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On some Properties of Weighted Hilbert Spaces

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We describe the weighted Hilbert spaces $L_{2,w}(\Omega)$ with positive weight functions $w(x)$ which are summable on every bounded interval. We give sufficient condition for $L_{2,w_1}(\Omega)$ space to be extension of $L_{2,w_2}(\Omega)$ space. We also describe how to use given result in statistical probability density estimation.

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Introduction

Let $L_{2,w}(\Omega)$, where $w(x)$ is measurable positive function, $\Omega \subseteq \mathbb{R}$ is measurable subset, be a space of real functions $f : \Omega \to \mathbb{R}$ for which the integral

$$\int_\Omega f^2(x)w(x)dx$$

is finite. The measure and the integral are comprehended in Lebesgue sense. Given space is often described (e.g. [1, 2]) as a case of $L_2(\Omega, \Sigma, \mu)$ space, where Σ is a σ-algebra of measurable subsets of Ω, μ is a measure on Σ, which is defined by:

$$\mu(X) = \int_X w(x)dx. \quad (1)$$

Particularly, the space $L_{2,w}(\Omega)$ is Euclidean space with scalar product

$$(f, g)_w = \int_\Omega f(x)g(x)w(x)dx,$$

which induces the norm

$$\|f\|_{2,w} = \sqrt{(f, f)_w}.$$

It is also known that if the measure (1) has countable basis then the space $L_{2,w}(\Omega)$ is separable.

If $w(x) \equiv 1$ then the space $L_{2,w}(\Omega)$ is denoted as $L_2(\Omega)$, and if $\Omega = \mathbb{R}$ then as $L_{2,w}$.

In [3, 7.1.3] they consider a weighted space $L_p(R_n, \rho(x))$, where $p \in (0; +\infty)$, $R_n = \mathbb{R}^n$ is n-dimensional arithmetic space, and $\rho(x) > 0$ is Borel measurable function on R_n. The paper [4] gives conditions on weight function $w(x)$, which makes wavelet spline system be a conditional

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or unconditional basis in $L_{p,w}(\mathbb{R})$ space, $p \in [1; +\infty)$. In the paper [5] they are studying similar problem for Haar wavelet system.

The $L_{2,w}(\Omega)$ space finds its application in the problem of statistical estimating of probability density function $f_\xi(x)$ of continuous random variate ξ. Indeed, in a case of completeness and separability of the space $L_{2,w}(\Omega)$ there is countable complete orthonormal system $\{\varphi_j(x)\}_{j=0}^{\infty}$, i.e. the system which for all function $f \in L_{2,w}(\Omega)$ satisfies the limiting relation

$$\lim_{l\to\infty} \left\| \sum_{j=0}^{l} (f, \varphi_j)_w \varphi_j - f \right\|_{2,w} = 0.$$

Thus, if $f_\xi \in L_{2,w}(\Omega)$ then its *projective estimate* $f_l(x)$ defined by

$$f_l(x) = \sum_{j=0}^{l} \alpha_j \varphi_j(x) = \sum_{j=0}^{l} (f_\xi, \varphi_j)_w \varphi_j(x)$$

converges to $f_\xi(x)$ in norm of the space $L_{2,w}(\Omega)$.

The paper [6] shows that for each continuous random variate ξ and for appropriate weight function $w(x)$ there exists the space $L_{2,w}(\mathbb{R})$ including it. In so doing, the choice of the function $w(x)$ is important for convergence speed of projective estimate. In connection with it there is a necessity to investigate the properties of the $L_{2,w}(\Omega)$ spaces in the context of weight function choice.

1. Main results

By virtue of σ-additivity of Lebesgue integral, for each measurable positive function $w(x)$ the measure defined by (1) is also σ-additive. But if the function $w(x)$ is not summable, i.e.

$$\int_\Omega w(x)dx = +\infty,$$

then it is possible the pathological behavior of the measure μ in particular cases.

For instance, let $\Omega = [0; 1]$, Σ is a σ-algebra of measurable subsets of Ω and $w(x) = \dfrac{1}{x}$. Then each segment from Σ containing 0 has infinite measure:

$$\mu[0; a] = \int_0^a w(t)dt = \lim_{x\to+0} \ln t\big|_x^a = +\infty, \quad a > 0.$$

Now we take a sequence $a_n \downarrow 0$. Then

$$\lim_{n\to\infty} \mu[0; a_n] = +\infty.$$

On the other hand,

$$\mu\left(\bigcap_{n=1}^{\infty} [0; a_n]\right) = \mu\{0\} = \int_{\{0\}} w(x)dx = 0.$$

We receive that

$$\mu\left(\bigcap_{n=1}^{\infty} [0; a_n]\right) \neq \lim_{n\to\infty} \mu[0; a_n],$$

i.e. built measure μ is not continuous.

Further we assume that the function $w(x)$ is summable on each bounded interval $X \subset \mathbb{R}$:

$$\int_X w(x)dx < +\infty.$$

Now, it is obvious that the measure μ induced by the function $w(x)$ is σ-finite. Then the theorem about completeness of $L_p(\Omega, \Sigma, \mu)$ spaces with $p \in [1; +\infty)$ and σ-finite measure μ [2, IV, 3.3] leads to completeness of the space $L_{2,w}(\Omega)$.

Besides the measure μ has a countable basis consisted of, for example, elements from σ-ring generated by semiring of half-intervals on real axis with rational endpoints. This leads to separability of the space $L_{2,w}(\Omega)$.

Thus, for each positive function $w(x)$ which is summable on every bounded interval the $L_{2,w}(\Omega)$ space is separable Hilbert space.

Present paper considers relationship between $L_{2,w}(\Omega)$ spaces with common set Ω and different weight functions $w(x)$.

Definition. *The space $L_{2,w_1}(\Omega)$ is called an* ***extension*** *of the space $L_{2,w_2}(\Omega)$ if the strict inclusion*

$$L_{2,w_2}(\Omega) \subset L_{2,w_1}(\Omega)$$

holds.

Let us denote obvious proposition.

Proposition 1. *If the inequality $w_1(x) \leqslant w_2(x)$ holds for all x from Ω, then*

$$L_{2,w_2}(\Omega) \subseteq L_{2,w_1}(\Omega).$$

Particularly, if $w(x) \leqslant 1$ then the space $L_{2,w}(\Omega)$ includes the space $L_2(\Omega)$.

Proof. It follows from $w_1(x) \leqslant w_2(x)$ that for all functions $f : \Omega \to \mathbb{R}$ we have

$$f^2(x)w_1(x) \leqslant f^2(x)w_2(x),$$

and

$$\int_\Omega f^2(x)w_1(x)dx \leqslant \int_\Omega f^2(x)w_2(x)dx.$$

Now, convergence of the integral at the left side follows from convergence of the integral at the right one. Thus, for all function $f : \Omega \to \mathbb{R}$ we have that $f \in L_{2,w_2}(\Omega)$ involves $f \in L_{2,w_1}(\Omega)$, i.e.

$$L_{2,w_2}(\Omega) \subseteq L_{2,w_1}(\Omega).$$

□

Remark 1. *The conclusion of the Proposition 1 remains true even when the inequality $w_1(x) \leqslant w_2(x)$ holds almost everywhere on Ω.*

Proposition 2. *With introduced assumptions on weighted functions $w_1(x)$ and $w_2(x)$ it is true that*

$$L_{2,w_1}(\Omega) \cap L_{2,w_2}(\Omega) = L_{2,w_{\max}}(\Omega),$$

where $w_{\max}(x) = \max\{w_1(x), w_2(x)\}$.

Proof. It is obvious that the function $w_{\max}(x)$ is also summable on each bounded interval. Then the space $L_{2,w_{\max}}(\Omega)$ is defined and separable Hilbert. At the same time $w_{\max}(x) \geqslant w_1(x)$ and $w_{\max}(x) \geqslant w_2(x)$. Then from Proposition 1 we have that

$$L_{2,w_{\max}}(\Omega) \subseteq L_{2,w_1}(\Omega) \quad \text{and} \quad L_{2,w_{\max}}(\Omega) \subseteq L_{2,w_2}(\Omega),$$

i.e.

$$L_{2,w_{\max}}(\Omega) \subseteq L_{2,w_1}(\Omega) \cap L_{2,w_2}(\Omega).$$

To prove inverse inclusion we can take arbitrary function $f(x)$ from the set $L_{2,w_1}(\Omega) \cap L_{2,w_2}(\Omega)$. From definition of the space $L_{2,w}(\Omega)$ we will have:

$$\int_\Omega f^2(x)w_1(x)dx < +\infty \quad \text{and} \quad \int_\Omega f^2(x)w_2(x)dx < +\infty.$$

Let us split the space Ω by two subsets Ω_1 and Ω_2, where

$$\Omega_1 = \{x \in \Omega \mid w_1(x) \geqslant w_2(x)\},$$

$$\Omega_2 = \Omega \setminus \Omega_1 = \{x \in \Omega \mid w_1(x) < w_2(x)\}.$$

Then

$$\int_\Omega f^2(x)w_1(x)dx = \int_{\Omega_1} f^2(x)w_1(x)dx + \int_{\Omega_2} f^2(x)w_1(x)dx.$$

We have got that both of the integrals

$$\int_{\Omega_1} f^2(x)w_1(x)dx \quad \text{and} \quad \int_{\Omega_2} f^2(x)w_1(x)dx$$

exist and are finite.

Similarly, folowing integrals exist and are finite:

$$\int_{\Omega_1} f^2(x)w_2(x)dx \quad \text{and} \quad \int_{\Omega_2} f^2(x)w_2(x)dx.$$

Now we will consider the sum of the integrals $\int_{\Omega_1} f^2(x)w_1(x)dx$ and $\int_{\Omega_2} f^2(x)w_2(x)dx$:

$$\int_{\Omega_1} f^2(x)w_1(x)dx + \int_{\Omega_2} f^2(x)w_2(x)dx =$$

$$= \int_{\Omega_1} f^2(x)w_{\max}(x)dx + \int_{\Omega_2} f^2(x)w_{\max}(x)dx = \int_\Omega f^2(x)w_{\max}(x)dx < +\infty.$$

We have from this that $f \in L_{2,w_{\max}}(\Omega)$. So the inclusion

$$L_{2,w_1}(\Omega) \cap L_{2,w_2}(\Omega) \subseteq L_{2,w_{\max}}(\Omega),$$

is proved and the conclusion of the proposition as well. □

The paper [6] gives necessary condition on weight functions $w_1(x)$ and $w_2(x)$ to spaces $L_{2,w_1}(\Omega)$ and $L_{2,w_2}(\Omega)$ not be equal. We express here a stronger proposition.

Proposition 3. *If $L_{2,w_1}(\Omega) \neq L_{2,w_2}(\Omega)$, then at least one of the inequalities holds:*

$$\operatorname*{ess\,inf}_{x\in\Omega} \frac{w_1(x)}{w_2(x)} = 0 \qquad \text{or} \qquad \operatorname*{ess\,sup}_{x\in\Omega} \frac{w_1(x)}{w_2(x)} = +\infty. \tag{2}$$

Proof. On the contrary we assume that all inequalities (2) do not hold. Then

$$\operatorname*{ess\,inf}_{x\in\Omega}\frac{w_1(x)}{w_2(x)} = m > 0, \qquad \operatorname*{ess\,sup}_{x\in\Omega}\frac{w_1(x)}{w_2(x)} = M < \infty;$$

i.e. almost everywhere on Ω

$$0 < m \leqslant \frac{w_1(x)}{w_2(x)} \leqslant M < +\infty.$$

It follows from the given inequalities that almost everywhere on Ω

$$w_1(x) \leqslant M w_2(x), \qquad w_2(x) \leqslant \frac{1}{m} w_1(x).$$

Then

$$\int_\Omega f^2(x) w_1(x) dx \leqslant \int_\Omega f^2(x) M w_2(x) dx = M \int_\Omega f^2(x) w_2(x) dx; \tag{3}$$

$$\int_\Omega f^2(x) w_2(x) dx \leqslant \int_\Omega f^2(x) \frac{1}{m} w_1(x) dx = \frac{1}{m} \int_\Omega f^2(x) w_1(x) dx. \tag{4}$$

We have now that (3) leads to inclusion $L_{2,w_1}(\Omega) \subseteq L_{2,w_2}(\Omega)$, and (4) leads to $L_{2,w_2}(\Omega) \subseteq L_{2,w_1}(\Omega)$. ☐

It follows from the Proposition 3 that if $L_{2,w_1}(\Omega)$ is an extension for $L_{2,w_2}(\Omega)$, then

$$\operatorname*{ess\,inf}_{x\in\Omega}\frac{w_1(x)}{w_2(x)} = 0.$$

Let we give sufficient condition for $L_{2,w_1}(\Omega)$ to contain elements which are outside of $L_{2,w_2}(\Omega)$.

Theorem 1. *Let $\Omega \subseteq \mathbb{R}$ contains right-side or left-side neighborhood of some point $a \in \mathbb{R}$, $w_1(x)$ and $w_2(x)$ are positive on Ω functions which are summable on every bounded interval and for which at least one of one-sided limits*

$$\lim_{x\to a+0}\frac{w_1(x)}{w_2(x)} \qquad \text{or} \qquad \lim_{x\to a-0}\frac{w_1(x)}{w_2(x)}$$

is equal 0*. Then*

$$L_{2,w_1}(\Omega) \setminus L_{2,w_2}(\Omega) \neq \varnothing.$$

Proposition 1 and Theorem 1 lead to convenient sufficient condition for extension of the space $L_{2,w}(\Omega)$. Let

1) Ω contains right-side or left-side neighborhood of some point $a \in \mathbb{R}$;

2) $w_1(x) \leqslant w_2(x)$ holds almost everywhere on Ω;

3) $\lim_{x\to a+0}\frac{w_1(x)}{w_2(x)} = 0$ or $\lim_{x\to a-0}\frac{w_1(x)}{w_2(x)} = 0$.

Then

$$L_{2,w_2}(\Omega) \subset L_{2,w_1}(\Omega).$$

2. Proof of the Theorem 1

We have to prove some intermediate propositions before we prove the Theorem 1.

Lemma 1. *Let $\Omega = (A; +\infty)$, where $A \in [-\infty; +\infty)$ and $f(x)$ is differentiable positive non-increasing on Ω function which satisfies*

$$\lim_{x\to+\infty} f(x) = 0.$$

Then there exists non-negative on Ω function $g(x)$, for which

$$\int_\Omega g(x)dx = +\infty \qquad \text{and} \qquad \int_\Omega f(x)g(x)dx < +\infty.$$

Proof. We define the function $g(x)$ on Ω in this way:

$$g(x) = -\frac{f'(x)}{f(x)}.$$

Because of $f(x) > 0$ and $f'(x) \leqslant 0$ then $g(x) \geqslant 0$. Further,

$$\int_\Omega g(x)dx = -\int_A^{+\infty} \frac{f'(x)}{f(x)}dx = \ln f(A) - \lim_{x\to+\infty} \ln f(x) = +\infty;$$

$$\int_\Omega f(x)g(x)dx = -\int_A^{+\infty} f'(x)dx = f(A) - \lim_{x\to+\infty} f(x) = f(A) < +\infty.$$

Thus, function $g(x)$ satisfies the conclusion of the lemma. □

Lemma 2. *The conclusion of the lemma 1 remains true if in the condition we change differentiability of the function $f(x)$ by its piecewise constancy on Ω.*

Proof. Let the function $f(x)$ is piecewise constant, positive and does not increase on Ω. Then Ω can be split by points

$$A = x_0 < x_1 < \cdots < x_n < \cdots$$

to intervals

$$(x_0; x_1), (x_1; x_2), \ldots, (x_{n-1}; x_n), \ldots \tag{5}$$

in which the function $f(x)$ is constant:

$$f(x) = y_n, \quad x \in (x_{n-1}; x_n), \quad n = 1, 2, \ldots$$

In this case

$$y_1 > y_2 > \cdots > y_n > \cdots$$

and

$$\lim_{n\to\infty} y_n = 0.$$

We are going to prove that for the function $f(x)$ there exists a majorizing function $f_0(x)$, i.e.

$$f(x) \leqslant f_0(x), \qquad x \in \Omega, \tag{6}$$

which satisfies the condition of the Lemma 1.

We can build the function $f_0(x)$ in the form of 2nd order infinity spline passing through the points (x_1, y_1), (x_2, y_2), ... :

$$f_0(x) = \begin{cases} s_0(x), & x \in (x_0; x_1] \\ s_1(x), & x \in (x_1; x_2] \\ \vdots \\ s_n(x), & x \in (x_n; x_{n+1}] \\ \vdots \end{cases}$$

Each of the functions $s_n(x)$ is a 2nd order polynomial:

$$s_n(x) = a_n x^2 + b_n x + c_n, \qquad x \in (x_n; x_{n+1}].$$

To reach a continuity and smoothness of the function $f_0(x)$ over all set Ω we submit the functions $s_n(x)$ to next conditions:

$$\begin{cases} s_n(x_n) = y_n \\ s_n(x_{n+1}) = y_{n+1} \\ s_n'(x_n) = s_{n-1}'(x_n) \end{cases}, \qquad n = 1, 2, \dots \tag{7}$$

At that for $s_0(x)$ we can take

$$s_0(x) \equiv y_1.$$

We are going to show that the system (7) defines unique 2nd order polynomial $s_n(x)$ for all x_n, x_{n+1}, y_n, y_{n+1} and $s_{n-1}'(x_n) = y_n'$ satisfying the conditions:

$$x_n < x_{n+1}, \quad y_n > y_{n+1}.$$

Indeed, the system (7) leads to system of linear equations with variable coefficients a_n, b_n and c_n:

$$\begin{cases} a_n x_n^2 + b_n x_n + c_n = y_n \\ a_n x_{n+1}^2 + b_n x_{n+1} + c_n = y_{n+1} \\ 2a_n x_n + b_n = y_n' \end{cases}$$

The determinant of basic matrix of this system is

$$\begin{vmatrix} x_n^2 & x_n & 1 \\ x_{n+1}^2 & x_{n+1} & 1 \\ 2x_n & 1 & 0 \end{vmatrix} = (x_2 - x_1)^2 > 0,$$

so the system has a unique solution:

$$a_n = \frac{\Delta_1}{(x_2 - x_1)^2}, \quad b_n = \frac{\Delta_2}{(x_2 - x_1)^2}, \quad c_n = \frac{\Delta_3}{(x_2 - x_1)^2}, \tag{8}$$

where

$$\Delta_1 = \begin{vmatrix} y_n & x_n & 1 \\ y_{n+1} & x_{n+1} & 1 \\ y_n' & 1 & 0 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} x_n^2 & y_n & 1 \\ x_{n+1}^2 & y_{n+1} & 1 \\ 2x_n & y_n' & 0 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} x_n^2 & x_n & y_n \\ x_{n+1}^2 & x_{n+1} & y_{n+1} \\ 2x_n & 1 & y_n' \end{vmatrix}.$$

Further, in order to make the spline $f_0(x)$ satisfy the condition of majority (6) it is necessary and sufficient to satisfy

$$s_n(x) \geqslant y_{n+1}, \quad x \in (x_n; x_{n+1}), \quad n = 0, 1, ...$$

Last condition will hold if $s'_n(x_{n+1}) \leqslant 0$, i.e. $2a_n x_{n+1} + b_n \leqslant 0$.

When we substitute in this inequality the solution (8), we will have

$$y'_n \geqslant \frac{2(y_{n+1} - y_n)}{x_{n+1} - x_n}.$$

In the case

$$y'_n < \frac{2(y_{n+1} - y_n)}{x_{n+1} - x_n}, \tag{9}$$

we will build the function $s_n(x)$ by this way:

$$s_n(x) = \begin{cases} s_n^{(1)}(x) & x \in (x_n; t] \\ s_n^{(2)}(x) & x \in (t; x_{n+1}] \end{cases},$$

where $s_n^{(1)}(x)$ and $s_n^{(2)}(x)$ are 2nd order polynomials

$$s_n^{(1)}(x) = a_n^{(1)} x^2 + b_n^{(1)} x + c_n^{(1)},$$

$$s_n^{(2)}(x) = a_n^{(2)} x^2 + b_n^{(2)} x + c_n^{(2)},$$

which are defined by this conditions:

$$\begin{cases} s_n^{(1)}(x_n) = y_n \\ \dfrac{d}{dx} s_n^{(1)}(x)\Big|_{x=x_n} = y'_n \\ s_n^{(1)}(t) = \dfrac{1}{2}(y_n + y_{n+1}) \\ \dfrac{d}{dx} s_n^{(1)}(x)\Big|_{x=t} = 0 \end{cases}, \quad \begin{cases} s_n^{(2)}(t) = \dfrac{1}{2}(y_n + y_{n+1}) \\ s_n^{(2)}(x_{n+1}) = y_{n+1} \\ \dfrac{d}{dx} s_n^{(2)}(x)\Big|_{x=t} = 0 \end{cases}, \quad x_n < t < x_{n+1} \tag{10}$$

(see Fig. 1).

The second system in (10) is similar to the system (7), therefore it defines unique function $s_n^{(2)}(x)$.

If we substitute the expression for $s_n^{(1)}(x)$ in (10), we will get (after exclusion t):

$$\begin{cases} a_n^{(1)} x_n^2 + b_n^{(1)} + c_n^{(1)} = y_n \\ 2a_n^{(1)} x_n + b_n^{(1)} = y'_n \\ 4a_n^{(1)} c_n^{(1)} - \left(b_n^{(1)}\right)^2 = \frac{1}{2} a_n^{(1)} (y_n + y_{n+1}) \end{cases}.$$

The last system is not linear but we can get unique solution by elementary simplifying:

$$\begin{cases} a_n^{(1)} = \dfrac{(y'_n)^2}{2(y_n - y_{n+1})} \\ b_n^{(1)} = y'_n - 2a_n^{(1)} x_n \\ c_n^{(1)} = y_n + a_n^{(1)} x_n^2 - x_n y'_n \end{cases}.$$

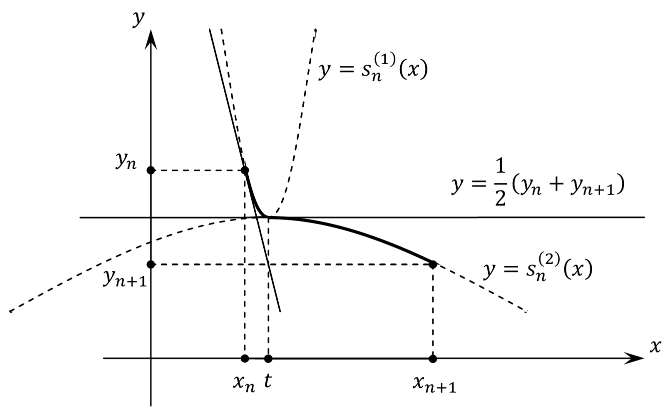

Fig. 1. Building of the function $s_n(x)$ in the case (9)

Now we check whether found solution satisfies to inequality in (10). From the first system we find t:

$$t = -\frac{b_n^{(1)}}{2a_n^{(1)}} = x_n - \frac{y_n - y_{n+1}}{y_n'}.$$

At the same time because of $y_n - y_{n+1} > 0$ and $y_n' < \dfrac{2(y_{n+1} - y_n)}{x_{n+1} - x_n} < 0$, then $t > x_n$. Further,

$$t = x_n - \frac{y_n - y_{n+1}}{y_n'} < x_n - \frac{y_n - y_{n+1}}{\frac{2(y_{n+1} - y_n)}{x_{n+1} - x_n}} = \frac{x_n + x_{n+1}}{2} < x_{n+1}.$$

Thus, part $s_n(x)$ of the spline $f_0(x)$ in the case of (9) is also built. We have that whole spline $f_0(x)$ is smooth on Ω, passes through the points (x_1, y_1), (x_2, y_2), ... and satisfies (6).

We will show that the function $f_0(x)$ satisfies the condition of the Lemma 1. First, $f_0(x)$ is differentiable on Ω. Second, $f_0(x)$ is positive because of $f_0(x) \geqslant f(x) > 0$. Third, according to building we have $f_0'(x) \leqslant 0$, therefore the function $f_0(x)$ does not increase.

Last, for all $x \in (x_n; x_{n+1})$ the following holds: $f_0(x) \leqslant y_n$, and so

$$0 \leqslant \lim_{x \to +\infty} f_0(x) \leqslant \lim_{n \to \infty} y_n = 0;$$

$$\lim_{x \to +\infty} f_0(x) = 0.$$

Then it follows from Lemma 1 that there exists non-negative function $g(x)$, for which

$$\int_\Omega g(x)dx = +\infty \qquad \text{and} \qquad \int_\Omega f_0(x)g(x)dx < +\infty.$$

Finiteness of the first integral and the inequality (6) lead to that the integral

$$\int_\Omega f(x)g(x)dx$$

is finite.

Thus, the function $g(x)$ satisfies the conclusion of the Lemma 2. □

Lemma 3. *The conclusion of the Lemma 1 is true for all positive function $f(x)$, for which*

$$\lim_{x\to+\infty} f(x) = 0.$$

Proof. Let function $f(x)$ satisfies to the condition of the Lemma 3. According to definition of limit of function, for all $\varepsilon > 0$ there exists $M \in \Omega$ for which for all $x > M$ following holds:

$$f(x) < \varepsilon.$$

Now we take a sequence $\varepsilon_n = \dfrac{1}{n}$. Some sequence M_n corresponds to it. Let us to consider a function

$$f_0(x) = \begin{cases} 1, & x \in [M_1; M_2) \\ \frac{1}{2}, & x \in [M_2; M_3) \\ \vdots & \\ \frac{1}{n}, & x \in [M_n; M_{n+1}) \\ \vdots & \end{cases}$$

This function satisfies the condition of the Lemma 2. Therefore, there exists non-negative function $g(x)$, for which

$$\int_\Omega g(x)dx = +\infty \qquad \text{and} \qquad \int_\Omega f_0(x)g(x)dx < +\infty.$$

It is obvious that on Ω the inequality $f(x) \leqslant f_0(x)$ holds. Then the integral

$$\int_\Omega f(x)g(x)dx$$

is finite. □

Lemma 4. *Let $\Omega = (a; b)$ and $f(x)$ is positive on Ω function for which at least one of single-sided limit*

$$\lim_{x\to a+0} f(x) \qquad \textit{or} \qquad \lim_{x\to b-0} f(x)$$

is equal 0. Then there exists non-negative on Ω function $h(x)$, for which

$$\int_\Omega h(x)dx = +\infty, \qquad \int_\Omega f(x)h(x)dx < +\infty.$$

Proof. Let us to consider the case of right-sided limit. We define a variable $y = \dfrac{1}{x-a}$. Then

$$x \to a+0 \quad \text{is equivalent to} \quad y \to +\infty;$$

$$x = b \quad \text{is equivalent to} \quad y = \frac{1}{b-a};$$

$$f(x) = f\left(a + \frac{1}{y}\right),$$

and the function $f\left(a + \dfrac{1}{y}\right)$ (from variable y) defined on $\Omega' = \left(\dfrac{1}{b-a}; +\infty\right)$ satisfies the condition of the Lemma 3. Then there exists the function $g(x)$, for which

$$\int_{\Omega'} g(y)dy = +\infty \qquad \text{and} \qquad \int_{\Omega'} f\left(a + \frac{1}{y}\right) g(y)dy < +\infty.$$

Because of

$$\int_{\Omega'} g(y)dy = \int_{\Omega'} g\left(\frac{1}{x-a}\right)\frac{1}{(x-a)^2}\,dx = +\infty,$$

$$\int_{\Omega'} f\left(a+\frac{1}{y}\right)g(y)dy = \int_{\Omega'} f(x)g\left(\frac{1}{x-a}\right)\frac{1}{(x-a)^2}\,dx < +\infty,$$

we can take for function $h(x)$

$$h(x) = g\left(\frac{1}{x-a}\right)\frac{1}{(x-a)^2}.$$

The case of left-sided limit is considered similar: $y = \dfrac{1}{b-x}$, and

$$h(x) = -g\left(\frac{1}{b-x}\right)\frac{1}{(b-x)^2}.$$

□

Proof of Theorem 1. We are going to prove that in the $L_{2,w_1}(\Omega)$ space there is a function f which does not belong the $L_{2,w_2}(\Omega)$ space, i.e.

$$\int_\Omega f^2(x)w_1(x)dx < +\infty \qquad \text{and} \qquad \int_\Omega f^2(x)w_2(x)dx = +\infty.$$

The function $\dfrac{w_1(x)}{w_2(x)}$ satisfies the condition of the Lemma 4. Then there exists non-negative on Ω function $h(x)$, for which

$$\int_\Omega h(x)dx = +\infty, \qquad \int_\Omega h(x)\frac{w_1(x)}{w_2(x)}dx < +\infty.$$

We define the required function $f(x)$ by this way: $f(x) = \sqrt{\dfrac{h(x)}{w_2(x)}}$.

We get:

$$\int_\Omega f^2(x)w_2(x)dx = \int_\Omega h(x)dx = +\infty;$$

$$\int_\Omega f^2(x)\,w_1(x)\,dx = \int_\Omega h(x)\,\frac{w_1(x)}{w_2(x)}dx < +\infty.$$

□

Conclusion

Present paper describes the properties of weighted functional Hilbert spaces of $L_{2,w}(\Omega)$ kind in the context of building probability density function estimate for continuous random variable ξ. Proposition about convergence of probability density function projective estimate is true in assumption that the probability density belongs to the space $L_{2,w}(\Omega)$ with appropriate weight function $w(x)$. However, the situations when that information is absent can appear in applications. The Theorem 1 of present paper suggests particularly the method of choice required function $w(x)$. For instance, if according to the received values of random variate ξ being investigated we have reasons to assume that for the chosen weight function $w_2(x)$ the equality

$$\|f\|^2_{w_2} = \int_\Omega f^2(x)w_2(x)dx = +\infty,$$

holds, i.e. $f \notin L_{2,w_2}(\Omega)$, then we can try to extend the space $\int_\Omega f^2(x)w_2(x)dx$ to space $\int_\Omega f^2(x)w_1(x)dx$ by taking the function $w_1(x)$ satisfied condition:

$$\lim_{x\to a}\frac{w_1(x)}{w_2(x)}=0,$$

e.g.

$$w_1(x)=\begin{cases}|x-a|^\alpha w_2(x), & x\in(a-\varepsilon;a+\varepsilon)\\ w_2(x), & \text{else}\end{cases},\ \alpha>0,\ \varepsilon\in(0;1],$$

and the point a is chosen from the condition

$$\int_{a-\delta}^{a+\delta} f^2(x)w_2(x)dx=+\infty \quad \text{for all } \delta>0.$$

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О некоторых свойствах весовых гильбертовых пространств

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В работе рассматриваются весовые гильбертовы пространства $L_{2,w}(\Omega)$ при положительных и суммируемых на любом ограниченном интервале весовых функциях $w(x)$. Приводится достаточное условие, при котором пространство $L_{2,w_1}(\Omega)$ является расширением пространства $L_{2,w_2}(\Omega)$. Описывается применение полученного результата при статистическом оценивании функции плотности вероятности случайной величины.

Ключевые слова: пространства интегрируемых функций, гильбертовы пространства, весовые функциональные пространства, сплайны второго порядка, оценивание функции плотности вероятности.