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On the Cauchy Problem for Operators with Injective Symbols in Sobolev Spaces

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Let D be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with a smooth boundary ∂D. We describe necessary and sufficient solvability conditions (in Sobolev spaces in D) of the ill-posed non-homogeneous Cauchy problem for a partial differential operator A with injective symbol and of order $m \geq 1$. Moreover, using bases with the double orthogonality property we construct Carleman's formulae for (vector-) functions from the Sobolev space $H^s(D)$, $s \geq m$, by their Cauchy data on Γ and the values of Au in D where Γ is an open (in the topology of ∂D) connected part of the boundary.

Key words: *ill-posed Cauchy problem, Carleman's formula, bases with double orthogonality.*

It is well-known that the Cauchy problem for an elliptic system A is ill-posed (see, for instance, [1]). However it naturally appears in applications: in hydrodynamics (as the Cauchy problem for holomorphic functions), in geophysics (as the Cauchy problem for the Laplace operator), in elasticity theory (as the Cauchy problem for the Lamé system) etc., see, for instance, the book [2] and its bibliography. The problem was actively studied through the XX-th century (see, for instance, [3], [4], [5], [6], [7], [8], [9], [10] and many others); it stimulated the development of the theory of conditionally stable problems.

In this paper we present the approach developed in [9] for the *homogeneous* Cauchy problem for overdetermined elliptic partial differential operators. However we consider the *non-homogeneous* Cauchy problem. Of course, it is easy to see that these problems are equivalent (at least, locally) for systems with the invertible principal symbol. But, if the system is overdetermined, the equivalence takes place only if we have information on the solvability of the equation $Au = f$ in a domain where we look for a solution of the problem. Therefore, even for operators with constant coefficients, the problems are not equivalent in domains which have no convexity properties with respect to the operator A (see, for example, [11]). Moreover, if the coefficients of the operator A are C^∞-smooth (and not real analytic) then there are no general results even on the local solvability of the equation $Au = f$ (see, for instance, [12, §0.0.2, §1.3.13]).

We emphasize that in the present paper we impose no convexity conditions on the domain D.

1. The Problem

Let X be a C^∞-manifold of dimension n with a smooth boundary ∂X. We tacitly assume that it is enclosed into a smooth closed manifold $\tilde{X}$ of the same dimension.

For any smooth $\mathbb{C}$-vector bundles E and F over X, we write $\mathrm{Diff}_m(X; E \to F)$ for the space of all the linear partial differential operators of order $\leq m$ between sections of E and F. Then, for an open set $O \subset \mathring{X}$ (here $\mathring{X}$ is the interior of X) over which the bundles and the manifold are trivial, the sections over O may be interpreted as (vector-) functions and $A \in \mathrm{Diff}_m(X; E \to F)$ is given

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as $(l \times k)$-matrix of scalar differential operators, i.e. we have $A = \sum\limits_{|\alpha| \le m} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \quad x \in O$, where $a_\alpha(x)$ are $(l \times k)$-matrices of $C^\infty(O)$-functions, $k = \mathrm{rank}(E)$, $l = \mathrm{rank}(F)$.

Denote E^* the conjugate bundle of E. Any Hermitian metric $(.,.)_x$ on E gives rise to a sesquilinear bundle isomorphism (the Hodge operator) $\star_E \colon E \to E^*$ by the equality $\langle \star_E v, u\rangle_x = (u,v)_x$ for all sections u and v of E; here $\langle .,.\rangle_x$ is the natural pairing in the fibers of E^* and E.

Pick a volume form dx on X, thus identifying the dual and conjugate bundles. For $A \in \mathrm{Diff}_m(X; E \to F)$, denote by $A' \in \mathrm{Diff}_m(X; F^* \to E^*)$ the transposed operator and by $A^* \in \mathrm{Diff}_m(X; F \to E)$ the formal adjoint operator. We obviously have $A^* = \star_E^{-1} A' \star_F$, cf. [2, 4.1.4] and elsewhere.

Write $\sigma(A)$ for the principal homogeneous symbol of the order m of the operator A, $\sigma(A)$ living on the (real) cotangent bundle T^*X of X. From now on we assume that $\sigma(A)$ is injective away from the zero section of T^*X. Then we will say that A is *elliptic* if $\mathrm{rank}(E) = \mathrm{rank}(F)$ and *overdetermined elliptic* otherwise. Hence it follows that the Laplacian A^*A is an elliptic differential operator of the order $2m$ on X.

We always assume that A satisfies the so-called uniqueness condition in the small on $\overset{\circ}{X}$.

(i) *if u is a distribution in a domain $D \Subset \overset{\circ}{X}$ with $Au = 0$ in the sense of distributions and $u \equiv 0$ on an open subset O of D then $u \equiv 0$ in D.*

It holds true if, for instance, all the objects under consideration are real analytic.

For an open set $O \subset X$, we write $L^2(O,E)$ for the Hilbert space of all the measurable sections of E over O with a finite norm $(u,u)_{L^2(O,E)} = \int_O (u,u)_x dx$. We also denote $H^s(O,E)$ the Sobolev space of the distribution sections of E over O, whose weak derivatives up to the order $s \in \mathbb{N}$ belong to $L^2(O,E)$. Let D be a bounded domain in $\overset{\circ}{X}$, and Γ be a C^∞-smooth open (in the topology of ∂D) connected part of ∂D. As usual, let $H^s_{loc}(D \cup \Gamma, E)$ be the set of sections in D belonging to $H^s(\sigma, E)$ for every measurable set σ in D with $\overline{\sigma} \subset D \cup \Gamma$. For $u \in H^s_{loc}(O,E)$, we always understand Au in the sense of distributions in O. Given any open set O in $\overset{\circ}{X}$ we let $\mathrm{Sol}_A(O)$ stand for the space of all the weak solutions to the equation $Au = 0$ in O.

Further, for non-integer positive s we define Sobolev spaces $H^s(O,E)$ with the use of the proper interpolation procedure (see, for example, [2, §1.4.11]). In the local situation we can use other (equivalent) approach. For instance, if $X \subset \mathbb{R}^n$ and the bundles E and F are trivial, we may we denote $H^{1/2}(O,E)$ the closure of $C^\infty(\overline{O},E)$ functions with respect to the norm (see [13]):

$$\|u\|_{H^{1/2}(O,E)} = \sqrt{\|u\|^2_{L^2(O,E)} + \int\limits_O \int\limits_O \frac{|u(x)-u(y)|^2 dx\, dy}{|x-y|^{2n+1}}}.$$

Then, for $s \in \mathbb{N}$, let $H^{s-1/2}(O,E)$ be the space of functions from $H^{s-1}(O,E)$ such that weak derivatives of the order $(s-1)$ belong to $H^{1/2}(O,E)$.

It is well-known that if ∂D is sufficiently smooth then the functions from the Sobolev space $H^s(D)$, $s \in \mathbb{N}$, have traces on the boundary in the Sobolev space $H^{s-1/2}(\partial D)$ and the corresponding trace operator $tr : H^s(D) \to H^{s-1/2}(\partial D)$ is bounded and surjective (see, for instance, [13]). In particular, this means that for every $u \in H^s_{loc}(D \cup \Gamma)$, $s \in \mathbb{N}$, there is a trace $tr_\Gamma(u)$ on Γ belonging to $H^{s-1/2}_{loc}(\Gamma)$.

Fix a Dirichlet system B_j, $j = 0,1,\dots,m-1$, of the order $(m-1)$ on the boundary of D. More precisely, each B_j is a differential operator of the type $E \to F_j$ and order $m_j \le m-1$, $m_j \ne m_i$ for $j \ne i$, in a neighbourhood U of ∂D. Moreover, the symbols $\sigma(B_j)$, if restricted to the conormal bundle of ∂D, have ranks equal to the dimensions of F_j. From now on we assume that $m_j = j$ and set $t(u) = \oplus_{j=0}^{m-1} B_j u \in \oplus_{j=0}^{m-1} H^{s-j-1/2}(\partial D, F_j)$ for $u \in H^s(D,E)$, $s \ge m$.

Problem 1. *Let $\mathbb{N} \ni s \ge m$. Given boundary data $\oplus_{j=0}^{m-1} u_j \in \oplus_{j=0}^{m-1} H^{s-j-1/2}_{loc}(\Gamma, F_j) \cap L^2(\Gamma, F_j)$ and $f \in H^{s-m}_{loc}(D \cup \Gamma, F) \cap L^2(D,F)$, find a section $u \in H^s_{loc}(D \cup \Gamma, E)$ such that*

$$Au = f \ \text{in } D, \tag{1}$$

$$t(u) = \oplus_{j=0}^{m-1} u_j \ \ on \ \Gamma. \tag{2}$$

As usual, we say that the problem is *homogeneous* if $f \equiv 0$ in D and *non-homogeneous* otherwise. It is well known that problem 1 has no more than one solution under the Uniqueness condition (i) (see, for instance, [9, theorem 2.8]). We reduce this problem to the problem of the extension as a solution to an elliptic system from a small domain to a bigger one. In this way we generalize [9, theorems 5.2 and 10.3] related to the homogeneous Cauchy problem. We also construct formulae for the approximate and exact solutions of the problem.

2. Necessary Solvability Conditions

As far as we consider the *overdetermined* systems, it is natural to assume that operator A is included into an *elliptic* differential complex

$$0 \to C^\infty(E) \stackrel{A}{\to} C^\infty(F) \stackrel{A_1}{\to} C^\infty(G).$$

This means that $A_1 \circ A = 0$ and the corresponding symbolic complex is exact away from the zero section of T^*X. It is possible, for instance, if the operator A is *sufficiently regular* (see, for instance, [12, Definition 1.3.7]). For example, every operator with constant coefficients is sufficiently regular. Also the operators with real analytic coefficients and injective symbol may be included into an elliptic complex under mild assumptions (see [14]). Of course, if A is elliptic then $A_1 \equiv 0$.

Now due to the properties of the complex, $A_1 f = 0$ in D if the Cauchy problem is solvable. Besides, for $l > k$ the operator A induces tangential operator A_τ on ∂D (see, for instance, [12, §3.1.5]). This means that the Cauchy data $\oplus_{j=0}^{m-1} u_j$ and f should be coherent.

More exactly, it is well-known that under our assumptions on the domain D there exists a real valued C^∞-smooth function ρ with $|\nabla\rho| \neq 0$ on ∂D and such that $D = \{x \in X : \ \rho(x) < 0\}$. Without loss of a generality we can always choose the function ρ in such a way that $|\nabla\rho| = 1$ on a neighborhood of ∂D.

Fix a Green operator G_A attached to A, i.e. an operator $G_A(.,.) \in \mathrm{Diff}_{m-1}(X; (F^*, E) \to \Lambda^{n-1})$ such that

$$dG_A(g, v) = (\langle g, Av\rangle_y - \langle A' g, v\rangle_y)\, dy \text{ for all } g \in C^\infty(X, F^*), \quad v \in C^\infty(X, E);$$

here Λ^p is the bundle of the exterior differential forms of the degree $0 \leq p \leq n$ over X.

The Green operator always exists (see [12, Proposition 2.4.4]) and (as ∂D is not characteristic for A in our sutuation) it may be written in the following form:

$$G_A(g, v) = \sum_{j=0}^{m-1} \langle C_j g, B_j v\rangle_y ds(y) + d\rho \wedge G_\nu(g, v) \tag{3}$$

in a neighbourghood U of ∂D, where ρ is a defyning function of D, $G_\nu(g, v) \in \mathrm{Diff}_{m-1}(U; (F^*, E) \to \Lambda^{n-2}|U)$ and $\{C_j\}_{j=0}^{m-1}$ is a Dirichlet system of the order $(m-1)$ on ∂D, with operators $C_j \in \mathrm{Diff}_{m-j-1}(U; F^*_{|U} \to F^*_j)$ (see [15, Lemma 8.3.2]); here ds is the volume form on ∂D induced from X.

Now let $C^\infty_{comp}(D \cup \Gamma, E)$ stand for the set of $C^\infty(\overline{D}, E)$-functions with compact support in $D \cup \Gamma$. Then for the solvability of problem 1 it is necessary that

$$\int_\Gamma \sum_{j=0}^{m-1} \langle C_j A_1' \beta, u_j\rangle_y ds(y) = \int_D \langle A_1' \beta, f\rangle_y dy \text{ for all } \beta \in C^\infty_{comp}(D \cup \Gamma, G^*). \tag{4}$$

In fact, $d\rho = 0$ on ∂D. Hence, if problem 1 is solvable and u is its solution then, by Stokes' formula, we have for each section $\beta \in C^\infty_{comp}(D \cup \Gamma, G^*)$:

$$\int_\Gamma \sum_{j=0}^{m-1} \langle C_j A_1' \beta u_j, \rangle_y ds(y) = \int_{\partial D} G_A(A_1'\beta, u) = \int_D \langle A_1'\beta, Au\rangle_y dy = \int_D \langle A_1'\beta, f\rangle_y dy$$

where G is a domain in D with a smooth boundary such that supp $v \subset \overline{G}$.

3. Solvability Criterion

From now on we assume that the Laplacian A^*A satisfies the Uniqueness condition (i). Then it has a two-sided (i.e. left and right) pseudo-differential fundamental solution, say, Φ, on $\overset{\circ}{X}$ (see, for instance, [2, §4.4.2]). In particular, $\mathcal{L} = \Phi A^*$ is a left pseudo-differential fundamental solution for A.

Let $M_\Gamma v$ be the Green integral with a density $v = \oplus_{j=0}^{m-1} v_j \in \oplus_{j=0}^{m-1} L^2(\Gamma, F_j)$:

$$M_\Gamma v(x) = -\int_\Gamma \sum_{j=0}^{m-1} \langle C_j(y)\mathcal{L}(x,y)), v_j\rangle_y \, ds(y), \qquad x \notin \Gamma \tag{5}$$

(here $\mathcal{L}(x,y)$ is the Schwartz kernel of $\mathcal{L}$ (see, for instance, [12, 1.5.4]). It is known that if ∂D is smooth enough (e.g. $\partial D \in C^\infty$) then the Green integral induces a bounded linear operator

$$M_{\partial D} : \oplus_{j=0}^{m-1} H^{s-j-1/2}(\partial D, F_j) \to H^s(D, E), \qquad s \in \mathbb{Z}_+, \quad s \geq m$$

(see, for instance, [16, 2.3.2.5]). In particular, we easily see that in our case $M_\Gamma(\oplus_{j=0}^{m-1} u_j) \in H^s_{loc}(D \cup \Gamma, E)$.

Further, for a section $f \in L^2(D, F)$ we denote by $T_D f$ the following volume potential:

$$T_D f = \mathcal{L}\chi_D f$$

where χ_D is the characteristic function of the domain D. If ∂D is smooth enough (e.g. $\partial D \in C^\infty$) then the potential T_D induces a bounded linear operator

$$T_D : H^p(D, F) \to H^{p+m}(D, E), \qquad p \in \mathbb{Z}_+$$

(see, for example, [16, 1.2.3.5]). Moreover, for $p = 0$ we can extend f by zero onto X obtaining thus a form $f \in L^2(X)$ and therefore the potential T_D induces actually a continuous linear operator

$$T_D : L^2(D, F) \to H^m_{loc}(\overset{\circ}{X}, E). \tag{6}$$

In particular, in our case we easily see that $T_D f \in H^s_{loc}(D \cup \Gamma, E) \cap H^m_{loc}(\overset{\circ}{X}, E)$.

Further, if ∂D is smooth then for every section $u \in H^m(D, E)$ we have the Green formula:

$$M_{\partial D}(\oplus_{j=0}^{m-1} B_j u) + T_D A u = \chi_D u \tag{7}$$

(see [2, lemma 10.2.3])).

It is clear that the integrals $M_\Gamma v$ and $T_D f$ satisfy $A^*A(M_\Gamma v) = 0$ and $A^*A(T_D f) = 0$ everywhere outside $\overline{D}$ as parameter dependent integrals. Hence the section

$$F = M_\Gamma(\oplus_{j=0}^{m-1} u_j) + T_D f$$

belongs to $\mathrm{Sol}_{A^*A}(\overset{\circ}{X} \setminus \overline{D})$. The Green formula (7) shows that the potential F contains a lot of information on solvability conditions of problem 1.

Now we would like to obtain necessary and sufficient conditions for the solvability of the Cauchy problem 1 with the use of function F. For this purpose we choose a set $D^+ \subset \overset{\circ}{X}$ in such a way that $\Omega = D \cup \Gamma \cup D^+$ is a bounded domain with piece-wise smooth boundary ∂D^+ in $\overset{\circ}{X}$.

Denote by $F^\pm$ the restrictions of F onto $D^\pm$ (here $D^- = D$). By the definition, F^+ belongs to $\mathrm{Sol}_{A^*A}(D^+)$. Besides, defining v in formula (5) by zero on the boundary of a large enough domain $\tilde\Omega \supset D$, we see that, if ∂D is smooth enough (e.g. $\partial D \in C^\infty$) then the Green integral $M_{\partial D}$ induces a bounded linear operator

$$M^+_{\partial D} : \oplus_{j=0}^{m-1} H^{s-j-1/2}(\partial D, F_j) \to H^s(\tilde\Omega \setminus \overline{D}, E), \qquad s \in \mathbb{N}$$

(see, for instance, [16]). In particular, we easily see that in our situation $M^+_\Gamma(\oplus_{j=0}^{m-1} u_j) \in H^s_{loc}(D^+ \cup \Gamma, E)$. Thus, $F^\pm \in H^s_{loc}(D^\pm \cup \Gamma, E)$.

Let $A^* \oplus A_1$ be the standard differential operator of type $F \to (E, G)$ mapping g to the pair $(A^*g, A_1 g)$.

Theorem 1. *Let both A^*A and $A^* \oplus A_1$ satisfy the Uniqueness condition* (i). *Then the Cauchy problem* 1 *is solvable if and only if condition* (4) *holds true, and there is $\mathcal{F} \in \mathrm{Sol}_{\mathrm{A^*A}}(\Omega)$ coinciding with F^+ on D^+.*

Proof. Let problem 1 be solvable and u be its solution. The necessity of condition (4) is already proved. Set

$$\mathcal{F} = F \, - \, \chi_D u.$$

By the definition, the function $\mathcal{F}$ satisfies $A^*A\mathcal{F} = 0$ in D^+ and belongs to $H^s_{loc}(D^\pm \cup \Gamma, E)$.

Take a domain $G \subset D$ with a smooth boundary such that $\overline{G} \cap \partial D \subset \Gamma$. Then according to the Green formula (7) we have in $D^+ \cup G$:

$$\mathcal{F} = M_\Gamma(\oplus_{j=0}^{m-1} u_j) + T_D f - \chi_G u =$$

$$= M_\Gamma(\oplus_{j=0}^{m-1} B_j u) + T_G A u + T_{D\setminus G} f - M_{\partial G}(\oplus_{j=0}^{m-1} B_j u) - T_G A u =$$

$$= -M_{\partial G \setminus \Gamma}(\oplus_{j=0}^{m-1} B_j u) + T_{D\setminus G} f.$$

This identity implies that $\mathcal{F}$ extends from D^+ to $D^+ \cup G \cup (\Gamma \cap \partial G)$ as a solution to the operator A^*A since the integrals $M_{\partial G\setminus\Gamma}(\oplus_{j=0}^{m-1} B_j u)$ and $T_{D\setminus G} f$ are solutions to this operator everywhere outside the integration sets as parameter depending integrals.

Finally, since for every point $x \in D$ there is a domain $G \ni x$ with the described properties, we see that in fact$\mathcal{F}$ belongs to $\mathrm{Sol}_{A^*A}(\Omega)$ and coincides with F^+ on D^+.

Back, let there be a section $\mathcal{F} \in \mathrm{Sol}_{A^*A}(\Omega)$ coinciding with F^+ on D^+. Set

$$u = F^- - \mathcal{F}^-. \tag{8}$$

By the construction, the section u belongs to $H^s_{loc}(D \cup \Gamma, E)$. Moreover, since the section $\mathcal{F}$ is C^∞-smooth in Ω and the potential $T_D f$ belong to $H^m_{loc}(\Omega)$ (see (6)), we see that $t^+_\Gamma(\mathcal{F}^+) = t_\Gamma(\mathcal{F}^-)$ and $t^+_\Gamma(T^+_D f) = t_\Gamma(T^-_D f)$; here $t^+_\Gamma : H^s_{loc}(D^+ \cup \Gamma, E) \to \oplus_{j=0}^{m-1} H^{s-j-1/2}_{loc}(\Gamma, F_j)$ is the corresponding trace operator. Hence the jump theorem for the Green integral (see [9, lemma 2.7]) gives:

$$t_\Gamma(u) = t_\Gamma(M^-_\Gamma \oplus_{j=0}^{m-1} u_j) - t^+_\Gamma(M^+_\Gamma \oplus_{j=0}^{m-1} u_j) + t_\Gamma(T^-_D f) - t^+_\Gamma(T^+_D f) = \oplus_{j=0}^{m-1} u_j.$$

In order to finish the proof we need to check that $Au = f$ in D. For this purpose we consider the section $g = f - Au$ belonging to $H^{s-m}_{loc}(D \cup \Gamma, F)$. Condition (4), in particular, means that f satisfies $A_1 f = 0$ in D, and therefore the section g has the same property.

Moreover, g satisfies $A^* g = 0$ in D. Indeed, as Φ is a two-sided fundamental solution of the Laplacian A^*A, we have

$$A^*(\chi_D f - A T_D f) = A^*(\chi_D f - A\Phi A^* \chi_D f) = 0 \text{ in } \mathring{X}, \tag{9}$$

$$A^* g = A^* f - A^* A M_\Gamma(\oplus_{j=0}^{m-1} u_j) - A^* A T_D f = 0 \text{ in } D.$$

Thus we have proved that $(A^* \oplus A_1) g = 0$ in D.

Now let $\nabla_E \in \mathrm{Diff}_1(X; E \to E \otimes (T^*X)_c)$ and $\nabla_G \in \mathrm{Diff}_1(X; G \to G \otimes (T^*X)_c)$ be connections in the bundles E and G respectively compatible with the corresponding Hermitian metrics (see [17, Ch. III, Proposition 1.11]). Let m_1 be the order of A_1. Set,

$$Q_E = \begin{cases} \nabla_E(\nabla^*_E \nabla_E)^{\frac{m_1-m-1}{2}}, \text{ if } (m_1 - m) \text{ is positive and odd;} \\ (\nabla^*_E \nabla_E)^{(m_1-m)/2}, \text{ if } (m_1 - m) \text{ is positive and even;} \\ I, \text{ if } m_1 \le m, \end{cases}$$

$$Q_G = \begin{cases} \nabla(\nabla^*_G \nabla_G)^{\frac{m-m_1-1}{2}}, \text{ if } (m - m_1) \text{ is positive and odd;} \\ (\nabla^*_G \nabla_G)^{(m-m_1)/2}, \text{ if } (m - m_1) \text{ is positive and even;} \\ I, \text{ if } m \le m_1. \end{cases}$$

Denote $\tilde m = max(m, m_1)$. Clearly, $Q_E \in \mathrm{Diff}_{\tilde m - m}(X; E \to B_E)$ and $Q_G \in \mathrm{Diff}_{\tilde m - m_1}(X; G \to B_G)$ have injective symbols; here B_E and B_G are the corresponding vector bundles. Then, the ellipticity of the complex means that

$$P = Q_E A^* \oplus Q_G A_1$$

belongs to $\mathrm{Diff}_{\tilde m}(X; F \to (B_E, B_G))$ and has the injective symbol (cf. [12, §2.1.4]).

Since $P(f - AT_D f) = Pg = 0$ in D, we conclude that both g and $(f - AT_D f)$ are smooth in D. As $g \in L^2_{loc}(D \cup \Gamma, F) \cap \mathrm{Sol}_P(D)$, it has a finite order of growth near Γ (see [9, theorems 2.6 and 4.4]).

Set $D_\varepsilon = \{x \in D : \ \rho(x) < -\varepsilon\}$. Then for all the sufficiently small $\varepsilon > 0$ the sets $D_\varepsilon \subset\subset D \subset\subset D_{-\varepsilon}$ are domains with smooth boundaries $\partial D_{\pm\varepsilon}$ and vectors $\mp\varepsilon\nu(x)$ belong to $\partial D_{\pm\varepsilon}$ for every point $x \in \partial D$ (here $\nu(x)$ is the outward unit normal vector to ∂D at the point x).

Now using Stokes' formula, we easily obtain

$$\int\limits_{\partial D_\varepsilon} G_{A_1}(\beta, g) = \int\limits_{D_\varepsilon} \langle A_1' \beta, (Au - f)\rangle_y \, dy =$$

$$= -\int\limits_{D_\varepsilon} \langle A_1'\beta, f\rangle_y \, dy + \int\limits_{\partial D_\varepsilon} G_A(A_1'\beta, u) \text{ for all } \beta \in C^\infty_{comp}(D \cup \Gamma, G^*) \tag{10}$$

Then, using (3) and condition (4), we get for all $\beta \in C^\infty_{comp}(D \cup \Gamma, G^*)$:

$$\lim_{\varepsilon \to +0} \left(-\int\limits_{D_\varepsilon} \langle A_1'\beta, f\rangle_y \, dy + \int\limits_{\partial D_\varepsilon} G_A(A_1'\beta, u) \right) =$$

$$= -\int\limits_{D} \langle A_1'\beta, f\rangle_y \, dy + \int\limits_{\Gamma} \sum_{j=0}^{m-1} \langle C_j A_1' \beta, u_j\rangle_y ds(y) = 0. \tag{11}$$

Combining (10) and (11), we obtain:

$$\lim_{\varepsilon \to +0} \int\limits_{\partial D_\varepsilon} G_{A_1}(\beta, g) = 0 \text{ for all } \beta \in C^\infty_{comp}(D \cup \Gamma, G^*). \tag{12}$$

Similarly, using Stokes' formula and [12, Proposition 2.4.5], we get for all $h \in C^\infty_{comp}(D \cup \Gamma, E^*)$:

$$\int\limits_{\partial D_\varepsilon} G_{A^*}(h, g) = -\int\limits_{D_\varepsilon} \langle (A^*)'h, f\rangle_y \, dy + \int\limits_{D_\varepsilon} \langle (A^*)'h, AT_D f\rangle_y \, dy +$$

$$+ \int\limits_{\partial D_\varepsilon} \overline{\sum_{j=0}^{m-1} \langle C_j \star_F (AM_\Gamma(\oplus_{j=0}^{m-1} u_j) - A\mathcal{F}), B_j \star_E^{-1} h\rangle_y} ds_\varepsilon(y). \tag{13}$$

Let $\tilde h \in C_0(\Omega, E^*)$ such that $\tilde h = h$ in D. Then, according to (9), we have

$$-\int\limits_D \langle (A^*)'h, f\rangle_y \, dy + \int\limits_D \langle (A^*)'h, AT_D f\rangle_y \, dy = -\int\limits_{\Omega \setminus D} \langle (A^*)'\tilde h, AT_D f\rangle_y \, dy. \tag{14}$$

Moreover, since $T_D f \in Sol_{A^*A}(D^+)$, and $\mathcal{F} \in C^\infty(\Omega, E)$, Stokes' formula implies:

$$-\lim_{\varepsilon \to +0} \int\limits_{\partial D_\varepsilon} \overline{\sum_{j=0}^{m-1} \langle C_j \star_F A\mathcal{F}, B_j \star_E^{-1} h\rangle_y} ds_\varepsilon(y) = \int\limits_{\Omega \setminus D} \langle (A^*)'\tilde h, AT_D f\rangle \, dy +$$

$$-\lim_{\varepsilon \to +0} \int\limits_{\partial D_{-\varepsilon}} \sum_{j=0}^{m-1} \overline{\langle C_j \star_F (AM_\Gamma(\oplus_{j=0}^{m-1} u_j), B_j \star_E^{-1} h\rangle_y} ds_{-\varepsilon}(y). \tag{15}$$

Hence, using (13), (14), and (15), we obtain:

$$\lim_{\varepsilon\to+0}\int\limits_{\partial D_\varepsilon} G_{A^*}(h,g)=\lim_{\varepsilon\to+0}\left(\int\limits_{\partial D_\varepsilon}\sum_{j=0}^{m-1}\overline{\langle C_j\star_F(AM_\Gamma(\oplus_{j=0}^{m-1}u_j),B_j\star_E^{-1}h\rangle_y}ds_\varepsilon(y)-\right.$$

$$\left.-\int\limits_{\partial D_{-\varepsilon}}\sum_{j=0}^{m-1}\overline{\langle C_j\star_F(AM_\Gamma(\oplus_{j=0}^{m-1}u_j),B_j\star_E^{-1}h\rangle_y}ds_{-\varepsilon}(y)\right)=0$$

for all $h\in C^\infty_{comp}(D\cup\Gamma,E^*)$, because of the lemma on the weak jump of Green integrals (see [9, Lemma 2.7]). Thus,

$$\lim_{\varepsilon\to+0}\int\limits_{\partial D_\varepsilon} G_{A^*}(h,g)=0 \text{ for all } h\in C^\infty_{comp}(D\cup\Gamma,E^*). \tag{16}$$

Choose a Dirichlet system $\{\tilde B_j\}_{j=0}^{\tilde m-1}$ of the order $(\tilde m-1)$ in a neighbourhood of ∂D and denote by $\{\tilde C_j\}_{j=0}^{\tilde m-1}$ a dual Dirichlet system for it, i.e. such that the Green operator G_P is presented in the form

$$G_P(\phi,\psi)=\sum_{j=0}^{\tilde m-1}\langle\tilde C_j\phi,\tilde B_j\psi\rangle_y ds(y)_\varepsilon+d\rho\wedge\tilde G_\nu(g,f),\quad \psi\in C^\infty(F),\quad \phi\in C^\infty((B_E^*,B_G^*))$$

in a neighbourhood of ∂D (see [15, Lemma 8.3.2] and the discussion in § above).

Using [12, Proposition 2.4.5], (12), (16), and the fact that $(A^*\oplus A_1)g=0$ in D, we see:

$$\lim_{\varepsilon\to+0}\int\limits_{\partial D_\varepsilon}\sum_{j=0}^{\tilde m-1}\langle\tilde C_j\phi,\tilde B_j g\rangle_y ds_\varepsilon(y)=\lim_{\varepsilon\to+0}\int\limits_{\partial D_\varepsilon} G_P(\phi,g)=$$

$$=\lim_{\varepsilon\to+0}\left(\int\limits_{\partial D_\varepsilon} G_{Q_GA_1}(\phi_G,g)+\int\limits_{\partial D_\varepsilon} G_{Q_EA^*}(\phi_E,g)\right)=$$

$$=\lim_{\varepsilon\to+0}\left(\int\limits_{\partial D_\varepsilon} G_{Q_G}(\phi_G,A_1g)+G_{A_1}\big(Q_1'\phi_G,g\big)+G_{Q_E}\big(\phi_E,A^*g\big)+G_{A^*}\big(Q_E'\phi_E,g\big)\right)=$$

$$=\lim_{\varepsilon\to+0}\left(\int\limits_{\partial D_\varepsilon} G_{A_1}\big(Q_G'\phi_G,g\big)+G_{A^*}\big(Q_E'\phi_E,g\big)\right)=0,$$

for all $\phi\in C^\infty_{comp}(D\cup\Gamma,(B_G^*,B_E^*))$; here $\phi=(\phi_E,\phi_G)$, $\phi_E\in C^\infty(D\cup\Gamma,B_E^*)$, $\phi_G\in C^\infty(D\cup\Gamma,B_G^*)$.

Hence,

$$\lim_{\varepsilon\to+0}\int\limits_{\partial D_\varepsilon} G_P(\phi,g)=0 \text{ for all } \phi\in C^\infty_{comp}(D\cup\Gamma,(B_E^*,B_G^*)). \tag{17}$$

As $\{\tilde C_j\}_{j=0}^{\tilde m-1}$ is a Dirichlet system on ∂D, for every $\psi_j\in C^\infty_0(\Gamma,F_j^*)$ there is $\phi\in C^\infty_{comp}(D\cup\Gamma,(B_E^*,B_G^*))$ with $\tilde C_i\phi=0$ for $i\neq j$, $\tilde C_j\phi=\psi_j$ on ∂D and therefore the famous theorem by Banach and Steinhaus yields that is equivalent to the following:

$$\lim_{\varepsilon\to+0}\int\limits_{\partial D}\langle\psi_j,\tilde B_j g(y-\varepsilon\nu(y))\rangle_y\,ds(y)=\text{ for every }\psi_j\in C^\infty_0(\Gamma,F_j^*)\text{ and for each }0\le j\le\tilde m-1,$$

i.e. $\oplus_{j=0}^{\tilde m-1}\tilde B_j g=0$ on Γ in the sense of the weak boundary values (see [9, Definition 2.2]).

Now the uniqueness theorem [9, theorem 2.8] for the Cauchy problem for systems with injective symbols implies that the section $g=f-Au$ equals to zero in D identically because the Uniqueness condition (i) for the operator $A^*\oplus A_1$ holds true in $\mathring{X}$. □

For $f=0$ and the operators with real analytic coefficients, theorem 1 was obtained in [9, theorem 10.3].

Remark 1. *Theorem 1 easily implies conditions of local solvability of the Cauchy problem. Indeed, fix a point* $x_0 \in \Gamma$*. Let* V *be a (one-sided) neighbourhood of* x_0 *in* D *and* $\hat{\Gamma} = \partial V \cap \Gamma$*. Set* $\hat{F} = M_{\hat{\Gamma}}(\oplus_{j=0}^{m-1} u_j) + T_V f$*. As*

$$F = \hat{F} + M_{\Gamma \setminus \hat{\Gamma}}(\oplus_{j=0}^{m-1} u_j) + T_{D\setminus V} f$$

we see that F^+ *extends as a solution to the Laplacian* A^*A *in* $\hat{\Omega} = V \cup \hat{\Gamma} \cup D^+$ *if and only if the potential* $\hat{F}^+$ *does. Hence, under condition* (4)*, the solution of the Cauchy problem exists in the neighbourhod* V *where the extention of the potential* F^+ *does.*

Also we would like to note that theorem 1 gives not only the solvability conditions to problem 1 but the solution itself, of course, if it exists (see (8)). It is clear that we can use the theory of functional series (Taylor series, Laurent series, etc.) in order to get information about extendability of the potential F^+ (cf. [8], [2]). However in this paper we will use the theory of Fourier series with respect to the bases with the double orthogonality property (cf. [18], [2] or elsewhere). Moreover, using formula (8) we can construct approximate solutions of problem 1 (see below).

4. Bases with Double Orthogonality in the Cauchy Problem and Carleman's Formula

It is often important in applications to look for a solution of problem 1 in the class $H^s(D, E)$. For this purpose in the present section we assume that $u_j \in H^{s-j-1/2}(\Gamma, F_j)$, $f \in H^{s-m}(D, F)$. Then Whitney's theorem implies that for each $0 \le j \le m-1$ there is a section $v_j \in H^{s-j-1/2}(\partial D, F_j)$ coinciding with u_j on Γ. We can always choose such a section v_j vanishing outside a given neighborhood of $\overline{\Gamma}$. Now fix such functions $\oplus_{j=0}^{m-1} v_j$.

Set

$$\tilde{F} = M_{\partial D}(\oplus_{j=0}^{m-1} v_j) + T_D f.$$

The boundedness theorems for potential operators in Sobolev spaces (see [16, 1.2.3.5 and 2.3.2.5]) imply that $\tilde{F}^{\pm} \in H^s(D^{\pm}, E)$.

Corollary 1. *Let both* A^*A *and* $A^* \oplus A_1$ *satisfy the Uniqueness condition* (i) *and let* $\partial\Omega$ *be piecewise smooth. In addition, let* $u_j \in H^{s-j-1/2}(\Gamma, F_j)$*,* $f \in H^{s-m}(D)$*. Then the Cauchy problem* 1 *is solvable in* $H^s(D, E)$ *if and only if condition* (4) *is fulfilled and there is a function* $\tilde{\mathcal{F}} \in H^s(\Omega, E) \cap \mathrm{Sol}_{A^*A}(\Omega)$ *coinciding with* $\tilde{F}^+$ *in* D^+*.*

Proof. Let problem 1 be solvable in $H^s(D, E)$. Then theorem 1 implies that condition (4) holds and there is a function $\mathcal{F} \in \mathrm{Sol}_{A^*A}(\Omega)$ coinciding with F^+ in D^+. Clearly,

$$\tilde{F} = F + M_{\partial D \setminus \Gamma}(\oplus_{j=0}^{m-1} v_j). \tag{18}$$

Since the potential $M_{\partial D\setminus\Gamma}(\oplus_{j=0}^{m-1} v_j)$ belongs to $\mathrm{Sol}_{A^*A}(\Omega)$ we conclude that the function F^+ extends to a solution from $\mathrm{Sol}_{A^*A}(\Omega)$ if and only if the function $\tilde{F}^+$ does. Therefore, the function

$$\tilde{\mathcal{F}} = \mathcal{F} + M_{\partial D\setminus\Gamma}(\oplus_{j=0}^{m-1} v_j) = F + M_{\partial D\setminus\Gamma}(\oplus_{j=0}^{m-1} v_j) - \chi_D u = \tilde{F} - \chi_D u \tag{19}$$

belongs to $\mathrm{Sol}_{A^*A}(\Omega)$ and coincides with $\tilde{F}^+$ in D^+. Moreover, as $\tilde{\mathcal{F}} \in H^s_{loc}(\Omega, E) \cap H^s(D^{\pm}, E)$ we easily see that $\tilde{\mathcal{F}} \in H^s(\Omega, E)$.

Back, formula (18) and theorem 1 imply that, under the hypothesis of the corollary, problem 1 is solvable. In order to finish the proof we will show that its solution u, given by (8), is, in fact, the solution of problem 1 in $H^s(\Omega, E)$. However, using (8), (18) and (19) we immediately obtain that

$$u = \tilde{F}^- - \tilde{\mathcal{F}}^-. \tag{20}$$

Since $\tilde{F} \in H^s(D, E)$ and $\tilde{\mathcal{F}} \in H^s(\Omega, E)$ we see that $u \in H^s(D, E)$. □

Now recall the notion of bases with the double orthogonality property in spaces of solutions of elliptic systems (cf. [18], [2] or [9]).For this purpose we denote by $h^s(\Omega)$ the space $\mathrm{Sol}_{A^*A}(\Omega) \cap H^s(\Omega, E)$.

Lemma 1. *If $\omega \Subset \Omega$ is a domain with a piece-wise smooth boundary and $\Omega \setminus \omega$ has no compact (connected) components then there exists an orthonormal basis $\{b_\nu\}_{\nu=1}^\infty$ in $h^s(\Omega)$ such that $\{b_{\nu|\omega}\}_{\nu=1}^\infty$ is an orthogonal basis in $h^s(\omega)$.*

Proof. These $\{b_\nu\}_{\nu=1}^\infty$ are eigen-functions of compact self-adjoint operator $R(\Omega,\omega)^* R(\Omega,\omega)$, where $R(\Omega,\omega) : h^s(\Omega) \to h^s(\omega)$ is the natural inclusion operator (see [2] or [9, theorem 3.1]). □

Now we can use the basis $\{b_\nu\}$ in order to simplify corollary 1. For this purpose fix domains $\omega \Subset D^+$ and Ω as in lemma 1 and denote by $c_\nu(\tilde{F}^+) = \dfrac{(\tilde{F}^+, b_\nu)_{H^s(\omega,E)}}{\|b_\nu\|^2_{H^s(\omega,E)}}$, $\nu \in \mathbb{N}$, the Fourier coefficients of $\tilde{F}^+$ with respect to the orthogonal basis $\{b_{\nu|\omega}\}$ in $h^s(\omega)$.

Corollary 2. *Let both A^*A and $A^* \oplus A_1$ satisfy the Uniqueness condition* (i)*. In addition, let $u_j \in H^{s-j-1/2}(\Gamma, F_j)$, $f \in H^{s-m}(D)$. The Cauchy problem 1 is solvable in $H^s(D,E)$ if and only if condition* (4) *is fulfilled and the series $\sum\limits_{\nu=1}^{\infty} |c_\nu(\tilde{F}^+)|^2$ converges.*

Proof. Indeed, if problem 1 is solvable in $H^s(D,E)$ then, according to corollary 1 condition (4) is fulfilled, and there exists a function $\tilde{\mathcal{F}} \in h^s(\Omega)$ coinciding with $\tilde{F}^+$ in ω.

By lemma 1 we conclude that

$$\tilde{\mathcal{F}}(x) = \sum_{\nu=1}^{\infty} k_\nu(\tilde{\mathcal{F}}) b_\nu(x), \quad x \in \Omega, \tag{21}$$

where $k_\nu(\tilde{\mathcal{F}}) = (\tilde{\mathcal{F}}, b_\nu)_{H^s(\Omega,E)}$, $\nu \in \mathbb{N}$, are the Fourier coefficients of $\tilde{\mathcal{F}}$ with respect to the orthonormal basis $\{b_\nu\}$ in $h^s(\Omega)$. Now Bessel's inequality implies that the series $\sum\limits_{\nu=1}^{\infty} |k_\nu(\tilde{\mathcal{F}})|^2$ converges.

Finally, the necessity of the corollary holds true because

$$c_\nu(\tilde{F}^+) = \frac{(R(\Omega,\omega)\tilde{\mathcal{F}}, R(\Omega,\omega) b_\nu)_{H^s(\omega,E)}}{(R(\Omega,\omega) b_\nu, R(\Omega,\omega) b_\nu)_{H^s(\omega,E)}} = \frac{(\tilde{\mathcal{F}}, R(\Omega,\omega)^* R(\Omega,\omega) b_\nu)_{H^s(\Omega,E)}}{(b_\nu, R(\Omega,\Omega)^* R(\Omega,\omega) b_\nu)_{H^s(\omega,E)}} = k_\nu(\tilde{\mathcal{F}}).$$

Back, if the hypothesis of the corollary holds true then we invoke the Riesz-Fisher theorem. According to it, in the space $h^s(\Omega)$ there is a section

$$\tilde{\mathcal{F}}(x) = \sum_{\nu=1}^{\infty} c_\nu(\tilde{F}^+) b_\nu(x), \qquad x \in \Omega. \tag{22}$$

By the construction, it coincides with $\tilde{F}^+$ in ω. Therefore, using theorem 1, we conclude that problem 1 is solvable in $H^s(D,E)$. □

The examples of bases with the double orthogonality property be found in [9], [2], [18].

Let us obtain Carleman's formula for the solution of problem 1. For this purpose we introduce the following Carleman's kernels:

$$\mathfrak{C}_N(y,x) = \mathcal{L}(y,x) - \sum_{\nu=1}^{N} c_\nu(\mathcal{L}(y,\cdot)) b_\nu(x),\ N \in \mathbb{N},\ x \in \Omega,\ y \notin \overline{\omega}, x \neq y.$$

Corollary 3. *Let both A^*A and $A^* \oplus A_1$ satisfy the Uniqueness condition* (i)*. Then, for every section $v \in H^s(D,E)$, $s \in \mathbb{N}$, the following Carleman's formula holds true:*

$$\lim_{N\to\infty} \left\| v - v^{(N)} \right\|_{H^s(D,E)} = 0, \tag{23}$$

$$v^{(N)}(x) = -\int\limits_{\partial D} \sum_{j=0}^{m-1} \langle C_j \mathfrak{C}_N(.,x)), v_j\rangle_y ds(y) + \int\limits_D \langle \mathfrak{C}_N(.,x), Au\rangle_y dy$$

and $v_j \in H^{s-j-1/2}(\partial D, F_j)$ are (arbitrary) sections coinciding with $B_j v$ on Γ for each $0 \leq j \leq m-1$.

Proof. Indeed for the Cauchy data $f = Av$ and $\oplus_{j=0}^{m-1} u_j = (B_j v)_{|\Gamma}$ the Cauchy problem 1 is solvable in $H^s(D,E)$. Hence corollary 1 implies that a solution of this problem u is given by formula (20). Then the Uniqueness theorem for the problem (see, for instance [9, theorem 2.8]) gives $u = v$ in D.

As $\overline{\omega} \cap \overline{D} = \emptyset$ we may use Fubini theorem and obtain for all $\nu \in \mathbb{N}$:

$$k_\nu(\tilde{F}^+) = \left(-\int\limits_{\partial D} \sum_{j=0}^{m-1} \langle C_j(y) c_\nu(\mathcal{L}(y,.)), v_j\rangle_y ds(y) + \int\limits_D \langle c_\nu(\mathcal{L}(y,.)), f\rangle dy \right). \tag{24}$$

Moreover (see the proof of corollary 2) we know that the function $\tilde{\mathcal{F}}$ is given by formula (21) with the coefficients (24), the series converges in $H^s(\Omega,E)$ to $\tilde{\mathcal{F}}$ and hence in $H^s(\Omega,D)$ to $\tilde{F}^- - u$, i.e. we have:

$$\lim_{N\to\infty} \Big\| v - M_{\partial D}(\oplus_{j=0}^{m-1} v_j) - T_D A v -$$

$$- \sum_{\nu=1}^{N} \left(-\int\limits_{\partial D} \sum_{j=0}^{m-1} \langle C_j(y) c_\nu(\mathcal{L}(y,\cdot)), v_j\rangle_y ds(y) + \int\limits_D \langle c_\nu(\mathcal{L}(y,\cdot)), f\rangle_y dy \right) b_\nu \Big\|_{H^s(D,E)} = 0.$$

This exactly gives identity (23) after regrouping the summands. □

Remark 2. *Formula (20) means that $v = M_{\partial D}(\oplus_{j=0}^{m-1} v_j) + T_D A v - \tilde{\mathcal{F}}$. As $\tilde{\mathcal{F}}$ and each function b_ν are solutions of the elliptic system A^*A in Ω, the Stiltjes-Vitali theorem implies that the series (22) converges in $C^\infty_{loc}(\Omega,E)$ Therefore, if $Av \in H^p(D,F)$, $s \leq p+m$, then $T_D A v \in H^{p+m}(D,E)$, $M_{\partial D}(\oplus_{j=0}^{m-1} v_j) \in C^\infty_{loc}(D,E)$ and we additionally have: 1) Av_N converges to Av in $H^p_{loc}(D \cup \Gamma, F)$; 2) v_N converges to v in $H^{p+m}_{loc}(D,E)$.*

It is worth emphasizing that in fact we obtain the same type of Carleman kernel as for $f = 0$ (cf. [9, theorem 12.6]). In particular, if A is a Dirac type operator and D is a part of a unit ball in $\mathbb{R}^n$ cut off by smooth hypersurface $\Gamma \not\ni 0$ we easily construct both exact and approximate solutions of the Cauchy problem 1 by using the decomposition for harmonic functions with respect to spherical harmonics (see [9, §13]). For the Cauchy-Riemann operator on the complex plane this formula for exact solution is the well-known formula by Goluzin and Krylov (see, for instance, [6, Theorem 1.1]).

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