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Construction of Series of Perfect Lattices by Layer Superposition

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We construct a new series of perfect lattices in n dimensions by the layer superposition method of Delaunay-Barnes.

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Introduction

This paper stems out of our discussions at the beginning of the 1990 s of those polytopes in $\mathbb{R}^n$ whose parallel displacements allow one to form a close partition of the space without open overlaps. The problem was motivated by constructions of optimal numerical schemes the first author elaborated at that time. We continued the discussions in 2009 and prepared a joint project to the German Research Society (DFG) devoted to construction of perfect lattices in the space. Unfortunately, an accident with the first author broke off our cooperation in 2012. This paper is written by the second author and it actually summarises the results of our joint work.

Construction of lattices by the layer superposition method goes back at least as far as [5, 6]. It consists in the following. Let ℓ^n be a lattice in $\mathbb{R}^n$ determined by a frame $f^n = \{e_1, \ldots, e_n\}$ of this space. By a Delaunay polytope of ℓ^n is meant any convex polytope with vertices at lattice points, such that there is a solid ball circumscribed about the polytope whose closure does not contain any other points of the lattice except for vertices of the polytope. Pick a Delaunay polytope P^n of ℓ^n. Let $B(P^n)$ be a solid ball described around P^n. Denote by c the centre of $B(P^n)$ and by r its radius. Set $e_{n+1} := c + h$, where h is a vector of $\mathbb{R}^{n+1}$ orthogonal to $e_1, \ldots, e_n$ and satisfying $|h| = \sqrt{1 - r^2}$. The vectors $e_1, \ldots, e_n, e_{n+1}$ constitute a frame f^{n+1} of $\mathbb{R}^{n+1}$. The $(n+1)$-dimensional lattice determined by the frame f^{n+1} is said to be constructed by the layer superposition method from ℓ^n. It is denoted by $\ell(\ell^n, P^n)$. Obviously, this procedure goes through only if $r < 1$.

If the lattice ℓ^n is perfect, then so is $\ell(\ell^n, P^n)$. Even if ℓ^n is not perfect, the lattice $\ell(\ell^n, P^n)$ may be perfect. The layer superposition method is used successfully for constructing perfect lattices.

By a series of lattices is usually meant an infinite number of lattices corresponding to all n beginning with some dimension $n = n_0$. As developed in [2], the layer superposition method fails to lead to series of perfect lattices, for it does not explain how to find a Delaunay polytope P^{n+1} of the lattice $\ell(\ell^n, P^n)$ lying in a ball $B(P^{n+1})$ of radius $r < 1$. The present paper is aimed at finding such a Delaunay polytope, which allows one to construct series of perfect lattices by the layer superposition method.

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1. Background

As usual, $\mathbb{R}^n$ stands for the n-dimensional Euclidean space whose elements are interpreted as n-columns of real numbers. Choose a system $e_1, \ldots, e_n$ of n linearly independent vectors $e_1, \ldots, e_n$ in $\mathbb{R}^n$. It forms a (non-necessarily orthogonal) frame f^n of $\mathbb{R}^n$. We can think of f^n as the $(n \times n)$-matrix whose columns just amount to $e_1, \ldots, e_n$.

The frame f^n determines an n-dimensional lattice $\ell^n = \ell(f^n)$ in $\mathbb{R}^n$ which consists of all points (or vectors) of the form $v = x^1 e_1 + \ldots + x^n e_n$ with $x^1, \ldots, x^n$ being integer numbers. This can be written as $v = f^n x$, where x is the n-column with entries $x^1, \ldots, x^n$. By the very definition, the origin $0 =: e_0$ is a point of each lattice.

The length of v is evaluated by

$$|v|^2 = \sum_{j,k=1}^{n} a_{j,k}\, x^j x^k,$$

where $a_{j,k} = (e_j, e_k)$ is the inner product of e_j and e_k in $\mathbb{R}^n$. This associates the quadratic form

$$q(x) = ((f^n)^T f^n\, x, x) = x^T A x \tag{1.1}$$

to f^n, where $A = (f^n)^T f^n$ is the Gram matrix of the frame vectors. By construction, the quadratic form $q(x)$ is positive definite and it allows one to recover the frame f^n up to translation and similarity. In that sense one says that $q(x)$ determines the same lattice $\ell^n = \ell(q)$.

The convex hull of the vectors $e_0, e_1, \ldots, e_n$ is called the basic simplex S^n of the lattice ℓ^n. It actually defines the lattice as the set of all points $v \in \mathbb{R}^n$ whose barycentric coordinates $\lambda^0, \lambda^1, \ldots, \lambda^n$ with respect to the simplex S^n are integer, i.e. $v = \lambda^0 e_0 + \lambda^1 e_1 + \lambda^n e_n$ where $\lambda^0, \lambda^1, \ldots, \lambda^n$ are integer numbers satisfying $\lambda^0 + \lambda^1 + \ldots + \lambda^n = 1$. In other words, $v = (e_0, f^n)\lambda$, where λ is the $(n+1)$-column with entries $\lambda^0, \lambda^1, \ldots, \lambda^n$.

With the simplex S^n there is associated the quadratic form

$$Q(\lambda) = -l_{0,1}\lambda^0\lambda^1 - l_{0,2}\lambda^0\lambda^2 - \ldots - l_{n-1,n}\lambda^{n-1}\lambda^n,$$

where $l_{j,k} = |e_j - e_k|^2$. The collection of $N = n(n+1)/2$ data $l_{j,k}$ with $0 \leqslant j < k \leqslant n$ determines both S^n and ℓ^n.

It is known [1] that $Q(\lambda) = d^2 - r^2$, where r is the radius of the ball circumscribed around S and d is the distance of v and the centre of the ball. Given any two vectors $v, w \in \mathbb{R}^n$ with barycentric coordinates λ and μ, respectively, the quadrat of the distance of v and w just amounts to $Q(\mu - \lambda)$, see *ibid.*

There are merely a finite number of vectors in the lattice, for which the square of the length is equal to the minimal value m of the form $q(x)$. The lattices which can be obtained from each other by rescaling are said to be equivalent. Hence, one can assume that $m = 1$. Among all vectors of the lattice the vectors of length one have the least length except for the vector $e_0 = 0$. Two lattice vectors of length one are called equivalent if they differ only in the sign.

Definition 1.1. *The quadratic form $q(x)$ and the lattice $\ell(q)$ are said to be perfect if the matrix A is uniquely determined by the lattice vectors for which $q(x)$ is minimal.*

This just amounts to saying that the matrix A of the form is uniquely determined from the system of equations

$$q(x_i) = 1 \tag{1.2}$$

for $i = 1, \ldots, I$, where x_i are lattice vectors of length one.

The lattice ℓ^n is called a limit lattice if on ℓ^n the density of packing of the space by equal disjoint balls with centres at lattice points takes on its local maximum. Each limit lattice is perfect but the inverse is not true, i.e., a perfect lattice is not necessarily limit. For a thorough treatment we refer the reader to [3–7].

By a Delaunay polytope of the lattice is meant any convex polytope with vertices at lattice points which is inscribed in a solid ball whose closure does not contain any lattice points in the interior but the vertices of the polytope on its boundary (such a ball is called "empty ball"). The family of all Delaunay polytopes forms a normal partitioning (Delaunay partitioning) of the space $\mathbb{R}^n$, i.e., a partitioning with the property that any two Delaunay polytopes of the family meet each other at most at a common face of some dimension.

2. General description of the method

Construction of lattices by the layer superposition method goes back at least as far as [5, 6]. We follow [2] in presenting it. Let $B(P^n)$ be a solid ball described around a Delaunay polytope P^n of the lattice $\ell(f^n)$. Write c for the centre of $B(P^n)$ and r for its radius. Set $e_{n+1} := c + h$, where h is a vector of $\mathbb{R}^{n+1}$ orthogonal to $e_1, \ldots, e_n$ and satisfying $|h| = \sqrt{1-r^2}$. It is clear that the vectors $e_1, \ldots, e_n, e_{n+1}$ constitute a frame f^{n+1} of $\mathbb{R}^{n+1}$. The $(n+1)$-dimensional lattice $\ell^{n+1} = \ell(f^{n+1})$ determined by the frame f^{n+1} is said to be constructed by the layer superposition method from $\ell(f^n)$. This lattice is also denoted by $\ell(\ell^n, P^n)$. Obviously, this procedure goes through only if $r < 1$.

If the lattice $\ell(f^n)$ is perfect, then so is $\ell(\ell^n, P^n)$. If $\ell(f^n)$ is a limit lattice, then $\ell(\ell^n, P^n)$ might be a limit lattice as well as it may fail to be limit. Even if $\ell(f^n)$ is not perfect, the lattice $\ell(\ell^n, P^n)$ may still prove to be perfect and even limit. The layer superposition method has been used successfully for constructing perfect lattices.

By a series of lattices is usually meant an infinite number of lattices corresponding to all n beginning with some dimension. Sometimes the very construction yields merely one series of perfect lattices. As described above, the layer superposition method fails to lead to series of perfect lattices, for it does not explain how to find a Delaunay polytope P^{n+1} of the lattice $\ell(\ell^n, P^n)$ lying in a ball $B(P^{n+1})$ of radius $r < 1$. The paper is aimed at finding such a Delaunay polytope, which allows one to readily construct series of perfect lattices by the layer superposition method.

The convex hull of the point $e_{n+1} = c + h$ and the vertices of the polytope P^n is an $(n+1)$-dimensional pyramid C^{n+1} whose base is P^n. The presence of the ball $B(P^n)$ gives an evidence to the existence of a ball $B(C^{n+1})$. It remains to show that $B(C^{n+1})$ is "empty."

By construction, the lattice $\ell(\ell^n, P^n)$ is constituted by the n-dimensional layers $\ell(f^n) + i\, e_{n+1}$, for $i = 0, \pm 1, \ldots$, which lie in n-dimensional planes Π_i, and so we get

$$\ell(\ell^n, P^n) = \bigcup_{i=-\infty}^{\infty} (\ell(f^n) + i\, e_{n+1}) .$$

The distance between the planes Π_i and Π_{i+1} is equal to $|h| = \sqrt{1-r^2}$. The lattice $\ell(f^n)$ just amounts to the layer $\ell(f^n) + 0\, e_{n+1}$. The centre of the ball $B(C^{n+1})$ belongs to the line $l(t) = c + t\, h$ which satifies $l(0) = c$, $l(1) = e_{n+1}$, $l(i) \in \Pi_i$ and $l(t)$ is orthogonal to Π_i for all t and $i = 0, \pm 1, \ldots$. The plane Π_1 is tangent to $B(C^{n+1})$ at the point e_{n+1}, hence the ball $B(C^{n+1})$ has no common points with layers $\ell(f^n) + i\, e_{n+1}$, for $i \geqslant 2$, and it has the only common point e_{n+1} with the layer $\ell(f^n) + e_{n+1}$ lying on the boundary surface of $B(C^{n+1})$. The vertices of the polytope P^n lie in the layer $\ell(f^n) + 0\, e_{n+1}$ and on the boundary surface of $B(C^{n+1})$. In the interior of $B(C^{n+1})$ there is no points of the layer $\ell(f^n) + 0\, e_{n+1} \subset \Pi_0$, for the intersection $B(C^{n+1}) \cap \Pi_0 = B(P^n)$ is empty. We have thus proved

Lemma 2.1. *If there is a point of the lattice $\ell(\ell^n, P^n)$ in the interior of the ball $B(C^{n+1})$, then this point belongs to a layer $\ell(f^n) + i\, e_{n+1}$ with $i \leqslant -1$.*

Let R be the radius of the ball $B(C^{n+1})$. It is easy to show that R is related to r by

$$R^2 = \frac{1}{4}\frac{1}{1-r^2}.$$

This equality gives rise to the recurrence formula

$$r_0 = r, \quad r_i^2 = \frac{1}{4}\frac{1}{1-r_{i-1}^2}$$

for $i = 1, 2, \ldots$. The sequence $\{r_i\}$ satisfies

$$\begin{array}{lll} r_i^2 < r_{i+1}^2 < 0,5 \quad \text{and} & \lim\limits_{i\to\infty} r_i^2 = 0,5, & \text{if} \quad r^2 < 0,5, \\ r_i^2 = 0,5 \quad \text{for all} & i = 1, 2, \ldots, & \text{if} \quad r^2 = 0,5, \end{array} \tag{2.1}$$

and $\{r_i\}$ is not defined for i large enough, if $r^2 > 0,5$.

Suppose that the Delaunay polytope P^n satisfies not only the condition $r < 1$ but also the stronger condition $r^2 < 0,5$. Then from (2.1) it follows that $R^2 < 0,5$ and $\sqrt{1-r^2} > R$. The diameter $2R$ of the ball $B(C^{n+1})$ is less than the distance $2\,|h|$ between the planes Π_1 and Π_{-1}. The ball $B(C^{n+1})$ is tangent to the plane Π_1, hence it has no common points with the planes Π_i, for $i \leqslant -1$. By Lemma 2.1, $B(C^{n+1})$ is "empty", i.e. the interior of the ball contains no points of the lattice $\ell(\ell^n, P^n)$. It follows that the pyramid C^{n+1} is a Delaunay polytope P^{n+1} of the lattice $\ell^{n+1} := \ell(\ell^n, P^n)$ satisfying not only the condition $r < 1$ but also $r^2 < 0,5$. The layer superposition method can therefore be applied infinitely many times. As a result we construct an infinite series of lattices ℓ^{n+m} and their Delaunay polytopes P^{n+m}, for $m = 1, 2, \ldots$.

Let P^n satisfy $r^2 = 0,5$. Then (2.1) yields $R = |h|$. The ball $B(C^{n+1})$ is tangent to the plane Π_{-1} at the point $l(-1)$ and it has no common points with the planes Π_i for $i \leqslant -2$, i.e., $B(C^{n+1})$ is "empty." If $l(-1) \notin \ell(f^n) - e_{n+1}$, then the pyramid C^{n+1} is a Delaunay polytope P^{n+1} of the lattice ℓ^{n+1} which satisfies $r^2 = 0,5$. In the opposite case C^{n+1} is a part of the Delaunay polytope P^{n+1}, the vertices of P^{n+1} being those of the pyramid C^{n+1} and the point $l(-1)$. Moreover, the balls $B(P^{n+1})$ and $B(C^{n+1})$ coincide, their common centre c, as the middle of the segment connecting two lattice points e_{n+1} and $l(-1)$, is a symmetry centre of the lattice ℓ^{n+1}. Hence, c is a symmetry centre of the Delaunay polytopes P^{n+1} and P^n. The length of Delaunay diagonals of these polytopes through the centre c satisfies $l^2 = 2$. It is easy to show that these polytopes are actually cross-polytopes. Recall that by a cross-polytope in $\mathbb{R}^n$ is meant the convex hull of n segments with common middle point which do not belong to any hyperplane. The layer superposition method can be recurrently applied infinitely many times to yield a series of lattices and their Delaunay polytopes, the polytopes being either all pyramids or all cross-polytopes.

If P^n satisfied $0,5 < r^2 < 1$, then $R > |h|$. The ball $B(C^{n+1})$ intersects the plane Π_{-1} and there may be a point of the lattice ℓ^{n+1} in the interior of $B(C^{n+1})$. Then the pyramid C^{n+1} fails to be a Delaunay polytope or a part of a Delaunay polytope of the lattice ℓ^{n+1}, in which case no series of lattices ℓ^{n+1} and their Delaunay polytopes P^{n+1} may occur. Even if one succeeds in constructing a series of lattices, the series proves to be finite, for by (2.1) the radius of the ball $B(P^{n+m})$ becomes larger than 1 in a finite number m of steps, and so the method no longer works. In particular, for $r^2 > 0,75$ no series of lattices exists, for it consists of one lattice. We thus arrive at the following result.

Theorem 2.1. *Assume that the radius r of the ball circumscribed around the Delaunay polytope P^n of the lattice $\ell^n = \ell(f^n)$ satisfies $r^2 \leqslant 0,5$. Then on recurrently applying the layer superposition method one is led to an infinite series of lattices $\ell^{n+m} = \ell(\ell^{n+m-1}, P^{n+m-1})$ and their Delaunay polytopes P^{n+m}. If P^n is a cross-polytope, so are all of P^{n+m}. Otherwise every P^{n+m} is a pyramid C^{n+m} over P^n for all $m = 1, 2, \ldots$. For $r^2 > 0,5$ the method does not work to produce infinite series of lattices.*

Corollary 2.2. *If the lattice ℓ^n of Theorem 2.1 is perfect, then the corresponding series of lattices is also perfect.*

3. Pyramid structure

The structure of the pyramid C^{n+m} is simple. All its edges emanating from the vertices $e_{n+1}, e_{n+2}, \ldots, e_{n+m}$ have length 1. Moreover, C^{n+m} is a right pyramid over the face P^n which is its base. Let $(f_0^{n+m-1}, f_1^{n+m-1}, \ldots, f_{n+m-2}^{n+m-1})$ be the Euler-Poincaré data of the pyramid C^{n+m-1}. Then C^{n+m} has the Euler-Poincaré data $(f_0^{n+m}, f_1^{n+m}, \ldots, f_{n+m-1}^{n+m})$, where

$$\begin{array}{rcl} f_0^{n+m} &=& f_0^{n+m-1} + 1, \\ f_j^{n+m} &=& f_j^{n+m-1} + f_{j-1}^{n+m-1}, \end{array}$$

for $j = 1, \ldots, n+m-2$, and $f_{n+m-1}^{n+m} = 1 + f_{n+m-2}^{n+m-1}$.

If the Delaunay polytope P^n of the lattice ℓ^n is basic or of double, tripled, etc. volume, then each pyramid C^{n+m} for $m = 1, 2, \ldots$ has the same property with respect to the corresponding lattice ℓ^{n+m}.

The quadratic forms $q(x)$ of (1.1) which correspond to the lattice series of Theorem 2.1 are more conveniently constructed in the frame whose elements are the edges of the pyramid C^{n+m} emanating from the vertex e_{n+1}. Such a frame $f'^{n+m} = \{e'_1, \ldots, e'_{n+m}\}$ demonstrates rather strikingly common properties of the lattices of the same series. However, it exists provided that the Delaunay polytope P^n of ℓ^n is basic.

We now consider this case in more detail. Let the vertices $v_1, \ldots, v_{n+1}$ of the basic Delaunay polytope P^n are end points of the vectors $e'_1, \ldots, e'_{n+1}$, respectively. The end points of vectors $e'_{n+2}, \ldots, e'_{n+m}$ are the vertices $e_{n+2}, \ldots, e_{n+m}$ of the pyramid C^{n+m}. The matrix of the quadratic form $q(x)$ with respect to the frame f'^{n+m} has the entries

$$\begin{array}{rcll} a_{j,j} &=& 1, & \text{if } \ j = 1, \ldots, n+i, \\ a_{j,k} &=& \dfrac{1}{2}\,(2 - |v_j - v_k|^2), & \text{if } \ j,k \leqslant n+1,\ j \neq k, \\ a_{j,k} &=& 0,5 & \text{otherwise.} \end{array}$$

It is convenient to write such a form as difference $q^{n+m}(x) = q_0^{n+m}(x) - r^{n+1}(x)$, where the form

$$q_0^{n+m}(x) = (x^1)^2 + \ldots + (x^{n+m})^2 + x^1x^2 + \ldots + x^{n+m-1}x^{n+m}$$

determines the first perfect lattice and the matrix of the quadratic form $r^{n+1}(x)$ has zero diagonal entries and entries $(|v_j - v_k|^2 - 1)/2$ for paarweise different $j, k \leqslant n+1$. The form $r^{n+1}(x)$ contains information on the length of edges and diagonals of the polytope P^n connecting the vertices $v_1, \ldots, v_{n+1}$ (and so r^{n+1} describes the geometry of P^n).

If P^n fails to be a basic Delaunay polytope of the lattice ℓ^n, then the frame f'^{n+m} determines merely a sublattice of the lattice ℓ^{n+m}. Any frame f''^{n+m} determining the lattice ℓ^{n+m} and maximally close to the frame f'^{n+m} contains at least one vector emanating from e_{n+1} to the layer ℓ^{n+m}, which is not any edge of the pyramid C^{n+m}.

We now evaluate the number of vectors of the lattice ℓ^{n+1} which have length one. We restrict our attention to those vectors of length one which emanate from the point $e_{n+1} \in \ell(f^n) + e_{n+1}$. The only vectors of length one of the lattice ℓ^{n+1} emanating from e_{n+1} and lying in the layer $\ell(f^n) + e_{n+1}$ are those of the lattice ℓ^n. Each vector of length one emanating from e_{n+1} and ending in the layer $\ell(f^n) + 0\, e_{n+1}$ amounts to an edge of the pyramid C^{n+1} emanating from the vertex e_{n+1}. There are no vectors of length one which end in a layer $\ell(f^n) + i\, e_{n+1}$ with $i = -1, -2, \ldots$, provided that $r^2 \leqslant 0,5$, for the distance of $e_{n+1} \in \ell(f^n) + e_{n+1}$ to the plane Π_i

is equal to $(1+|i|)|h|$, what is greater than 1. The same is true concerning the vectors of length one which end in a layer $\ell(f^n) + i\, e_{n+1}$ with $i = 2, 3, \ldots$, for they differ only by sign from those already considered. It follows that the number of vectors of length one of the lattice ℓ^{n+1} just amounts to $N(\ell^n) + f_0$, where $N(\ell^n)$ is the number of vectors of length one of the lattice ℓ^n and f_0 the number of vertices of the polytope P^n. Hence the number of vectors of length one of the lattice ℓ^{n+m} is equal to $N(\ell^n) + f_0\, m + (m-1)m/2$. Note that $f_0 \geqslant n+1$. We have thus proved

Lemma 3.1. *The lattice* ℓ^{n+m} *has*

$$N(\ell^{n+m}) = N(\ell^n) - \frac{n(n+1)}{2} + (f_0 - n - 1)\, m + \frac{(n+m)(n+m+1)}{2}$$

vectors of length 1.

It is worth pointing out that for constructing a series of pyramids C^{n+m} for $m = 1, 2, \ldots$ it suffices to have a Delaunay polytope P^n of the lattice ℓ^n with the property $r^2 \leqslant 0,5$. It gives rise to a pyramid series which can be constructed independently of the lattice. In this sense the pyramid series is of independent interest. Moreover, P^n may appear to be a Delaunay polytope simultaneously for two lattices ℓ'^n and ℓ''^n. For one of them P^n may be basic and for the other it may be, e.g., of double volume. Then the pyramid series C^{n+m} will be a series of Delaunay polytopes for two lattice series ℓ'^{n+m} and ℓ''^{n+m}. For the first series ℓ'^{n+m} the pyramids will be basic Delaunay polytopes, for the second series ℓ''^{n+m} the pyramids will be Delaunay polytopes of double volume. This shows that to a series of pyramids there may correspond many series of lattices.

When constructing series of perfect lattices and their Delaunay pyramids, the question on the first pyramid and first lattice series is not treated in a unified way. If the generating Delaunay polytope P^n of a perfect lattice ℓ^n has no vertex v connected with other its vertices by edges of length one, then the first pyramid of the series is C^{n+1} and the first lattice of the series is $\ell^{n+1} = \ell(\ell^n, P^n)$. If P^n has several vertices $v_1, \ldots, v_k$, each of them being connected with all other vertices of P^n by edges of length one, then the series generating Delaunay polytope is the face P^{n-k} of P^n which lies opposite to the vertices $v_1, \ldots, v_k$. The first pyramid of the series is the face C^{n-k+1} containing the face P^{n-k}.

If P^n is the basic Delaunay polytope of a perfect lattice ℓ^n, then ℓ^n can be constructed from some lattice ℓ^{n-k} and its Delaunay polytope P^{n-k} by the layer superposition method over k steps. Under this approach the lattice ℓ^{n-k} need not be perfect. It suffices that in a finite number $j < k$ of steps the lattice ℓ^{n-k+j} would become perfect. Then it will be the first lattice of a series of perfect lattices $\ell^{n-k+j+m}$ with $m = 1, 2, \ldots$. If P^n fails to be a basic Delaunay polytope of a perfect series ℓ^n, the general approach to the choice of the first lattice of the series still remains the same. However, one should take into account that lattices of dimension less than n may fail to exist.

4. Examples

In [2] one described all Delaunay polytopes of perfect lattices in the space $\mathbb{R}^n$ with $n \leqslant 6$. We first consider the case $n = 2$ in detail.

In $\mathbb{R}^2$ there is only the first perfect lattice ℓ_0^2, which is usually given by the first quadratic form q_0^2. All Delaunay polytopes of this lattice are equilateral triangles, one of these (the convex span of frame vectors) is denoted by P_1^2. The edges of the triangle are of length one. The quadrat of the radius of the described circle just amounts to $r^2 = 1/3$. The lattice has three vectors of length one.

Since $r^2 < 0,5$, the triangle P_1^2 generates a series of Delaunay polytopes of perfect lattices P_1^{2+m} for $m = 1, 2, \ldots$, each P^{2+m} being a regular simplex with edges of length one. Since one-dimensional lattices are hardly of interest, P_1^2 can be thought of as the first polytope of the series.

The lattice ℓ_0^2 and its Delaunay polytope P_1^2 generate a series of perfect lattices ℓ_0^n determined by the form q_0^n. To our best knowledge this quadratic form was first studied by G. F. Voronoy and called the first perfect form. The first perfect lattice ℓ^n is known to have $n(n+1)/2$ vectors of length one, as is also seen from Lemma 3.1.

The regular simplex with edges of length one in $\mathbb{R}^7$ is a Delaunay polytope of double volume (denoted S^7) of a perfect lattice ℓ'^7 in $\mathbb{R}^7$. This lattice is given by the frame

$$f^7 = \Big\{e_1, \ldots, e_6, \frac{1}{2}\,(-e_1 - e_2 - e_3 + e_4 + e_5 + e_6 + e_7)\Big\},$$

where $e_1, \ldots, e_7$ are edges of the simplex S^7 emanating from one vertex. To the frame f^7 there corresponds the quadratic form $q^7(x) = q_0^7(x) - x^1x^7 - x^2x^7 - x^3x^7$. The number of vectors of length one in the lattice is 63. The lattice ℓ'^7 and its Delaunay polytope S^7 generate a series of perfect lattices which are given by the form

$$q^n(x) = q_0^n(x) - x^1x^7 - x^2x^7 - x^3x^7 - \frac{1}{2}\,(x^7x^8 + \ldots + x^7x^n)$$

for $n = 8, 9, \ldots$. The Delaunay polytopes of double volume of lattices of this series are precisely regular simplices of the corresponding dimension. By Lemma 3.1, the number of vectors of length one of an n-dimensional lattice is $35 + n(n+1)/2$.

A Delaunay partitioning of the lattice ℓ'^8 contains regular simplices of tripled volume with edges of length one. Denote by S^8 one of them. The lattice can be determined by the frame $f^8 = \{e_1, \ldots, e_7, e'_8\}$, where $e'_8 = \frac{1}{3}\,(-2e_1 - 2e_2 + e_3 + \ldots + e_8)$ and $e_1, \ldots, e_8$ are the edges of the simplex S^8 emanating from one vertex. To the frame f^8 there corresponds the quadratic form $q^8(x) = q_0^8(x) - x^1x^8 - x^2x^8$. The lattice contains 120 vectors of length one. Together with its Delaunay polytope S^8 the lattice determines a series of perfect lattices. The regular simplex S^n is the Delaunay polytope of tripled volume of the n-dimensional lattice of this series, and the simplex S^8 is the 8-dimensional face of S^n. The frame of the n-dimensional lattice is $f^n = \{e_1, \ldots, e_7, e'_8, e_9, \ldots, e_n\}$, where $e_9, \ldots, e_n$ are the edges of the simplex S^n emanating from a vertex of S^8. To this frame there corresponds the form

$$q^n(x) = q_0^n(x) - x^1x^8 - x^2x^8 - \frac{1}{3}\,(x^8x^9 + \ldots + x^8x^n)$$

for $n = 9, 10, \ldots$. By Lemma 3.1 the number of vectors of length one of the n-dimensional lattice is $84 + n(n+1)/2$.

On the lattices ℓ'^7 and ℓ'^8 the density of packing the spaces $\mathbb{R}^7$ and $\mathbb{R}^8$, respectively, by equal disjoint balls with centres at lattice points takes on its global maximum.

When the dimension of the space increases, one derives always new series of perfect lattices whose Delaunay polytopes are regular simplices. Along with finding new perfect forms, the construction of series of perfect lattices by the layer superposition method allows one to partially order already existing perfect quadratic forms and reduce notation.

In the space $\mathbb{R}^3$ there is only the first perfect lattice ℓ_0^3 whose partitioning consists of regular simplices P_1^3 and octahedra P_2^3. The simplex P_1^3 belongs to the series of polytopes considered above. The octahedron P_2^3 gives rise to a series of n-cross-polytopes with edges of length one which are Delaunay polytopes of perfect lattices, where $n \geqslant 3$.

The lattice ℓ_0^3 and its Delaunay polytope P_2^3 give rise to a series of perfect lattices ℓ_1^n given by the second perfect form $q_1^n(x) = q_0^n(x) - x^1x^2$, where $n \geqslant 4$. The lattice ℓ_1^n has $n^2 - n$ vectors of length one. The n-cross-polytope is a basic Delaunay polytope of this lattice.

The cross-polytope with edges of length one is a Delaunay polytope of double volume of the perfect lattice ℓ'^8. The frame f^8 of this lattice and the corresponding quadratic form are adduced above. The convex hull of the frame vectors $e_1, \dots, e_7$ constitutes the 7-dimensional face of the partitioning which is a regular simplex. Along this face the simplex S^8 borders on an 8-cross-polytope which we denote by P^8. The lattice ℓ'^8 and its Delaunay polytope P^8 generate a series of perfect lattices. The n-dimensional lattice of this series has an n-cross-polytope P^n as a Delaunay polytope of double volume and it is given by the frame $\{e_1, \dots, e_7, e'_8, e_9, \dots, e_n\}$. The vectors $e_1, \dots, e_7, e_9, \dots, e_n$ form an $(n-1)$-dimensional face of an n-cross-polytope P^n which is a regular simplex. Only the vector e'_8 of the frame does not belong to the cross-polytope P^n. The distance of the point e'_8 to each of the vertices $e_9, \dots, e_n$ of the cross-polytope P^n just amounts to 3/2. Hence, to the frame of the lattice of this series there corresponds the quadratic form

$$q_0^n(x) - x^1 x^8 - x^2 x^8 - \frac{1}{2}\, x^8 (x^9 + \ldots + x^n),$$

where $n \geqslant 9$. The number of lattice vectors of length one is equal to $n^2 - n + 64$, as is easy to see.

In the space $\mathbb{R}^4$ there are two perfect lattices ℓ_0^4 and ℓ_1^4. The Delaunay polytopes of the lattices ℓ_0^n and ℓ_1^n are well known. All Delaunay polytopes of these lattices, except for the regular simplex P_1^n of the lattice ℓ_0^n and the n-cross-polytope of the lattice ℓ_1^n, have circumscribed ball of radius greater than $0,5$, and so they give no rise to any countable series of perfect lattices. Therefore, the first and second perfect lattices are not considered in the sequel.

In addition to the first and second perfect lattices in $\mathbb{R}^5$ there is also the third perfect lattice ℓ_2^5. The partitioning of ℓ_2^5 contains Delaunay polytopes of three types whose representatives are U^5, S^5 and M^5 of [2]. The first polytope has circumscribed ball of radius exceeding $0,5$. The second one is a basic simplex of the lattice ($r^2 = 11/24$) and has no vertex connected with other vertices of the simplex by edges of length one. It gives rise to a series of regular simplices over S^5, which we denote by S_1^n for $n \geqslant 6$. The lattice ℓ_2^5 and simplex S^5 generate a series of perfect lattices ℓ_3^n, for which the simplices S_1^n are basic Delaunay polytopes. The vertex of the simplex S_1^6 facing S^5 is connected with other vertices of the simplex by edges of length one. On taking these edges as frame we get a quadratic form $q_3^6(x) = q_0^6(x) - 1/2\,(x^1x^2 + x^3x^4 + x^5x^6)$ which gives the first lattice of this series. Then the entire series of lattices ℓ_3^n is determined by the form

$$q_3^n(x) = q_0^n(x) - \frac{1}{2}\,(x^1x^2 + x^3x^4 + x^5x^6),$$

where $n \geqslant 6$. Several of the first forms and lattices given by them are known, however, they do not bear any common designation let alone name. The number of vectors of length one in the lattice ℓ_2^5 amounts to 15, and the simplex S^5 has 6 vertices. It follows that the number of vectors of length one in the lattice ℓ_3^n is equal to $n(n+1)/2$.

The polytope M^5 is a regular pyramid over a polytope $V_{2,2}^4$ which is a repartitioning body, see [2]. The radius of circumscribed ball of M^5 is $r = \sqrt{0,5}$. Therefore, $V_{2,2}^4$ gives rise to a series of pyramids M^n, $n \geqslant 5$, which are Delaunay polytopes of perfect lattices. The polytope $V_{2,2}^4$ is a basic Delaunay polytope of a lattice ℓ^4 which is not perfect. However, on applying the layer superposition method to the lattice ℓ^4 and its Delaunay polytope $V_{2,2}^4$ one obtains a perfect lattice ℓ_2^5. In this way ℓ^4 and $V_{2,2}^4$ generate a series of perfect lattices ℓ_2^n and a series of Delaunay polytopes M^n of these lattices, where $n \geqslant 5$. The lattice ℓ_2^5 is the first lattice of the series. The body $V_{2,2}^4$ has 6 vertices and the number of vectors of length one in the lattice ℓ^4 is equal to 9. Hence, the number of vectors of length one in the lattice ℓ_2^n just amounts to $-1 + 1(n-4) + n(n+1)/2$ or $n(n+3)/2 - 5$, which is due to Lemma 3.1. Any 5 of the 6 edges of the polytope M^5 emanating from the vertex facing $V_{2,2}^4$ can be taken as a frame of ℓ_2^5. To this frame there corresponds the quadratic form $q_2^5(x) = q_0^5(x) - 1/2\,(x^1x^2 + x^3x^4 + x^3x^5 + x^4x^5)$. Then the series ℓ_2^n is determined by

$$q_2^n(x) = q_0^n(x) - \frac{1}{2}\,(x^1x^2 + x^3x^4 + x^3x^5 + x^4x^5),$$

where $n \geqslant 5$. This series of perfect quadratic forms is known, see [8]. Unfortunately, G. F. Voronoy found and called the third perfect form only the first quadratic form of this series, cf. [9]. It seems to be more logical to think of the whole series as the third perfect form by analogy with the first and second perfect forms.

In addition to finding new perfect forms the construction of series of perfect lattices by the layer superposition method allows one to partially regulate already known quadratic forms, thus reducing the number of designations in use.

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Построение серий совершенных решеток посредством наложения слоев

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Построена новая серия совершенных решеток в n размерностях методом наложения слоев Делоне-Барнса.

Ключевые слова: решетчатая упаковка и покрытие, многогранники и политопы, правильные фигуры, деление пространств.