without the diagonal is called *elementary net.* A full or elementary net $\sigma = (\sigma_{ij})$ is called *irreducible* if all additive subgroups σ_{ij} are different from zero. An elementary net σ is *closed (admissible)* if the subgroup $E(\sigma)$ does not contain new transvections. If the diagonal of an elementary net can be supplemented by a subgroup to a full net then such elementary net is closed.

For a non-negative integer $n \in \mathbb{N} \cup 0$ consider the ideal

$$F_n[x] = \{c_n x^n + c_{n+1}x^{n+1} + \ldots + c_m x^m : \ m \geqslant n, \ c_i \in F\}$$

of ring $F[x] = F_0[x]$. It is obvious that $(n, s \in \mathbb{N} \cup 0)$

$$F_n[x]F_s[x] = F_{n+s}[x], \quad F_n[x] \supseteq F_{n+1}[x] \supseteq F_{n+2}[x] \ldots . \tag{2}$$

Let us consider the supplemented (in particular, closed) elementary net of order $n \geqslant 3$)

$$\sigma = \begin{pmatrix} * & F_2[x] & F_1[x] & \ldots & F_1[x] \\ F_2[x] & * & F_1[x] & \ldots & F_1[x] \\ F_1[x] & F_1[x] & * & \ldots & F_1[x] \\ . & . & . & \ldots & . \\ F_1[x] & F_1[x] & F_1[x] & \ldots & * \end{pmatrix} \tag{3}$$

of ideals of the ring $F[x]$: $\sigma_{12} = \sigma_{21} = F_2[x]$, $\sigma_{ij} = F_1[x]$ for other $i \neq j$. Taking into account (2), the table of σ is an elementary net over ring $F[x]$ (or over field $K = F(x)$) of order n. This elementary net is supplemented net (for example, $F_1[x]$ can be put in all positions of the diagonal).

For elementary net σ (3) consider the subgroup

$$H = \langle t_{ij}(\sigma_{ij}) : \ 1 \leqslant i \neq j \leqslant n; \ \{i, j\} \neq \{1, 2\}\rangle$$

of the elementary group $E(\sigma)$ generated by all root subgroups $t_{ij}(\sigma_{ij})$ except of $t_{12}(\sigma_{12})$ and $t_{21}(\sigma_{21})$.

Proposition. *Elementary net group $E(\sigma)$ is equal to the product of the group $E_{12}(\sigma)$ and the normal subgroup H: $E(\sigma) = E_{12}(\sigma) \cdot H$.*

Proof. If $a \in E(\sigma), a$ is the product of elementary transvections of $E(\sigma)$, then sequentially pulling elementary transvections $t_{12}(*)$ and $t_{21}(*)$ to the left, we get the inclusion $a \in E_{12}(\sigma) \cdot H$. Let us show that H is an normal subgroup of the group $E(\sigma)$. Taking into accoun the equality $E(\sigma) = E_{12}(\sigma) \cdot H$, it is sufficient to show that $shs^{-1} \in H$ for all $s \in E_{12}(\sigma),\ h \in H$. Let assume that $i \geqslant 3$. Then we have

$$t_{12}(\alpha)t_{2i}(\beta)t_{12}^{-1}(\alpha) = t_{1i}(\alpha\beta)t_{2i}(\beta), \quad t_{21}(\alpha)t_{1i}(\beta)t_{21}^{-1}(\alpha) = t_{2i}(\alpha\beta)t_{1i}(\beta).$$

□

Theorem. *The subgroup $H \cap \langle t_{12}(K), t_{21}(K)\rangle$ is not contained in the group $E_{12}(\sigma)$. In particular, the subgroup $E(\sigma) \cap \langle t_{12}(K), t_{21}(K)\rangle$ is not contained in the group $E_{12}(\sigma) = \langle t_{12}(\sigma_{12}), t_{21}(\sigma_{21})\rangle$.*

Proof. Let us assume that $\alpha, \beta, \gamma, \delta$ are elements of arbitrary commutative ring and $\alpha\delta + \gamma\beta = 0$, $([z, t] = ztz^{-1}t^{-1})$. Then

$$b = [t_{23}(\alpha)t_{13}(\beta),\, t_{31}(\gamma)t_{32}(\delta)] = \operatorname{diag}\left(\begin{pmatrix} 1+\beta\gamma & \beta\delta \\ \alpha\gamma & 1+\alpha\delta \end{pmatrix},\ e_{n-2}\right)$$

(general transvection). Applying this formula to our case, we put $\alpha = -x^1$, $\beta = \gamma = \delta = x^1$. Then matrix

$$b = [t_{23}(-x)t_{13}(x),\ t_{31}(x)t_{32}(x)]$$

has the form

$$b = \operatorname{diag}\left(\begin{pmatrix} 1+x^2 & x^2 \\ -x^2 & 1-x^2 \end{pmatrix},\ e_{n-2}\right).$$

Hence matrix b is contained in the group $H \cap \langle t_{12}(K), t_{21}(K)\rangle$. □

To prove the theorem one need to show that matrix b is not contained in the group $\langle t_{12}(\sigma_{12}), t_{21}(\sigma_{21})\rangle$. This follows from the following lemma.

Lemma. *If $c = (\delta_{ij} + c_{ij}) \in \langle t_{12}(\sigma_{12}),\ t_{21}(\sigma_{21})\rangle$ then $c_{12}, c_{21} \in F_2[x]$ and $c_{11}, c_{22} \in F_4[x]$.*

Proof. Let matrix c be the product of n elementary transvections of $t_{12}(F_2[x])$ and $t_{21}(F_2[x])$: $c = t_{12}(\alpha_1)t_{21}(\alpha_2)t_{12}(\alpha_3)\dots$. The proof of the Lemma is carried out by induction on n, and induction transition follows from (2) and inclusions $(\alpha, a_{12}, a_{21} \in F_2[x],\ a_{11}, a_{22} \in F_4[x])$

$$\begin{pmatrix} 1+a_{11} & a_{12} \\ a_{21} & 1+a_{22} \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \subseteq \begin{pmatrix} 1+F_4[x] & F_2[x] \\ F_2[x] & 1+F_4[x] \end{pmatrix},$$

$$\begin{pmatrix} 1+a_{11} & a_{12} \\ a_{21} & 1+a_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \subseteq \begin{pmatrix} 1+F_4[x] & F_2[x] \\ F_2[x] & 1+F_4[x] \end{pmatrix}.$$

□

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К вопросу об обобщенных конгруэнц-подгруппах

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Элементарная сеть (ковер) $\sigma = (\sigma_{ij})$ называется допустимой (замкнутой), если элементарная сетевая (ковровая) группа $E(\sigma)$ не содержит новых элементарных трансвекций. Работа связана с вопросом, поставленным Я.Н.Нужиным в связи с вопросом В.М.Левчука 15.46 из Коуровской тетради о допустимости (замкнутости) элементарной сети (ковра) $\sigma = (\sigma_{ij})$ над полем K. Приводится пример поля K и сети $\sigma = (\sigma_{ij})$ порядка n над полем K, для которой подгруппа $\langle t_{ij}(\sigma_{ij}),\ t_{ji}(\sigma_{ji})\rangle$ не совпадает с группой $E(\sigma) \cap \langle t_{ij}(K),\ t_{ji}(K)\rangle$.

Ключевые слова: ковры, ковровые группы, сети, элементарные сети, допустимые элементарные сети, замкнутые элементарные сети, элементарная сетевая группа, трансвекция.