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Embedding Theorems for Functional Spaces Associated with a Class of Hermitian Forms

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We prove embedding theorems into the scale of Sobolev-Slobodetskii spaces for functional spaces associated with a class of Hermitian forms. More precisely we consider the Hermitian forms constructed with the use of the first order differential matrix operators with injective principal symbol. The results are valid for both coercive and non-coercive forms.

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It is well known that integro-differential Hermitian forms are closely related to the generalized setting of mixed boundary value problems for strongly elliptic equations and systems, as well as the existence and uniqueness theorems for such problems (see, for example, [1–6], and other). We prove embedding theorems into the scale of Sobolev-Slobodetskii spaces for functional spaces associated with one class of Hermitian forms. More precisely, we consider the Hermitian forms constructed with the use of the first order matrix differential operators with the injective principal symbol. The results are valid for both coercive and non-coercive forms.

1. Function spaces

Let D be a bounded domain with Lipschitz boundary in Euclidean space $\mathbb{R}^n$, $n \geqslant 2$, with coordinates $x = (x_1, \dots, x_n)$. For some multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ we will write ∂^α for the partial derivative $\dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$. We consider the complex-valued functions defined over the domain D and its closure $\overline{D}$. We also fix an open (in the induced topology) connected set S with piecewise smooth boundary ∂S on the hypersurface ∂D.

Let $C^s(\overline{D}, S)$, $s \in \mathbb{Z}_+$, be the set of s-times continuously differentiable functions in $\overline{D}$, which are disappearing in some (one-sided) neighborhood $\overline{S}$ in D. We will write $L^q(D)$, $1 \leqslant q \leqslant +\infty$, for standard normed Lebesgue spaces of functions over D. We also write $H^s(D)$, $s \subset \mathbb{N}$, for the Sobolev space of functions whose weak derivatives up to the order s belong to $L^2(D)$. Similarly, $H^s(\partial D)$, $s \subset \mathbb{N}$, stand for the Sobolev space on the boundary of domain ∂D of functions whose weak derivatives up to the order s belong to $L^2(\partial D)$. Let the space $H^s_0(D)$ stand for the closure of space $C^\infty_0(D)$ in $H^s(D)$. For positive non-integer s we denote by $H^s(D)$ the Sobolev-Slobodetskii space, see, for instance, [7]. More precisely, in the case of $0 < s < 1$ the space $H^s(D)$ consists of

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functions $f \in L^2(D)$, $D \subset \mathbb{R}^n$ such that the functional

$$\|f\|_{H^s(D)} = \left(\|f\|^2_{L^2(D)} + \int_D \frac{|f(x) - f(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \right)^{1/2},$$

is finite, that is defining the norm in this space. Similarly, we define the function space $H^s(\partial D)$, $0 < s < 1$, on the space of functions defined on ∂D.

For non-integer $s > 1$ we set $s = [s] + \lambda$, where $[s]$ is integer part of the s. Then $H^s(D)$ consists of elements of $H^{[s]}(D)$ such that $\partial^\alpha f \in H^\lambda(D)$ for all multi-induces $|\alpha| = [s]$. This is a Banach space with the norm

$$\|f\|_{H^s(D)} = \Big(\|f\|^2_{H^{[s]}(D)} + \sum_{|\alpha|=[s]} \|\partial^\alpha f\|^2_{H^\lambda(D)} \Big)^{1/2},$$

and even a Hilbert space with the evident scalar product. The closure of $C^s(\overline{D}, \overline{S})$ in the space $H^s(D)$ is denoted by $H^s(D, S)$. In particular, $H^1(D, \partial D) = H^1_0(D)$.

As usual, for the function space B we denote by B^n the Cartesian product of n copies of this space. If B is a normed space, then we will provide B^n with the norm $\|u\|_{B^n} = \Big(\sum\limits_{j=1}^{n} \|u_j\|^2_B \Big)^{1/2}$. Therefore, B^n is a Hilbert space, if B is the one.

2. The embedding theorems for coercive forms

Let $A(x, \partial)$ be a homogeneous differential the first order matrix operator with injective symbol in a domain $X \subset \mathbb{R}^n$, i.e.

$$A = \sum_{j=1}^{n} A_j(x) \partial_j,$$

here $A_j(x)$ are $(l \times k)$-matrices, whose components are complex-valued $C^\infty(X)$-functions and

$$\text{rang}\Big(\sum_{j=1}^{n} A_j(x)\zeta_j \Big) = k \text{ for all } x \in X, \zeta \in \mathbb{R}^n \setminus \{0\}.$$

Moreover, we require that the following Uniqueness Property in the small on X holds:

$$\text{if } Au = 0 \text{ in a domain } U \subset X \text{ and } u \equiv 0 \text{ on an open subset } V \subset U, \text{ then } u \equiv 0 \text{ in } U. \quad (1)$$

Let $D \Subset X$. We consider the following Hermitian form on the space $[C^1(\overline{D}, \overline{S})]^k$:

$$(u, v)_+ = (Au, Av)_{[L^2(D)]^l} + (a_0 u, v)_{[L^2(D)]^k} + (b_0 u, v)_{[L^2(\partial D \setminus S)]^k},$$

where $a_0(x)$ is a Hermitian non-negative $(k \times k)$-matrix with entries $a_0^{(p,q)} \in L^\infty(D)$ and $(k \times k)$-matrix b_0 is a Hermitian non-negative $(k \times k)$-matrix with measurable bounded components on $\partial D \setminus S$. We denote by $H^+(D)$ the completion of the space $[C^1(\overline{D}, \overline{S})]^k$ with respect to the norm $\| \cdot \|_+$, which is induced by the inner product $(\cdot, \cdot)_+$ (in those cases where such a form is defined).

Let $A^*_j(x)$ be the adjoint matrix for the matrix $A_j(x)$ and

$$A^* = -\sum_{j=1}^{n} \partial_j (A^*_j(x) \cdot),$$

be the formal adjoint for A. Then the second order differential operator

$$A^*A = -\sum_{i,j=1}^{n} \partial_i (A_i^* A_j \partial_j)$$

is strongly elliptic in X, i.e. for all $x \in X$, $w \in \overline{D} \times (\mathbb{C}^k \setminus 0)$, $\zeta \in \overline{D} \times (\mathbb{R}^n \setminus 0)$

$$\det \sum_{i,j=1}^{n} A_i^* A_j(x)\zeta_i\zeta_j \neq 0; \quad \Re\left\{ \overline{w}^* \left(\sum_{i,j=1}^{n} A_i^* A_j(x)\zeta_i\zeta_j \right) w \right\} \geqslant 0.$$

Then the form $(\cdot,\cdot)_+$ is related to a mixed problem for the operator A^*A.

Lemma 2.1. *The Hermitian form $(\cdot,\cdot)_+$ defines a scalar product on $[C^1(\overline{D},\overline{S})]^k$ if one of the following conditions is fulfilled:*

1) $a_0 \geqslant c_0 I_k$ on $\overline{U}$ with a constant $c_0 > 0$ and a non-empty open subset U of D;

2) the relatively open set $S \subset \partial D$ is not empty;

3) $b_0 \geqslant c_1 I_k$ on $\overline{V}$ with a constant $c_1 > 0$ and a non-empty relatively open subset V of $\partial D \setminus S$.

Besides, in these cases we see that:

a) the embedding $[H^1(D,S)]^k \to H^+(D)$ is continuous, if the entries of the matrix b_0 belong $L^\infty(\partial D \setminus S)$;

b) the embedding $H^+(D) \to [L^2(D)]^k$ is continuous; moreover, in this case the elements of $H^+(D)$ belong $[H^1_{\text{loc}}(D \cup S)]^k$ and vanish on S.

Proof. To prove that it is a scalar product, we only need to check that $(u,u)_+ = 0$ for a function $u \in [C^1(\overline{D},\overline{S})]^k$ implies $u = 0$ in D.

From 1) and $a_0 u = 0$ it follows, that $u = 0$ on some subset V. Then, the Uniqueness Property (1) for the operator A and the assertion of $Au = 0$ imply that $u \equiv 0$.

From conditions 2) or 3) it follows, that any vector u from $[C^1(\overline{D},\overline{S})]^k$, satisfying $(u,u)_+ = 0$, vanishes in an open non-empty subset Γ of $S \subset \partial D$. Since u also satisfied $Au = 0$ in D, then we obtain a Cauchy problem:

$$\begin{cases} Au &= 0 \quad \text{in} \quad D, \\ u &= 0 \quad \text{on} \quad \Gamma. \end{cases}$$

It follows from the Uniqueness Theorem for the Cauchy problem for systems with an injective symbol, that $u \equiv 0$ in D (see, for instance, [8, Proposition 4.3.3]).

This proves that the Hermitian form $(\cdot,\cdot)_+$ defines a scalar product. Further, if the elements of b_0 belong to $L^\infty(\partial D \setminus S)$ then, by the Trace Theorem for the Sobolev spaces (see, for example, [7]), we find that for all $u \in [C^1(\overline{D},\overline{S})]^k$ we have following:

$$|(b_0 u, u)_{[L^2(\partial D \setminus S)]^k}| \leqslant c\|u\|^2_{[L^2(\partial D \setminus S)]^k} \leqslant \tilde{c}\|u\|^2_{[H^1(D)]^k},$$

where the positive constant c is independent of u. On the other hand, since the components of matrices a_0 and A_i ($1 \leqslant i \leqslant n$) belong to $L^\infty(D)$, we see that for all $u \in [C^1(\overline{D},\overline{S})]^k$ the following is true:

$$|(a_0 u, u)_{[L^2(D)]^k}| \leqslant c_0\|u\|^2_{[L^2(D)]^k} \leqslant c_0\|u\|^2_{[H^1(D)]^k},$$

$$|(A_i A_j \partial_j u, \partial_i u)_{[L^2(D)]^k}| \leqslant c_{i,j}\|\nabla u\|^2_{[L^2(D)]^k} \leqslant c_{i,j}\|u\|^2_{[H^1(D)]^k},$$

where the positive constants c_0, $c_{i,j}$ are independent of u. That is, the space $[H^1(D,S)]^k$ is continuously embedded into the $H^+(D)$.

If

$$a_0 \geqslant cI > 0 \text{ in } \overline{D} \tag{2}$$

then for all $u \in [C^1(\overline{D}, \overline{S})]^k$ we obtain

$$\|u\|_+^2 \geqslant (a_0 u, u)_{[L^2(D)]^k} \geqslant c\|u\|^2_{[L^2(D)]^k},$$

i.e. the space $H^+(D)$ is continuously embedded into $[L^2(D)]^k$.

Since the principal symbol of the operator A is injective then the solutions $Au = 0$ are infinitely differentiable functions in D. In addition, due to the Uniqueness property (1) the operator A is injective on $[C_0^\infty(D)]^k$. Then the Gårding inequality for the strongly elliptic operator A^*A yields

$$\|u\|_{[H^1(D)]^k} \leqslant c\,\|Au\|_{[L^2(D)]^k} \text{ for all } u \in [H_0^1(D)]^k, \tag{3}$$

with positive constant c does not depend of u. We take an domain $G \subset D$, such that $\overline{G} \subset D \cup S$ and we fix a function $\phi \in C^1(\overline{D})$ is vanishes outside the compact set $\overline{G}$ and also such that $\phi(x) = 1$ for all $x \in \overline{G}$. Then $\phi u \in [C^1(\overline{D}, \partial D)]^k$ for all $u \in [C^1(\overline{D}, \overline{S})]^k$ and, according to (3), we have:

$$\|u\|^2_{[H^1(G)]^k} \leqslant \|\phi u\|^2_{[H^1(D)]^k} \leqslant c\|A(\phi u)\|^2_{[L^2(D)]^l} \tag{4}$$

for all $u \in [C^1(\overline{D}, \overline{S})]^k$, with constant c does not depend of u.

Finally, as (2) is fulfilled, we obtain that

$$\|A(\phi u)\|^2_{[L^2(D)]^l} \leqslant 2\|\phi A u\|^2_{[L^2(D)]^l} + 2\|(A\phi)u\|^2_{[L^2(D)]^l} \leqslant c\|u\|_+^2 \tag{5}$$

for all $u \in [C^1(\overline{D}, \overline{S})]^k$, with constant c does not depend of u. In particular, inequalities (4), (5) mean that any sequence $\{u_\nu\} \subset [C^1(\overline{D}, \overline{S})]^k$, convergent in the space $H^+(D)$, converges in the $[H^1(G)]^k$, too. This proves the statement b). □

As we have seen in the proof of Lemma 2.1, if $S = \partial D$ then space $H^+(D)$ is embedded continuously to $[H^1(D, S)]^k$ (cf. [9]). Therefore, we are primarily interested in the case $S \neq \partial D$.

Everywhere below we assume that $H^+(D)$ is embedded continuously to $[L^2(D)]^k$, i.e.

$$\|u\|_{[L^2(D)]^k} \leqslant c\,\|u\|_+$$

for all $u \in [H^1(D, S)]^k$, where the constant c does not depend of u. It follows from Lemma 2.1, that this condition is not too restrictive. The Hermitian form $(\cdot, \cdot)_+$ is called coercive if there exists a constant $c > 0$ such that

$$\|u\|_{[H^1(D)]^k} \leqslant c\,\|u\|_+ \text{ for all } u \in [H^1(D, S)]^k$$

that is, the space $H^+(D)$ is continuously embedded into the space $[H^1(D, S)]^k$. Let ι be the natural (continuous) embedding:

$$\iota : H^+(D) \to [L^2(D)]^k. \tag{6}$$

Also, we will need Sobolev spaces with negative smoothness. More precisely, we denote by $H^-(D)$ the completion of the space $[H^1(D, S)]^k$ by the norm

$$\|u\|_- = \sup_{\substack{v \in H^+(D)\\ v \neq 0}} \frac{|(v, u)_{L^2(D)}|}{\|v\|_+}.$$

As is known, (see, for example, [10, Lemma 2.3]) the space $H^-(D)$ can be identified with the space dual to $H^+(D)$ with respect to pairing

$$\langle v, u \rangle = \lim_{\nu \to \infty} (v, u_\nu)_{[L^2(D)]^k}. \tag{7}$$

The space $[L^2(D)]^k$ is continuously embedded into $H^-(D)$; corresponding embedding we will denote by ι' (see, for instance, [10, Lemma 2.2]).

We define $[H^{-s}(D)]^k$ and $[\tilde{H}^{-s}(D)]^k$ as the dual spaces to $[H^s(D)]^k$ and $[H_0^s(D)]^k$ respectively, relatively to the pairing (7). By the construction, the following embedding is continuous:

$$[H^{-s}(D)]^k \hookrightarrow [\tilde{H}^{-s}(D)]^k,\ s > 0.$$

For the coercive case the following two statements on the embedding are, probably, well-known (cf. [5], [6]).

Lemma 2.2. *We assume that following estimate*

$$\overline{w}^* \left(\sum_{i,j=1}^{n} A_i^* A_j(x)\, \zeta_i \zeta_j \right) w \geqslant \tilde{c}\, |w|^2 |\zeta|^2 \tag{8}$$

is fulfilled for all $(x, w) \in \overline{D} \times (\mathbb{C}^n \setminus \{0\})$, $\zeta \in \overline{D} \times (\mathbb{R}^n \setminus \{0\})$, *where* $\tilde{c}$ *is positive constant independent on* (x, w) *and* ζ. *Then the embeddings*

$$H^+(D) \to [H^1(D, S)]^k, \quad [H^{-1}(D)]^k \to H^-(D)$$

are continuous, if one of the following conditions is fulfilled:

1) there is a positive constant $c > 0$ *such that inequality (2) is true;*

2) the set $\partial D \setminus S$ *has at least one interior point in the relative topology* ∂D *and*

$$\|b_0 u\|_{[L^2(\partial D \setminus S)]^k} \geqslant c_1 \, \|u\|_{[L^2(\partial D \setminus S)]^k} \ \textit{for all}\ u \in H^1(\partial D, S); \tag{9}$$

3) the set S *contains a not empty, open (in the relative topology of* ∂D*) subset.*

Proof. Suppose, that condition 1) is fulfilled. Then

$$\|u\|_+^2 \geqslant (a_0 u, u)_{[L^2(D)]^k} \geqslant c \|u\|^2_{[L^2(D)]^k}$$

for all $u \in [C^1(\overline{D}, \overline{S})]^k$ and from condition (8) it follows, that

$$|(Au, Au)_{[L^2(D)]^k}| \geqslant \tilde{c} \sum_{i=1}^{n} \|\partial_i u\|^2_{[L^2(D)]^k},$$

here positive constants c, $\tilde{c}$ are independent of u. But this means that for all $u \in [C^1(\overline{D}, \overline{S})]^k$

$$(u, u)_+ \geqslant c_0 \|u\|^2_{[H^1(D)]^k},$$

so, the statement of theorem is true, if condition 1).

Let the 2) be fulfilled. We suggest, that for any natural number m there is a such $u_m \in [C^1(\overline{D}, \overline{S})]^k$, that the inequality

$$(u_m, u_m)_+ < \frac{1}{m} \|u_m\|^2_{[H^1(D)]^k} \tag{10}$$

is true. Consider the bounded sequence $\left\{ \dfrac{u_m}{\|u_m\|} \right\}$. We can extract a weakly convergent subsequence $\left\{ v_{m_j} = \dfrac{u_{m_j}}{\|u_{m_j}\|} \right\}_{j \in \mathbb{N}}$ to an element $v_0 \in [H^1(D)]^k$. By the Trace Theorem for Sobolev-Slobodetskii spaces, the sequence of traces $\{v_{m_j|\partial D}\}_{j \in \mathbb{N}}$ converges weakly in $[H^{1/2}(\partial D)]^k$ to the trace $v_{0|\partial D} \in [H^{1/2}(\partial D)]^k$ and $\{Av_{m_j}\}_{j\in\mathbb{N}}$ converges weakly to the element Av_0 in $[L^2(D)]^k$. In addition, by Rellich Theorem, the space $[H^{1/2}(\partial D)]^k$ is compactly embedded in $[L^2(\partial D)]^k$ and the $[H^1(D)]^k$ is compactly embedded in $[L^2(D)]^k$. Hence, the sequence $\{v_{m_j}\}_{j\in\mathbb{N}}$ converges to v_0 in $[L^2(D)]^k$ and the sequence $\{v_{m_j|\partial D}\}_{j\in\mathbb{N}}$ converges to the trace $v_{0|\partial D}$ in $[L^2(\partial D)]^k$.

According to (8), (10) and 2)

$$\|v_{m_j}\|_{[L^2(\partial D)]^k} \leqslant \|b_0 v_{m_j}\|_{[L^2(\partial D)]^k} \leqslant \|v_{m_j}\|_+ \to 0,\ m_j \to +\infty,$$

$$\sum_{i=1}^{n} \|\partial_i v_{m_j}\|^2_{[L^2(D)]^k} \leqslant \|Av_{m_j}\|^2_{[L^2(D)]^l} \to 0,\ m_j \to +\infty.$$

Since the weak and strong limits are coincided (when both exist), then $Av_0 = 0$ in D and $v_0 = 0$ on S. More other the condition of uniqueness in the small implies that $v_0 \equiv 0$. So it means, that

$$\|v_{m_j}\|_{[L^2(D)]^k} \to 0,\ m_j \to +\infty; \quad \|v_{m_j}\|_{[H^1(D)]^k} \to 0,\ m_j \to +\infty,$$

which contradicts the equality $\|v_{m_j}\|_{[H^1(D)]^k} = 1$.

For conditions 3) the proof is similar. □

In particular, under the hypothesis of Lemma 2.2 the Hermitian form $(\cdot,\cdot)_+$ is coercive, and the embedding (6) is compact.

Lemma 2.3. *Under the assumption $S = \partial D$, the following embeddings are continuous:*

$$H^+(D) \to [H^1(D, \partial D)]^k, \quad ([H^1(D, \partial D)]^k)' \to H^-(D).$$

In particular, the Hermitian form $(\cdot,\cdot)_+$ is coercive and the embedding (6) is compact.

Proof. The statement is well known (see, for example, [4]). □

3. The embedding theorem for non-coercive forms

Now we obtain an embedding theorem for the space $H^+(D)$ under weaker assumptions than in the Lemmata 2.2, 2.3.

Theorem 3.1. *We assume that the coefficients of the matrix A_i are infinitely smooth in the closure of some neighborhood X of the compact set $\overline{D}$ and there is a constant $c > 0$ such that (9) is done. Then*

1) the space $H^+(D)$ is continuously embedded into $[L^2(D)]^k$, if there is a constant $c_1 > 0$ such that (2) is satisfied;

2) the space $H^+(D)$ is continuously embedded into $[H^{1/2-\epsilon}(D)]^k$ with an arbitrary $\epsilon > 0$, if for all $u \in [C^\infty(X)]^k$, the following is true

$$(Au, Au)_{[L^2(X)]^l} \geqslant m\, \|u\|^2_{[L^2(X)]^k} \tag{11}$$

with a constant $m > 0$ is independent of the function u.

Moreover, if $\partial D \in C^2$, then $H^+(D)$ is continuously embedded into $[H^{1/2}(D)]^k$.

Proof. The statement 1) follows immediately from the definition of the norm $\|\cdot\|_+$.

Let $\partial X \in C^\infty$. This assumption does not bear loss of generality, although for most of the set goals it will be enough to ∂X was a Lipschitz surface. Since the operator A^*A is strongly elliptic, then the classical Gårding inequality and the condition of uniqueness in the small mean, that there exists the Green function G of Dirichlet Problem (see, for example, [2, 11], [12, Th. 3.3] and [13, Theorem 2.26]). More precisely, as above, we define the space $[\tilde{H}^{-1}(X)]^k$ as the dual space to $[H^1(X, \partial X)]^k$ with respect to $[L^2(X)]^k$-pairing. Clearly, the space $[H^{-1}(X)]^k$ is continuously embedded into $[\tilde{H}^{-1}(X)]^k$. As usual, the space $[C^1(X, \partial X)]^k$ is dense in the space $[H^1(X, \partial X)]^k$, then the operator A^*A extends to a continuous mapping of $[H^1(X, \partial X)]^k$ into $[\tilde{H}^{-1}(X)]^k$ with using Hermitian form

$$\sum_{i,j=1}^{n} (A_i \partial_i u, A_j \partial_j v)_{[L^2(X)]^l} = (Au, Av)_{[L^2(X)]^l}, \quad u, v \in [H^1(X, \partial X)]^k. \tag{12}$$

In particular, form (12) induces a generalized setting of the Dirichlet Problem for the operator A^*A в X. Thus, there is a bounded linear operator

$$G : [\tilde{H}^{-1}(X)]^k \to [H^1(X, \partial X)]^k,$$

satisfying

$$GA^*A = I, \quad A^*AG = I \tag{13}$$

on $[H^1(X, \partial X)]^k$ and $[\tilde{H}^{-1}(X)]^k$ respectively (see [12, Th. 3.3] or [13, Th. 2.26]).

Applying the Trace Theorem for Sobolev spaces (see, for example, [1]), we introduce the so-called Poisson operator $P : [H^{1/2}(\partial X)]^k \to [H^1(X)]^k$, which satisfies

$$P \circ t_1 + GA^*A = I,$$

where t_1 denotes the operator of the trace $[H^1(X)]^k \to [H^{1/2}(\partial X)]^k$ (cf. [12, Lem. 4.3] or [13, Cor. 2.31]). If the boundary of X is a Lipschitz surface, then the Green and Poisson operators posses adequate regularity properties. More precisely,

$$G : [\tilde{H}^{s-1}(X)]^k \to [H^{s+1}(X)]^k, \ \partial_j G : [\tilde{H}^{s-1}(X)]^k \to [H^s(X)]^k,$$
$$P : [H^{s+1/2}(\partial X)]^k \to [H^{s+1}(X)]^k$$

for all $0 \leqslant s < 1/2$ (see, for example, [6, § 12]). In particular, if ∂X is C^2-smooth, then the mappings

$$G : [L^2(X)]^k \to [H^2(X)]^k, \quad P : [H^{3/2}(\partial X)]^k \to [H^2(X)]^k$$

are also continuous.

Let ν_A be so-called a co-normal derivative to ∂X with respect to the operator A:

$$\nu_A = -\sum_{i=1}^{n} A_i^*(x)\nu_i(x)A, \tag{14}$$

here $\nu(x) = (\nu_1(x), \ldots, \nu_n(x))$ is unit outward normal to ∂X at the point $x \in \partial X$. If X is a domain with Lipschitz boundary, then the normal $\nu(x)$ exists almost everywhere on ∂X.

Lemma 3.1. *Let X be a domain with Lipschitz boundary. If $\partial X \in C^2$, then the operator P is continuously mapping $[L^2(\partial X)]^k$ into $[H^{1/2}(X)]^k$.*

Proof. The proof of this statement is similar to the proof of the corresponding statement for weighted Sobolev spaces in [10, Lemma 7.7]. □

We continue the proof of the theorem. Let e^+ will be an operator of extension by zero of the domain D on X and by r^+ we denote an operator of restriction from the set X to domain D. Evidently, e^+ is the bounded linear operator from $[L^2(D)]^k$ into $[L^2(X)]^k$ and r^+ is bounded linear operator from $[H^s(X)]^k$ into $[H^s(D)]^k$ for all $s \in \mathbb{R}$.

If (11) is not valid, then according to the condition of the theorem, it follows, that $a_0 \geqslant c_1 I$ in D with some constant $c_1 > 0$, and hence $H^+(D)$ is continuously embedded into $[L^2(D)]^k$. Therefore, the norm $\| \cdot \|_+$ is not weaker than the norm $\| \cdot \|_p$ on $[H^1(D, S)]^k$

$$\|u\|_p = \left(\|Au\|^2_{[L_2(D)]^l} + \|u\|^2_{[L^2(\partial D)]^k} \right)^{1/2}.$$

Since the coefficients of the matrix $A_i(x)$ are continuous up to the boundary of the domain D, from Green's formula the next follows

$$\int_{\partial D} - \sum_{i=1}^n A_i^*(x)\nu_i(x) A u \overline{v}\, ds = \int_D \sum_{i,j=1}^n \left(A_i^* A_j \partial_j u \overline{\partial_i v} + \partial_i (A_i^* A_j \partial_j u) \overline{v} \right) dx \tag{15}$$

for all $u \in [H^2(D)]^k$ and $v \in [H^1(D)]^k$.

We denote by

$$G_D : [\tilde{H}^{-1}(D)]^k \to [H^1(D, \partial D)]^k,$$

Green's operator of the Dirichlet Problem for the operator A^*A in D. Properties of the operator G_D are similar to the properties of the operator G discussed above for the domain X. In a similar way we introduce the operator Poisson P_D.

Combining the formulas (13), (15) we get the following

$$\|u\|_p^2 \geqslant \sum_{i,j=1}^n (A_i^* A_j \partial_j G_D A^* A u, \partial_i G_D A^* A u)_{[L^2(D)]^k} + \\ + \sum_{i,j=1}^n (A_i^* A_j \partial_j P_D u, \partial_i P_D u)_{[L^2(D)]^k} + \|P_D u\|^2_{[L^2(\partial D)]^k} \tag{16}$$

if $u \in [H^1(D, S)]^k$. On the other hand, it follows from the Gårding inequality that for all $u \in [H^1(D, S)]^k$ we have

$$\|G_D A^* A u\|^2_{[H^1(D)]^k} \leqslant c \sum_{i,j=1}^n (A_i^* A_j \partial_j G_D A^* A u, \partial_i G_D A^* A u)_{[L^2(D)]^k}. \tag{17}$$

Using (13), (16) and (17) we see that any sequence $\{u_\mu\} \subset [H^1(D, S)]^k$, which converges to a function u in the space $H^+(D)$, it can be represented as

$$u_\mu = G_D A^* A u_\mu + P_D u_\mu, \tag{18}$$

where the sequence $\{G_D A^* A u_\mu\}$ converges in $[H^1(D, \partial D)]^k \subset [H^1(D, S)]^k$ to the element u_G. Therefore, the sequence $\{P_D u_\mu\}$ converges to some element u_P in $H^+(D)$ and

$$u = u_G + u_P = G_D A^* A u + P_D u, \tag{19}$$

here $P_D u$ is the integral of Poisson of the trace of the function $u|_{\partial D} \in [L^2(\partial D)]^k$ for $u \in H^+(D)$. Hence this theorem depends of the behavior of the element $u_P = P_D u$.

Since the coefficients of A_i are smooth in a neighborhood of $\overline{D}$, we can assume without loss of generality that X is a region with smooth boundary. In this way, if $A^* A u \in [L^2(X)]^k$ and $u = 0$ on ∂X, then the element u belongs to the $[H^2(X)]^k$. Therefore, from a priori estimates, it follows that G generates a bounded operator

$$r^+ G e^+ : [L^2(D)]^k \to [H^2(D)]^k,$$

where operators r^+ and e^+ are defined above.

Let $s \geqslant 0$. It is clear that any element $u \in [H^{-s}(D)]^k$ extends up to the element $U \in [H^{-s}(X)]^k$ by the equality

$$\langle U, v\rangle_X = \langle u, v\rangle_D$$

for all $v \in [H^s(X)]^k$. Since the element U vanishes in $X \setminus \overline{D}$, it is natural to denote it by e^+u. On this way we define a linear operator $e^+ : [H^{-s}(D)]^k \to [H^{-s}(X)]^k$, $s \geqslant 0$.

The support of the distribution e^+u belongs to the $\overline{D}$. Therefore, using the continuity of pseudo-differential operators on the compact closed manifolds, we conclude that r^+Ge^+ extends to a linear bounded operator

$$r^+Ge^+ : [H^{-1/2}(D)]^k \to [H^{3/2}(D)]^k.$$

Hence the operators

$$\partial_j \left(r^+Ge^+\right) : [H^{\epsilon-1/2}(D)]^k \to [H^{1/2+\epsilon}(D)]^k, \quad \nu_A \left(r^+Ge^+\right) : [H^{\epsilon-1/2}(D)]^k \to [H^{\epsilon}(\partial D)]^k \quad (20)$$

are also bounded by the Trace Theorem for Sobolev spaces in Lipschitz domains. Note that when $\epsilon = 0$ the statement becomes invalid, as the elements of space $[H^{1/2}(D)]^k$ may have no traces on $\partial D \subset X$.

From (15) and the continuity property (20) it follows that

$$(v, u)_{[L^2(D)]^k} = (A^*AG(e^+v), u)_{[L^2(D)]^k}(\nu_A(r^+Ge^+)v, u)_{[L^2(\partial D)]^k} + \\ + \sum_{i,j=1}^{n} \int_D A_i^* A_j \partial_j G(e^+v)\overline{\partial_i u}\, dx \quad (21)$$

for all $u \in [H^1(D, S)]^k$ and $v \in [L^2(D)]^k$.

Now we claim that the norm $\|\cdot\|_p$ is not weaker, than the norm $\|\cdot\|_{H^{[1/2-\epsilon}(D)]^k}$ on $[H^1(D, S)]^k$. Indeed,

$$\|u\|_{[H^{1/2-\epsilon}(D)]^k} = \sup_{\substack{v \in [H^{\epsilon-1/2}(D)]^k \\ v \neq 0}} \frac{|(v, u)_{[L^2(D)]^k}|}{\|v\|_{[H^{\epsilon-1/2}(D)]^k}} = \sup_{\substack{v \in [H^{\epsilon-1/2}(D)]^k \\ v \neq 0}} \frac{\lim\limits_{\nu \to \infty} |(v_\mu, u)_{[L^2(D)]^k}|}{\|v\|_{[H^{\epsilon-1/2}(D)]^k}} \quad (22)$$

for all $u \in [H^1(D, S)]^k$, where $\{v_\mu\}$ is any sequence of smooth functions in $\overline{D}$, which approximates v in the space $[H^{\epsilon-1/2}(D)]^k$. Using the formula (21) for u and $v = v_\mu$, the expression on the right side of (22) is equal to

$$\Big| \sum_{i,j=1}^{n} \int_D A_i^* A_j \partial_j G(e^+v)\overline{\partial_i u}\, dx + (\nu_A(r^+Ge^+)v, u)_{[L^2(\partial D)]^k} \Big|.$$

Then

$$|(\nu_A(r^+Ge^+)v, u)_{[L^2(\partial D)]^k}| \leqslant \|\nu_A(r^+Ge^+)v\|_{[H^{\epsilon}(D)]^k} \|u\|_{[H^{-\epsilon}(\partial D)]^k} \leqslant \\ \leqslant c\, \|v\|_{[H^{\epsilon-1/2}(D)]^k} \, \|u\|_{[H^{-\epsilon}(\partial D)]^k} \quad (23)$$

for all $u \in [H^1(D, S)]^k$ and $v \in [H^{\epsilon-1/2}(D)]^k$, where the last inequality follows from (20). Here c denotes a constant, which independents of u and v.

It follows from the generalized Cauchy inequality that

$$\Big| \sum_{i,j=1}^{n} A_i^* A_j(x)\overline{z}_i z_j \Big|^2 \leqslant \Big(\sum_{i,j=1}^{n} A_i^* A_j(x)\overline{z}_i z_j \Big)\Big(\sum_{i,j=1}^{n} A_i^* A_j(x)\overline{\zeta}_i \zeta_j \Big) \quad (24)$$

for all $z, \zeta \in \mathbb{C}^n$. The application (24) leads us to the following:

$$\Big| \sum_{i,j=1}^{n} \int_D A_i^* A_j \partial_j G(e^+ v) \overline{\partial_i u}\, dx \Big| \leqslant c \left(\sum_{i,j=1}^{n} \int_D A_i^* A_j \partial_j u \overline{\partial_i u}\, dx \right)^{1/2} \|v\|_{[H^{\epsilon - 1/2}(D)]^k} \tag{25}$$

with a constant c independent on u and v.

Using (22), (23) and (25) we conclude that there are positive constants c and C such that

$$c\, \|u\|_{[H^{1/2-\epsilon}(D)]^k} \leqslant \|u\|_p \leqslant C\, \|u\|_+$$

for all $u \in [H^1(D, S)]^k$. In particular, this provides a continuous embedding $H^+(D) \to [H^{1/2-\epsilon}(D)]^k$, as required.

Finally, let $\partial D \in C^2$. The sequence $\{u_\nu\}$ from (18) converges in $H^+(\partial D)$, and the norm in this space is not weaker than the norm $\| \cdot \|_p$, then it converges also in the $[L^2(\partial D)]^k$. Lemma 3.1 implies, that the sequence $\{P_D u_\nu\}$ convergence to some element u_P in $[H^{1/2}(D)]^k$. The continuously embedding of space $H^+(D)$ into $[H^{1/2}(D)]^k$ follows from (19). □

For the case where the operator A^*A is scalar, this theorem is similar to [10, Theorem 7.4]. In the work [14] a special case corresponding to the factorization A^*A for Lamé system of linear elasticity theory was considered. Actually the present proof follows the same general scheme, taking into the account the matrix nature of A^*A.

Example 3.1. Denote by $\mathcal{L}$ the Lamé type operator in $\mathbb{R}^2$:

$$\mathcal{L}_0(x, \partial) = -\mu(x) I \Delta - (\lambda(x) + \mu(x)) \nabla \text{div},$$

where μ and λ are real-valued functions in $L^\infty(D)$, with $\mu \geqslant \kappa$, $(2\mu + \lambda) \geqslant \kappa$ for a constant $\kappa > 0$. If the functions μ, λ belong to $C^{0,1}(D)$ then there is a formally non-negative self-adjoint operator $\mathcal{L}_A(x, \partial) = A^*A$, that differs from $\mathcal{L}_0(x, \partial)$ by lower order terms. Of course, there are several possible choices of A. For instance, one may consider

$$A = \begin{pmatrix} \sqrt{\mu}\ \text{rot} \\ \sqrt{2\mu + \lambda} \text{div} \end{pmatrix}.$$

If $b_0 > 0$ is a matrix with constant entries, then we have a continuous embedding $H^+(D) \to [H^{1/2}(D)]^2$ (see Theorem 3.1). Moreover, $H^+(D)$ is not embedded continuously into $[H^s(D)]^2$ for any $s \in (1/2, 1]$ (see, for instance, [14, Example 4.5]).

The operator ν_A for this case is responsible not for the stress/viscosity on the boundary but for a more large class of interactions with ∂D. Interpreting the Lamé system as a linearization of the stationary version of the Navier-Stokes' type equations for compressible fluids, we see that the operator $(\nu_A + b_0)$ reflects rather the vorticity and the source density on conormal directions to $\partial D \setminus S$. This means that the operator ν_A is more fit to study problems related to models with the turbulent flows than some others operators (see [14, Example 4.5]). This is useful when we consider the boundary value problems for the Lamé operator with boundary-type operator $(\nu_A + b_0)$. Then it is natural that the class of the possible solutions to these problem extends to $H^+(D)$ due to the loss of the regularity of solutions near $\partial D \setminus S$ (for more details see [14]).

Example 3.2. Let D be the unit circle in $\mathbb{R}^2 (\cong \mathbb{C})$ and S be the part of its boundary where $\arg(z) \in [0, 2\pi] \setminus [-\pi/2, \pi/2]$. Let $A = \bar\partial$ be the Cauchy-Riemann operator in $\mathbb{R}^2$, i.e. it is the (2×2)-matrix

$$\bar\partial = \begin{pmatrix} \frac{\partial}{\partial x} & -\frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix}. \tag{26}$$

The formal adjoint $\bar{\partial}^*$ of $\bar{\partial}$ is the (2×2)-matrix is conjugated to (26) with respect to the usual Hermitian structure in the space $L^2(\mathbb{R}^2)$. Then an easy computation shows that $\bar{\partial}^*\bar{\partial}$ amounts to the unit (2×2)-matrix I_2 multiple of the Laplace operator $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ in $\mathbb{R}^2$. In the space $\mathbb{C}$ the Cauchy-Riemann operator just is $\dfrac{\partial}{\partial \bar{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + \sqrt{-1}\dfrac{\partial}{\partial y}\right)$.

We assume that $a_0 = 0$ and $b_0 > 0$ is a (2×2)-matrix with constant elements. Thus, $H^+(D)$ is continuously embedded into $[H^{1/2}(D)]^2$ (see Theorem 3.1). Then the norm of the element $u \in H^+(D)$ will have the following form:

$$\|u\|_+ = \|\bar{\partial}u\|_{[L^2(D)]^2} + \|b_0 u\|_{[L^2(\partial D\setminus S)]^2}.$$

Fix a function ϕ of the class $C^\infty(\overline{D})$, which identically equals to zero in the neighborhood of S and equals to 1 on the part where $\arg(z) \in [-\pi/4, \pi/4]$. We define a new vector function $u_\varepsilon(x,y) = \phi(x,y)v_\varepsilon(x,y)$, where

$$v_\varepsilon(x,y) = \begin{pmatrix} \Re\left\{\displaystyle\sum_{\nu=0}^{\infty} \frac{(x+\sqrt{-1}y)^{4\nu}}{(4\nu+1)^{(1+\varepsilon)/2}}\right\} \\ \Im\left\{\displaystyle\sum_{\nu=0}^{\infty} \frac{(x+\sqrt{-1}y)^{4\nu}}{(4\nu+1)^{(1+\varepsilon)/2}}\right\} \end{pmatrix} \quad \varepsilon > 0.$$

For brevity, the further argument we will carry out in the space $\mathbb{C}$. In $\mathbb{C}$ the element $v_\varepsilon(x,y)$ is

$$v_\varepsilon(z) = \sum_{\nu=0}^{\infty} \frac{z^{4\nu}}{(4\nu+1)^{(1+\varepsilon)/2}},\ \varepsilon > 0.$$

In other words

$$u_\varepsilon = \begin{cases} v_\varepsilon & \text{if} \quad \arg z \in \left[\dfrac{-\pi}{4}; \dfrac{\pi}{4}\right], \\ 0, & \text{if} \quad \arg z \in U_S, \end{cases}$$

where U_S is some neighborhood S. As v_ϵ independent of $\bar{z}$, the Leibniz rule implies, that

$$\bar{\partial}v_\varepsilon = 0,\ \bar{\partial}u_\varepsilon = (\bar{\partial}\phi)v_\varepsilon. \tag{27}$$

We will show that the series u_ε converges in the space $H^+(D)$ and will find its norm. To the end, we recall that

$$(z^k, z^\nu)_{[L^2(D)]^2} = \begin{cases} 0, & k \neq \nu, \\ \pi/(\nu+1), & k = \nu. \end{cases} \tag{28}$$

Hence

$$\|v_\varepsilon\|^2_{[L^2(D)]^2} = \sum_{\nu=0}^{\infty} \frac{\pi}{(4\nu+1)^{(2+\varepsilon)}},$$

and therefore the series of v_ε converges in $L^2(D)$ and in $H^+(D)$, because $\overline{\partial}v_\varepsilon = 0$. Using (27) we see that u_ε converges in the space $H^+(D)$, too.

On the other hand, by direct calculation with the use of (28) (see, for instance, [15, Lemma 1.4]), we obtain:

$$\|v_\varepsilon\|^2_{[H^s(D)]^2} \geqslant \text{const} \sum_{\nu=0}^{\infty} \frac{\pi(4\nu+1)^{2s-1}}{(4\nu+1)^{(1+\varepsilon)}}, \qquad 0 < s \leqslant 1. \tag{29}$$

Hence the series v_ε converges in $H^s(D)$ for all $0 \leqslant s \leqslant 1/2$. Then the series u_ε converges in $H^s(D)$ for all $0 \leqslant s \leqslant 1/2$.

If $s > \dfrac{1}{2}$ we have $2s-1>0$. For any $s > \dfrac{1}{2}$ then it follows from (29) that there is a number $0 < \varepsilon_s \leqslant 2s-1$ such that the series v_{ε_s} and u_{ε_s} do not converge in $H^s(D)$. Thus, u_{ε_s} belongs to the space $H^+(D)$ but for any $s \in (1/2, 1]$ the element u_{ε_s} does not belong $H^s(D)$. Therefore, the space $H^+(D)$ is embedded continuously into $[H^{1/2}(D)]^2$ (see Theorem 3.1), but it is not embedded continuously into $[H^s(D)]^2$, with any $s \in (1/2, 1]$.

In the case when $S = \emptyset$ the similar result was proved in [16, Theorem 1].

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Теоремы вложения для функциональных пространств, ассоциированных с одним классом эрмитовых форм

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Мы докажем теоремы вложения в шкалу пространств Соболева-Слободецкого для функциональных пространств, ассоциированных с одним классом эрмитовых форм. Более точно мы рассматриваем эрмитовы формы, построенные с помощью матричных дифференциальных операторов первого порядка с инъективным главным символом. Конечные результаты получаются как для коэрцитивных, так и для некоэрцитивных форм.

Ключевые слова: некоэрцитивная эрмитова форма, теоремы вложения, эллиптические матричные операторы.