V_{JK 517.548.2} Morera's Theorem and Functional Series in the Class of A-analytic Functions

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The aim of this paper is to investigate A-analytic functions in a special case when the function A is an anti-analytic function in a domain. We prove that a continuous function satisfying the integral condition of the Cauchy theorem is A-analytic (an analog of Morera's theorem, Sec. 2). In Sec. 3 we prove an analog of the Weierstrass theorem for functional series of A-analytic functions and the expansion of A-analytic functions into functional series (Sec. 4).

Keywords: A-analytic functions, analog of Morera's theorem, analog of the Weierstrass theorem, expansion of A-analytic functions.

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1. Introduction and preliminaries

The paper is devoted to the theory of real-analytic solutions of the Beltrami equation

$$f_{\bar{z}}(z) = A(z) f_z(z) \tag{1}$$

which is directly related to theory of quasi-conformal mappings. The function A(z) is, in general, assumed to be measurable with $|A(z)| \leq C < 1$ almost everywhere in the domain $D \subset \mathbb{C}$. Solutions of equation (1) are often referred to as *A*-analytic functions in the literature.

The solutions of equation (1), as well as quasi-conformal homeomorphisms in the complex plane \mathbb{C} , have been studied in sufficient details. Here we confine ourselves to giving the references ([1,4,5,8–10]) and formulating the following three theorems:

Theorem 1.1 ([1]). For any measurable on the complex plane function A(z): $||A||_{\infty} < 1$ there exists a unique homeomorphic solution $\chi(z)$ of equation (1) which fixes the points $0, 1, \infty$.

Note that if the function $|A(z)| \leq C < 1$ is defined only in the domain $D \subset \mathbb{C}$, then it can be extended to the whole \mathbb{C} by setting $A \equiv 0$ outside D, so Theorem 1.1 holds for any domain $D \subset \mathbb{C}$.

Theorem 1.2 ([4,5]). All generalized solutions of equation (1) have the form $f(z) = \Phi[\chi(z)]$, where $\chi(z)$ is a homeomorphic solution in Theorem 1.1, and $\Phi(\xi)$ is a holomorphic function in the domain $\chi(D)$. Moreover, if a generalized solution f(z) has isolated singular points, then the holomorphic function $\Phi = f \circ \chi^{-1}$ also has isolated singularities of the same types.

Theorem 1.2 implies that an A-analytic function f carries out an internal (open) mapping, i.e. it maps an open set to an open set. It follows that the maximum principle holds for such functions: for any bounded domain $G \in D$ the maximum of the modulus is reached only on the boundary, i.e. $|f(z)| \leq \max_{z \in \partial G} |f(z)|, z \in G$. If the function is not zero, then the minimum principle also holds, i.e. $|f(z)| \geq \min_{z \in \partial G} |f(z)|, z \in G$.

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Theorem 1.3 ([8]). If a function A(z) belongs to the class $C^m(D)$, then every solution f of equation (1) also belongs, at least, to the same class $C^m(D)$.

The aim of this paper is to investigate A-analytic functions in a special case when the function A is an anti-analytic function in a domain. We prove that a continuous function satisfying the integral condition of the Cauchy theorem is A-analytic (an analog of Morera's theorem, Sec. 2). In Sec. 3 we prove an analog of the Weierstrass theorem for functional series of A-analytic functions and the expansion of A-analytic functions into functional series (Sec. 4).

The study of A-analytic functions was inspired by their applications in tomography problems. In a series of papers by A. Bukhgeim and S. G. Kazantsev (see [6,7]) the Radon problem is interpreted as a boundary value problem for an infinite-dimensional analog of the equation $f_{\bar{z}} - Af_z = 0$, where f is a function of complex argument z with values in some Banach space X, and A is a linear continuous operator $A: X \to X$, ||A|| < 1.

A-analytic functions can be applied in the theory of elliptic equations (see [11, 16]), when A is a continuous linear operator in a finite or infinite-dimensional space. In papers [11, 16] A is a linear continuous operator in X. In case when $X = \mathbb{C}$ the function A is a constant.

Let A be anti-analytic, i.e. $\frac{\partial A}{\partial z} = 0$ in $D \subset \mathbb{C}$, and such that $|A(z)| \leq C < 1$, $\forall z \in D$. We put

$$D_A = \frac{\partial}{\partial z} - \bar{A}(z) \frac{\partial}{\partial \bar{z}}, \quad \bar{D}_A = \frac{\partial}{\partial \bar{z}} - A(z) \frac{\partial}{\partial z}.$$

Then according to (1) the class $O_A(D)$ of A-analytic functions in D is characterized by the fact that $\overline{D}_A f = 0$. Since an anti-analytic function is smooth, Theorem 1.3 implies that $O_A(D) \subset C^{\infty}(D)$.

Theorem 1.4 (an analog of Cauchy's theorem, see [16]). If $f \in O_A(D) \cap C(\overline{D})$, where $D \subset \mathbb{C}$ is a domain with rectifiable boundary ∂D , then

$$\int_{\partial D} f(z)(dz + A(z)d\,\bar{z}) = 0$$

Now we assume that the domain $D \subset \mathbb{C}$ is convex, and $\xi \in D$ is a fixed point in it. Consider the function

$$K(z,\,\xi) = \frac{1}{2\pi i} \frac{1}{z - \xi + \int_{\gamma(\xi,\,z)} \bar{A}(\tau) d\tau},\tag{2}$$

where $\gamma(\xi, z)$ is a smooth curve which connects points ξ and z in D. Since the domain is simply connected and the function $\bar{A}(z)$ is holomorphic, the integral

$$I(z) = \int_{\gamma(\xi, z)} \bar{A}(\tau) d\tau$$

does not depend on a path of integration; it coincides with a primitive, i.e. $I'(z) = \overline{A}(z)$.

Theorem 1.5 ([14]). $K(z, \xi)$ is an A-analytic function outside of the point $z = \xi$, i.e. $K \in O_A(D \setminus \{\xi\})$. Moreover, at $z = \xi$ the function $K(z, \xi)$ has a simple pole.

Remark 1. If a simply connected domain $D \subset \mathbb{C}$ is not convex, then the function

$$\psi(z,\,\xi) = z - \xi + \overline{\int_{\gamma(\xi,\,z)} \overline{A}(\tau) d\tau},$$

although well defined in D, may have other isolated zeros except for $\xi: \psi(z, \xi) = 0$ for $z \in P = \{\xi, \xi_1, \xi_2, \ldots\}$. Consequently, $\psi \in O_A(D)$, $\psi(z, \xi) \neq 0$ when $z \notin P$, and $K(z, \xi)$ is an A-analytic function only in $D \setminus P$, **it has poles** at the points of P. Due to this fact we consider the class of A-analytic functions only in convex domains.

According to Theorem 1.2, the function $\psi(z, \xi) \in O_A(D)$ carries out an internal mapping. In particular, the set

$$L(\xi, r) = \left\{ z \in D \colon |\psi(z, \xi)| = \left| z - \xi + \overline{\int_{\gamma(\xi, z)} \overline{A}(\tau) d\tau} \right| < r \right\}$$

is open in D. For sufficiently small r > 0 it compactly belongs to D and contains the point ξ . This set is called an A-lemniscate with the center ξ and denoted by $L(\xi, r)$. It is a simply connected domain (see [13]).

Theorem 1.6 (the Cauchy formula, see [13]). Let $D \subset \mathbb{C}$ be a convex domain and $G \subset D$ be its subdomain with piecewise smooth boundary ∂G . Then for any function $f(z) \in O_A(G) \cap C(\overline{G})$ we have

$$f(z) = \int_{\partial G} K(\xi, z) f(\xi) (d\xi + A(\xi) d\bar{\xi}), \ z \in G.$$
(3)

Let A(z) be an anti-analytic function. The following theorem holds, which, as is not difficult to see, without the condition of anti-analyticity A(z) does not hold.

Theorem 1.7. If $f(z) \in O_A(G)$ then

$$\partial f = \frac{\partial f}{\partial z} \in O_A(G).$$

The proof of the theorem follows easily from the relation that $\bar{D}_A \partial f = \partial \bar{D}_A f$, where

$$\bar{D}_A = \frac{\partial}{\partial \bar{z}} - A(z) \frac{\partial}{\partial z} = \bar{\partial} - A(z) \partial.$$

In fact, direct calculation shows that $\bar{D}_A \partial = (\bar{\partial} - A\partial) \partial = \bar{\partial}\partial - A\partial^2, \partial \bar{D}_A = \partial (\bar{\partial} - A\partial) = \partial \bar{\partial} - \partial A \cdot \partial - A\partial^2 = \bar{\partial}\partial - A\partial^2$, since $\partial A = 0$ because of anti-analyticity of A(z).

Note that if A(z) is not identically a constant then other derivatives such as ∂f or $D_A f$ are not A-analytic functions.

2. An analog of Morera's theorem

As in the classical case, for A-analytic functions the inverse of the Cauchy theorem holds.

Theorem 2.1. Let f(z) be a continuous function in a simply-connected domain D and the integral of f(z) over any closed smooth curve Γ that belongs to the domain D be equal to zero, *i.e.*

$$\oint_{\Gamma} f(z)(dz + A(z)d\bar{z}) = 0.$$
(4)

Then f(z) is an A-analytic function in the domain D.

Proof. Let the function f(z) = u(x, y) + iv(x, y) and A(z) = a(x, y) + ib(x, y). Then condition (4) can be rewritten in the form of contour integrals of the 2nd type:

$$\begin{split} \oint_{\Gamma} f(z)(dz + A(z)d\bar{z}) &= \oint_{\Gamma} ((a+1)u - bv)dx + ((a-1)v + bu)dy + \\ &+ i \oint_{\Gamma} ((a+1)v + bu)dx + ((a-1)u - bv)dy = 0. \end{split}$$

Hence,

$$\oint_{\Gamma} ((a+1)u - bv)dx + ((a-1)v + bu)dy = 0,
\oint_{\Gamma} ((a+1)v + bu)dx + ((a-1)u - bv)dy = 0.$$
(5)

We fix a point $a \in D$ and consider the following integral

$$F(z) = \int_{\Gamma(a,z)} f(z)(dz + A(z)d\bar{z}), \tag{6}$$

where $\Gamma(a, z)$ is a smooth curve connecting the points a and $z \in D$. According to (4), the integrals (5) do not depend on the path of integration $\Gamma(a, z)$.

We write the function F(z) in the form

$$F(z) = U(z) + iV(z),$$
(7)

where

$$U(z) = \int_{\Gamma(a, z)} ((a+1)u - bv)dx + ((a-1)v + bu)dy,$$
$$V(z) = \int_{\Gamma(a, z)} ((a+1)v + bu)dx + ((a-1)u - bv)dy,$$

and according to (6), each of these integrals do not depend on the path of integration, and the following equalities hold

$$\frac{\partial U}{\partial x} = (a+1)u - bv, \quad \frac{\partial U}{\partial y} = (a-1)v + bu,$$
$$\frac{\partial V}{\partial x} = (a+1)v + bu, \quad \frac{\partial V}{\partial y} = (a-1)u - bv.$$

Hence,

$$\frac{\partial F}{\partial z} = \frac{1}{2} \left(\frac{\partial F}{\partial x} - i \frac{\partial F}{\partial y} \right) = \frac{1}{2} \left(\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) = u + iv = f,$$

$$\frac{\partial F}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial F}{\partial x} + i \frac{\partial F}{\partial y} \right) = \frac{1}{2} \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial U}{\partial y} - \frac{\partial V}{\partial x} \right) = au - bv + i(bu + av) = Af.$$
(8)

Since

$$\frac{\partial F}{\partial \, \bar{z}} - A \frac{\partial F}{\partial z} = A f - A f = 0,$$

the function F(z) is an A(z)-analytic function, i.e. $F \in O_A(D)$. In particular, $F \in C^{\infty}(D)$. According to (8) $f = \frac{\partial F}{\partial z}$ and by Theorem 1.7 $f \in O_A(D)$. The theorem is proved.

3. Functional series

Lemma 1. Let $D \subset \mathbb{C}$ be a bounded domain with a smooth boundary and $f, g \in C^1(\overline{D} \times \overline{D})$, then the function

$$F(z) = \int_{\partial D} f(\xi, z) d\xi + g(\xi, z) d\bar{\xi}$$

is differentiable with respect to z and the following equality holds

$$\partial F = \int_{\partial D} \partial f(\xi, z) d\xi + \partial g(\xi, z) d\bar{\xi}$$

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Proof. Let $f(\xi, z) = f_1(\xi, x, y) + if_2(\xi, x, y)$ and $g(\xi, z) = g_1(\xi, x, y) + ig_2(\xi, x, y)$, where $\xi = \zeta + i\eta$, z = x + iy. Then

$$F(x,y) = \int_{\partial D} (f_1 + if_2)(d\zeta + id\eta) + (g_1 + ig_2)(d\zeta - id\eta) = \\ = \int_{\partial D} (f_1 + g_1)d\zeta + (g_2 - f_2)d\eta + i \int_{\partial D} (f_2 + g_2)d\zeta + (f_1 - g_1)d\eta$$

The rule of differentiation of an integral depending on a parameter implies

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$$\frac{\partial F}{\partial x} = \int_{\partial D} \left(\frac{\partial f_1}{\partial x} + \frac{\partial g_1}{\partial x} \right) d\zeta + \left(\frac{\partial g_2}{\partial x} - \frac{\partial f_2}{\partial x} \right) d\eta + i \int_{\partial D} \left(\frac{\partial f_2}{\partial x} + \frac{\partial g_2}{\partial x} \right) d\zeta + \left(\frac{\partial f_1}{\partial x} - \frac{\partial g_1}{\partial x} \right) d\eta.$$

Moreover

$$\frac{\partial F}{\partial y} = \int_{\partial D} \left(\frac{\partial f_1}{\partial y} + \frac{\partial g_1}{\partial y} \right) d\zeta + \left(\frac{\partial g_2}{\partial y} - \frac{\partial f_2}{\partial y} \right) d\eta + i \int_{\partial D} \left(\frac{\partial f_2}{\partial y} + \frac{\partial g_2}{\partial y} \right) d\zeta + \left(\frac{\partial f_1}{\partial y} - \frac{\partial g_1}{\partial y} \right) d\eta$$

Using now

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \quad \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$$

we have

$$\frac{\partial F}{\partial z} = \frac{1}{2} \left(\frac{\partial F}{\partial x} + i \frac{\partial F}{\partial y} \right) = \frac{1}{2} \int_{\partial D} \left(\frac{\partial f_1}{\partial z} + \frac{\partial g_1}{\partial z} \right) d\zeta + \left(\frac{\partial g_2}{\partial z} - \frac{\partial f_2}{\partial z} \right) d\eta + \frac{i}{2} \int_{\partial D} \left(\frac{\partial f_2}{\partial z} + \frac{\partial g_2}{\partial y} \right) d\zeta + \left(\frac{\partial f_1}{\partial z} - \frac{\partial g_1}{\partial y} \right) d\eta = \int_{\partial D} \frac{\partial f}{\partial z} d\xi + \frac{\partial g}{\partial z} d\bar{\xi}$$

We can similarly prove that

$$\bar{\partial}F = \int_{\partial D} \bar{\partial}f(\xi, z)d\xi + \bar{\partial}g(\xi, z)d\bar{\xi}.$$

Now we consider an A(z)-analytic function f(z) in a simply-connected domain D. We fix a point $a \in D$ and a lemniscate $L(a, r) = \{\xi : |\psi(a, \xi)| < r\} \subseteq D$. Then we have

Lemma 2. In L(a, r) the following equality holds

$$\frac{\partial^n f(z)}{\partial z^n} = \frac{n!}{2\pi i} \int_{\partial L(a,r)} \frac{f(\xi)}{\left[\psi(\xi,z)\right]^{n+1}} (d\xi + A(\xi)d\,\bar{\xi}), \ n = 0, 1, \dots,$$
(11)

where we recall $\psi(\xi, z) = \xi - z + \overline{\int_{\gamma(z, \xi)} \overline{A(\tau)} d\tau}$.

Proof. By the Cauchy integral formula we have

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial L(a,r)} \frac{f(\xi)(d\xi + A(\xi)d\bar{\xi})}{\xi - z + \int_{\gamma(z,\,\xi)} \overline{A(\tau)} d\tau} = \frac{1}{2\pi i} \int\limits_{\partial L(a,r)} \frac{f(\xi)(d\xi + A(\xi)d\bar{\xi})}{\psi(\xi,\,z)}.$$

We use the obvious relation

$$\frac{\partial \psi^n(\xi, z)}{\partial z} = n\psi^{n-1}(\xi, z)\frac{\partial \psi(\xi, z)}{\partial z} = -n\psi^{n-1}(\xi, z)$$

and

$$\frac{\partial^n}{\partial z^n} \left(\frac{1}{\psi(\zeta, z)} \right) = \frac{\partial^{n-1}}{\partial z^{n-1}} \left(\frac{\partial}{\partial z} \frac{1}{\psi(\zeta, z)} \right) = \frac{\partial^{n-1}}{\partial z^{n-1}} \left(\frac{1}{\psi^2(\zeta, z)} \right) = \dots = \frac{n!}{\psi^{n+1}(\zeta, z)}$$

We have

$$\begin{split} \frac{\partial^n f}{\partial z^n} &= \partial^n f = \partial^n \frac{1}{2\pi i} \int\limits_{L(a,\,r)} \frac{f(\xi) \left(d\xi + A(\xi)d\bar{\xi}\right)}{\psi(\xi,\,z)} = \frac{1}{2\pi i} \int\limits_{L(a,\,r)} \partial^n \left(\frac{1}{\psi(\xi,\,z)}\right) f(\xi) \left(d\xi + A(\xi)d\bar{\xi}\right) = \\ &= \frac{1}{2\pi i} \int\limits_{L(a,\,r)} \frac{n!}{\psi^{n+1}(\zeta,\,z)} f(\xi) \left(d\xi + A(\xi)d\bar{\xi}\right) = \frac{n!}{2\pi i} \int\limits_{\partial L(a,\,r)} \frac{f(\xi)}{\left[\psi(\xi,\,z)\right]^{n+1}} \left(d\xi + A(\xi)d\bar{\xi}\right). \end{split}$$

Theorem 3.1 (an analog of the Weierstrass theorem). If a series of A-analytic functions in the domain D

$$f(z) = \sum_{n=1}^{\infty} f_n(z), \ f_n(z) \in \mathcal{O}_A(D),$$
 (11)

converges uniformly on any compact subset of this domain, then

1) $f(z) \in \mathcal{O}_A(D);$

2) the series (11) can be differentiated term by term:

$$\partial f(z) = \sum_{n=1}^{\infty} \partial f_n(z), \quad \bar{\partial} f(z) = \sum_{n=1}^{\infty} \bar{\partial} f_n(z), \quad D_A f(z) = \sum_{n=1}^{\infty} D_A f_n(z); \tag{12}$$

3) the series (12) converge uniformly on any compact subset of D.

Proof. We fix an arbitrary simply connected domain $G \in D$. By the hypothesis of the theorem the series (11) converges uniformly in \overline{G} , i.e. its sum f(z) is continuous in G. We can integrate the series term by term along any closed curve $\gamma \subset G$:

$$\int_{\gamma} f(z)(dz + A(z)d\,\overline{z}) = \sum_{n=1}^{\infty} \int_{\gamma} f_n(z)(dz + A(z)d\,\overline{z}).$$

Since $f_n(z)$ is an A-analytic function in G, then by the Cauchy theorem (Theorem 1.4) all the integrals on the right-hand side are zero. Therefore, the integral of f(z) along γ is also zero. Morera's theorem (Theorem 2.1) implies that f(z) is A-analytic, which proves statement 1.

We now prove statement 2. We choose an arbitrary point $a \in D$ and construct a lemniscate $L(a, r) = \{ |\psi(z, a)| < r \} \in D$. According to Lemma 2 we have

$$\partial f|_{z=a} = \frac{1}{2\pi i} \int_{\partial L(a,r)} \frac{f(z)(dz + A(z)d\,\bar{z})}{\left(z - a + \overline{\int_{\gamma(z,a)} \overline{A(\tau)} d\tau}\right)^2}.$$
(13)

Since the series

$$\frac{f(z)}{\left(z-a+\overline{\int_{\gamma(z,a)}\overline{A(\tau)}d\tau}\right)^2} = \sum_{n=0}^{\infty} \frac{f_n(z)}{\left(z-a+\overline{\int_{\gamma(z,a)}\overline{A(\tau)}d\tau}\right)^2}$$
(14)

converges uniformly on $\partial L(a, r)$, then we can substitute (14) into the integral (13) and interchange the sum and the integral:

$$\partial f|_{z=a} = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int\limits_{\partial L(a,r)} \frac{f_n(z)(dz + A(z)d\,\bar{z})}{\left(z - a + \int_{\gamma(z,a)} \overline{A(\tau)}d\tau\right)^2} = \sum_{n=0}^{\infty} \partial f_n|_{z=a}$$

i.e. $\partial f = \sum_{n=0}^{\infty} \partial f_n$. Uniform convergence of the series $\sum_{n=1}^{\infty} \partial f_n(z)$ on any compact subset of the domain D follows from Cauchy's formula and from the uniform convergence of the series (11). Similarly, we can prove

$$\bar{\partial}f(z) = \sum_{n=1}^{\infty} \bar{\partial}f_n(z), \quad D_A f(z) = \sum_{n=1}^{\infty} D_A f_n(z).$$

We have

$$\bar{\partial}f = \bar{A}(z)\partial f = \bar{A}(z)\partial \sum_{n=1}^{\infty} f_n(z) = \sum_{n=1}^{\infty} \bar{A}(z)\partial f_n(z) = \sum_{n=1}^{\infty} \bar{\partial}f_n(z)$$

and

$$D_A f = \partial f - A(z)\bar{\partial}f = \sum_{n=1}^{\infty} \partial f_n(z) - A(z)\sum_{n=1}^{\infty} \bar{\partial}f_n(z) =$$
$$= \sum_{n=1}^{\infty} \partial f_n(z) - \sum_{n=1}^{\infty} A(z)\bar{\partial}f_n(z) = \sum_{n=1}^{\infty} [\partial f_n(z) - A(z)\bar{\partial}f_n(z)] = \sum_{n=1}^{\infty} D_A f_n(z)$$

Since the series $\sum_{n=1}^{\infty} \bar{A}(z) \partial f_n(z)$ converges uniformly and absolutely inside D, then all the series participating in these relations also converge uniformly and absolutely inside D.

Here it is pertinent to note that from uniform convergence of the series, its differentiability in general does not follow. For this, the series of differentials must also be uniformly convergent.

4. Expansion of A-analytic functions into power series

First we note that the analog of power series for A-analytic functions are the following series

$$\sum_{j=0}^{\infty} c_j \psi^j(z, a), a \in D,$$
(15)

where c_j are constants. The domain of convergence of the series (15) is the lemniscate $L(a, R) = \{|\psi(z, a)| < R\}$, where the radius of convergence is given by the Cauchy-Hadamard formula:

$$\frac{1}{R} = \overline{\lim_{j \to \infty}} \sqrt[j]{|c_j|}.$$

We show that the series (15) converges absolutely and uniformly inside the lemniscate $|\psi(z, a)| = |z - a + \overline{\int_{\gamma(a, z)} \overline{A}(\tau) d\tau}| < R$. Let r < R. For $|\psi(z, a)| = \frac{R+r}{2}$ the series (15) converges, and therefore $\exists n_0 :$ for $n \ge n_0$ the following inequality holds

$$\sqrt[n]{|c_n|} \leqslant \frac{2}{r+R}$$

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Then for such $n \ge n_0$ and for $|\psi(z, a)| \le r$ we have

$$|c_n\psi(z, a)^n| \leq |c_n||\psi(z, a)|^n \leq \left(\frac{2r}{r+R}\right)^n.$$

Hence, the series (15) can be reduced to a convergent numerical series and it converges absolutely and uniformly in $\{|\psi(z, a)| \leq r\}$.

There is inverse

Theorem 4.1 (see [14]). If $f(z) \in O_A(L(a, r))$, where $L(a, r) = \{\xi \in D : |\psi(\xi, a)| < r\} \subseteq D$ is a lemniscate, then the function f(z) can be expanded into the series in L(a, r):

$$f(z) = \sum_{k=0}^{\infty} c_k \psi^k(z, a).$$
 (16)

Coefficients of the series are determined by the formula

$$c_{k} = \frac{1}{k!} \frac{\partial^{k} f(z)}{\partial z^{k}} \Big|_{z=a} = \frac{1}{2\pi i} \int_{\partial L(a,\rho)} \frac{f(\xi)}{\left[\psi(\xi, a)\right]^{k+1}} (d\xi + A(\xi)d\bar{\xi}), \ 0 < \rho < r, \ k = 0, 1, \dots$$

Theorem 4.2. The coefficient of a series $\sum_{k=0}^{\infty} c_j \psi^k(z, a)$ converging in a lemniscate L(a, r), r > 0, are uniquely determined by its sum

$$f(z) = \sum_{k=0}^{\infty} c_j \psi^k(z, a) \tag{17}$$

by the formulas

$$c_{k} = \frac{1}{k!} \frac{\partial^{k} f(z)}{\partial z^{k}} \Big|_{z=a} = \frac{1}{2\pi i} \int_{\partial L(a,\rho)} \frac{f(\xi)}{\left[\psi(\xi, a)\right]^{k+1}} (d\xi + A(\xi)d\bar{\xi}), \ 0 < \rho < r, \ k = 0, 1, \dots$$

Proof. We use formulas (11)

$$\frac{\partial^n f(z)}{\partial z^n} = \frac{n!}{2\pi i} \int_{\partial L(a,r)} \frac{f(\xi)}{\left[\psi(\xi,z)\right]^{n+1}} (d\xi + A(\xi)d\,\bar{\xi}), \, n = 0, 1, \dots,$$
(18)

and

$$\frac{\partial \psi^n(z,\,a)}{\partial z} = n\psi^{n-1}(z,\,a).$$

Substituting into (17) z = a, we find $f(a) = c_0$. We now take the partial derivative of the series (17) with respect to z:

$$\frac{\partial f(z)}{\partial z} = c_1 + 2c_2\psi(z, a) + 3c_3\psi(z, a)^2 + \dots$$
(19)

and then substite z = a, thus we find $\frac{\partial f(z)}{\partial z}\Big|_{z=a} = c_1$. The series (19) is a series converging in the lemniscate L(a, r). We take its partial derivative and substitute z = a again to obtain $c_2 = \frac{1}{2} \cdot \frac{\partial^2 f(z)}{\partial z^2}\Big|_{z=a}$. And so, in k-th step we get $c_k = \frac{1}{k!} \frac{\partial^k f(z)}{\partial z^k}\Big|_{z=a}$. The second part of the formula

$$\frac{1}{k!} \frac{\partial^k f(z)}{\partial z^k} \bigg|_{z=a} = \frac{1}{2\pi i} \int\limits_{\partial L(a,\rho)} \frac{f(\xi)}{\left[\psi(\xi, a)\right]^{k+1}} (d\xi + A(\xi)d\,\bar{\xi})$$

follows from (18).

For completeness of the presentation of the material, we give the expansion of functions into 'Laurent' series.

Theorem 4.3 (Laurent series expansion, see [14]). Let f(z) be A-analytic in a ring of lemniscates: $f \in O_A(L(a, R) \setminus L(a, r))$, r < R. Then f(z) admits a 'Laurent' series expansion in this ring:

$$f(z) = \sum_{k=-\infty}^{\infty} c_j \psi^k(z, a),$$
(20)

where the coefficients of the series are determined by the formulas

$$c_k = \frac{1}{2\pi i} \int_{\partial L(a,\,\rho)} \frac{f(\xi)}{\left[\psi(\xi,\,a)\right]^{k+1}} (d\xi + A(\xi)d\,\bar{\xi}), \quad r < \rho < R, \quad k = 0,\,\pm 1,\,\pm 2,\dots$$

The series (20) converges uniformly inside the ring

$$L(a, R) \setminus L(a, r) = \{ z \in D : r < |\psi(z, a)| < R \}.$$

The Cauchy inequalities (see [14]). For the coefficients of this series there hold the following inequalities

$$|c_k| \leqslant \frac{\max\{|f(z)| : z \in \partial L(a, \rho)\}}{\rho^k}, \quad r < \rho < R, \quad k = 0, \pm 1, \pm 2, \dots$$
 (21)

References

- [1] L.Ahlfors, Lectures on quasiconformal mappings, University Lecture Series, v. 38, 2006.
- [2] L.Bers, Mathematical aspects of subsonic and transonic gas dynamics, Wiley, 1958.
- [3] L.Bers, An outline of the theory of pseudoanalytic functions, Bull AMS, 62(1956), no. 4, 291–331.
- [4] B.Bojarski, Homeomorphic solutions of the Beltrami systems, Dokl. Akad. Nauk SSSR, 102(1955), 661–664 (in Russian).
- [5] B.Bojarski, Generalized solutions of a system of differential equations of the first order of the elliptic type with discontinuous coefficients, Mat. Sb. (N.S.), 43(85)(1957), no. 4, 451–503.
- [6] A.L.Bukhgeim, Inversion formulas in inverse problems, A supplement to M.M. Lavrent'ev and L.Ya. Savel'ev Linear Operators and Ill-Posed Problems. Translated from the Russian, Consultants Bureau, New York; Nauka, Moscow, 1995.
- [7] A.L.Bukhgeim, S.G.Kazantsev, Elliptic systems of Beltrami type and tomography problems, Dokl. Akad. Nauk SSSR, 42(1991), no. 3, 704–707 (in Russian).
- [8] I.N.Vekua, Generalized analytic functions, Nauka, Moscow, 1988 (in Russian).

- [9] L.I.Volkovysky, Quasiconformal mappings, Lvov, 1954 (in Russian).
- [10] V.Gutlyanski, V.Ryazanov, U.Srebro, E.Yakubov, The Beltrami equation: a geometric approach, Springer, 2011.
- [11] N.M.Jabborov, Kh.Kh.Imomnazarov, Some mixed problems of the mechanics of two-velocity media, Tashkent, 2012 (in Russian).
- [12] N.M.Jabborov, T.U.Otaboev, Cauchy's theorem for A(z)-analytic functions, Uzbek Mat. Zh., (2014), no. 1, 15–18 (in Russian).
- [13] N.M.Jabborov, T.U.Otaboev, An analog of the integral Cauchy theorem for A-analytic functions, Uzbek Mat. Zh., (2016), no. 4, 50–59 (in Russian)
- [14] A.Sadullaev, N.M.Jabborov, On a class of A-analitic functions, J. Sib. Fed. Univ. Math. Phys., 9(2016), no. 3, 374–383.
- [15] M.A.Lavrent'ev, B.V.Shabat, Methods of the Theory of Functions of a Complex Variable, Moscow, Fizmatgiz, 1958 (in Russian).
- [16] A.P.Soldatov, One-dimensional singular operators and boundary value problems of the theory of functions, Moscow, Vysshaya Shkola, 1991 (in Russian).

Теорема Морера и функциональные ряды в классе А-аналитических функций

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Цель данной статьи — исследование A-аналитических функций в частном случае, когда функция A является антианалитической функцией в области. Доказано, что непрерывная функция, удовлетворяющая интегральным условиям теоремы Коши, аналитическая функция (аналог теоремы Морера, § 2). В § 3 доказывается аналог теоремы Вейеритрасса для функционального ряда по A-аналитическим функциям и разложение A-аналитических функций в функциональные ряды (§ 4).

Ключевые слова: А-аналитическая функция, аналог теоремы Морера, аналог теоремы Вейеритрасса, разложение А-аналитических функций.