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On the Limit Distribution of Sums of Real Random Variables

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Considers centered sequence of absolutely continuous random variables with a non-trivial weak limit of the sums $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i$*. We found a general view of the limit distribution. It is shown that the form of the limit distribution depends only on the average mixed moments of the first order, describing the sequence of Rademacher random variables, into which can be decomposed the elements of the given sequence.*

Keywords: dependent random variables, the sums of dependent random variables, limit distribution of sums of random variables, the normality of the limit distribution of sums of random variables.
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Introduction

In this paper we continue the study initiated in [1]. The main purpose is to obtain a general form of the limit distribution sums of centered absolutely continuous random variables with a non-trivial weak limit of the sums $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i$.

1. Preliminary results

Let us consider a sequence of absolutely continuous centered real random variables $\xi = (\xi_t)_{t\in N}$. We assume that the random variables are defined at similar spaces of elementary events $\Omega_t = \Omega, t \in I$ with similar σ-algebras of events $\mathfrak{A}_t = \mathfrak{A}$. As previously, we assume $\Omega_I = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_t \times \ldots$ and $\mathfrak{A}_I = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots \times \mathfrak{A}_t \times \ldots$. The space of values of random variables, we assume the set of real numbers $\xi_t(\omega) \in \mathfrak{X}_t = \mathbf{R}$, $\omega \in \Omega_t$ with given on it Borel σ-algebra $\mathfrak{B}$. By analogy with previous unite into the set Ξ_3 such the sequence of random variables $\xi = (\xi_t)_{t\in N}$, for which exists a continuous random variable η_ξ – weak limit of sequence $S_{1/2}(\xi_{(n)}) = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\xi_t$

$$\eta_\xi \overset{\textbf{weak}}{=\!=} \lim_{n\to\infty} S_{1/2}(\xi_{(n)}).$$

We also denote $\hat{\Xi}_3 \subset \Xi_3$ subset of sequences with averaged links (shortly ***sal***) $\hat{\xi} = (\hat{\xi}_t)_{t\in N}$, the existence and construction of which is shown in Theorem 3.3 [2] and for distribution functions of this sequences the following property satisfied:

$$\mathbf{F}_{\eta_\xi}(y) = \mathbf{F}_{\eta_{\hat{\xi}}}(y), \quad \forall y \in \mathbf{R}.$$

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We study the limit distribution of the random variable η_ξ:

$$\eta_\xi \overset{\textbf{weak}}{=} \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \xi_t, \text{ where } \xi = (\xi_t)_{t\in N} \in \Xi_3.$$

In fact, to achieve this goal we needed to prove the existence of such a sequence $\hat{\gamma} = \hat{\gamma}(\xi) \in \hat{\Xi}_1$, that

$$\mathbf{F}_{\eta_\xi}(x) = \mathbf{F}_{\eta_{\hat{\gamma}}}(x), \ \forall x \in \mathbf{R}$$

and apply Theorem 3 [1] to it. Issues of existence and the construction of sequences of this type for a given distribution of sums of elements of the initial sequences are described in [2]. Therefore, we will be based on results of this work.

2. Main results

Consider the approximation sums $S_{1/2}(\xi_{(n)}) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \xi_t$ of random variables the investigated sequence ξ by sums $S_{1/2}(\pi_{(s,n)}) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \pi_{t,s,n}$ of lattice random variables $\pi_{t,s,n}$. For this purpose we divide the set of real numbers $\mathbf{R}$ as follows:

$$\Delta x_s(k) = \Big(\frac{2k-s-1}{\sqrt{s}}, \frac{2k-s+1}{\sqrt{s}}\Big]$$
$$\text{for } \ k = 1, \dots, s-1, \ \Delta x_s(0) = \ \Big(-\infty, -\frac{s-1}{\sqrt{s}}\Big], \ \Delta x_s(s) = \Big(\frac{s-1}{\sqrt{s}}, \infty\Big).$$

At that we put

$$\mathbf{P}\Big(S_{1/2}(\pi_{(s,n)}) = \frac{2k-s}{\sqrt{s}}\Big) = \mathbf{P}(S_{1/2}(\xi_{(n)}) \in \Delta x_n(k)) = \mathbf{P}_{S_{1/2}}(\Delta x_s(k)), \ k = 0, 1, \dots, s.$$

For $\pi_{(s,n)}$ in [2] shown (see Theorem 2.4) the existence of a finite ***sal*** $\hat{\pi}_{(s,n)}$, having the same distribution of sums. Out there it also shown the existence of a finite and Rademacher type ***sal*** $\hat{\gamma}_{(sn)}$ such that

$$\hat{\pi}_{t,s,n} = \frac{1}{\sqrt{s}} \sum_{i=0}^{s-1} \hat{\gamma}_{t+i\cdot n} \tag{1}$$

For it also is performed the relation:

$$\mathbf{F}_{S_{1/2}(\pi_{(n)})}(x) = \mathbf{F}_{S_{1/2}(\hat{\gamma}_{(sn)})} \ \forall x \in \mathbf{R}, \text{ where } S_{1/2}(\hat{\gamma}_{(sn)}) = \frac{1}{\sqrt{sn}} \sum_{t=1}^{sn} \hat{\gamma}_t.$$

Proceeding to the limit with $s \to \infty$, obtain the ***sal*** $\hat{\gamma}_{(n,N)}$ for which

$$\mathbf{F}_{S_{1/2}(\xi_{(n)})}(x) = \mathbf{F}_{S_{1/2}(\hat{\gamma}_{(nN)})} \ \forall x \in \mathbf{R}, \text{ where } S_{1/2}(\hat{\gamma}_{(nN)}) \overset{\text{weak}}{=} \lim_{s\to\infty} \frac{1}{\sqrt{sn}} \sum_{t=1}^{sn} \hat{\gamma}_t.$$

Note that $\hat{\gamma}_{(n,N)} \in \hat{\Xi}_1$. Further, proceeding to the limit with $n \to \infty$, obtain, on the one parties, ***sal*** $\hat{\gamma}$ for which

$$\mathbf{F}_{\eta_\xi}(x) = \mathbf{F}_{\eta_{\hat{\gamma}}}(x) \ \forall x \in \mathbf{R}, \text{ where } \eta_{\hat{\gamma}} \overset{\text{weak}}{=} \lim_{r\to\infty} \frac{1}{\sqrt{r}} \sum_{t=1}^{r} \hat{\gamma}_t.$$

But on the other parties, random variables of ***sal*** $\hat{\xi}$, constructed by the limit sum of the original sequence ξ, nothing else than

$$\hat{\xi}_t \overset{\text{weak}}{=} \lim_{n\to\infty} \lim_{s\to\infty} \hat{\pi}_{t,s,n} \text{ or a considering (1), we have } \hat{\xi}_t \overset{\text{weak}}{=} \lim_{n\to\infty} \lim_{s\to\infty} \frac{1}{\sqrt{s}} \sum_{i=0}^{s-1} \hat{\gamma}_{t+i\cdot n}.$$

Then

Lemma 1. *Let us assume that a sequence $\xi \in \Xi_3$ is given. Then there exists a sequence $\hat{\gamma} = \hat{\gamma}(\xi) \in \hat{\Xi}_1$ such that*

$$\mathbf{F}_{\eta_\xi}(x) = \mathbf{F}_{\eta_{\hat{\gamma}}}(x),\ \forall x \in \mathbf{R}.$$

Proof. Follows from the foregoing. □

Theorem 1. *Let us assume that a sequence $\xi \in \Xi_3$ is given. Then random variable η*

$$\eta \overset{\textbf{weak}}{=} \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \xi_t,$$

have a density distribution function μ as follows:

$$\mu(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum_{m=0}^{\infty} \ddot{v}_m(\hat{\gamma}) \cdot h_m(x),\ \forall x \in \mathbf{R},$$

where $\ddot{v}_m$ are mixed moments of the sequence $\hat{\gamma}(\xi) \in \hat{\Xi}_1$, constructed in Lemma 1.

Proof. Follows from Theorem 3 [1] and Lemma 1. □

Theorem 2. *Let us assume that a sequence $\xi \in \Xi_3$ is given. Then, if the density distribution function μ of the random variable η*

$$\eta \overset{\textbf{weak}}{=} \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \xi_t,$$

is continuous function, all moments for this random variable are exist and finite.

Proof. Similarly to the proof Corollary 1 [1]. □

Corollary 1. *Let us assume that a sequence $\xi \in \Xi_3$ is given. Then a random variable η*

$$\eta \overset{\textbf{weak}}{=} \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \xi_t$$

have a standard normal distribution then and only then when

$$\lim_{n\to\infty} \ddot{v}_m(\hat{\gamma}_{(n)}) = 0,\ \forall m \geqslant 2.$$

Proof. Obviously follows from the expression for density of limit distribution in Theorem 1. □

Concerning the moments of sequence of random variables $\hat{\xi}$ can be argued, considering relation (16) [1] and method of forming sequence $\hat{\gamma}$, that

- if
$$\ddot{v}_m(\hat{\xi}) = \lim_{n\to\infty} \lim_{s\to\infty} s^m \cdot \ddot{v}_m(\hat{\gamma}_{(n),s}) = 0, \ \forall m \geqslant 2,$$
then the sequence $\hat{\xi}$ has zero average mixed moments for $m = 2, 3 \ldots$, that is particularly true for independent random variables;

- if
$$\ddot{v}_m(\hat{\xi}) = \lim_{n\to\infty} \lim_{s\to\infty} s^m \cdot \ddot{v}_m(\hat{\gamma}_{(n),s}) < \infty, \ \forall m \geqslant 2,$$
then the sequence $\hat{\xi}$ has finite moments;

- and if beginning with some $m = m_0$
$$\ddot{v}_m(\hat{\xi}) = \lim_{n\to\infty} \lim_{s\to\infty} s^m \cdot \ddot{v}_m(\hat{\gamma}_{(n),s}) = \infty,$$
then moments ($m \geqslant m_0$) of sequence $\hat{\xi}$ does not exist.

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О предельном распределении сумм действительных случайных величин

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Рассмотрены центрированные последовательности абсолютно непрерывных случайных величин, имеющие нетривиальный слабый предел сумм $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i$. *Для них найден общий вид предельного распределения. Показано, что вид предельного распределения зависит лишь от усредненных смешанных моментов первого порядка, характеризующих случайные величины последовательности радемахеровских случайных величин, в которую можно разложить элементы рассматриваемой последовательности.*

Ключевые слова: зависимые случайные величины, сумма зависимых случайных величин, предельное распределение сумм случайных величин, нормальность предельного распределения сумм случайных величин.