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On the Limit Structure of Continuous-time Markov Branching Process

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We study the limiting probability function of continuous-time Markov Branching Processes conditioned to be never extinct. Hereupon we obtain a new stochastic population process called the Markov Q-Process. The principal aim is to investigate structural and asymptotic properties of the Markov Q-Process, also we study transition functions of this process and their convergence to stationary measures.

Keywords: Markov Branching Process, Markov Q-process, transition function; invariant measures.
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1. Introduction and preliminaries

We consider a population of monotypic individuals that are capable of perishing and transformation into a random number of individuals of the same type. These individuals may be of biological kind, molecules in chemical reactions etc. Suppose that the population size changes according to a random reproduction law as follows. Each individual existing at the epoch $t \in \mathcal{T} = [0; +\infty)$, independently of its history and of each other for a small time interval $(t; t+\varepsilon)$ transforms into $j \in \mathbb{N}_0 \backslash \{1\}$ individuals with probability $a_j\varepsilon + o(\varepsilon)$ and, with probability $1 + a_1\varepsilon + o(\varepsilon)$ each individual survives or makes evenly one descendant (as $\varepsilon \downarrow 0$). Where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\mathbb{N} = \{1, 2, \ldots\}$. The numbers $\{a_j\}$ are intensities of individuals' transformation that $a_j \geqslant 0$ for $j \in \mathbb{N}_0 \backslash \{1\}$ and $0 < a_0 < -a_1 = \sum_{j \in \mathbb{N}_0 \backslash \{1\}} a_j < \infty$. Newly appeared individuals undergo transformations under the same way as above. Letting $Z(t)$ be the population size at the moment t, we have a homogeneous continuous-time Markov Branching Process (MBP), which was first considered by Kolmogorov and Dmitriev [13]. The process $Z(t)$ is a Markov chain with the state space on $\mathbb{N}_0$. Its transition functions

$$P_{ij}(t) = \mathbb{P}_i \{Z(t) = j\} := \mathbb{P}\{Z(t+\tau) = j \,| Z(\tau) = i\},$$

satisfy the branching property

$$P_{ij}(t) = \sum_{j_1 + \cdots + j_i = j} P_{1j_1}(t) \cdot P_{1j_2}(t) \cdots P_{1j_i}(t). \tag{1.1}$$

The probabilities $P_{1j}(t)$ in (1.1) are calculated using the local densities $\{a_j\}$ by the relation

$$P_{1j}(\varepsilon) = \delta_{1j} + a_j\varepsilon + o(\varepsilon), \quad \text{as} \ \ \varepsilon \downarrow 0, \tag{1.2}$$

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where δ_{ij} is the Kronecker delta function. A probability Generating Functions (GF) version of the relation (1.2) is

$$F(\varepsilon; s) = s + f(s) \cdot \varepsilon + o(\varepsilon), \quad \text{as} \quad \varepsilon \downarrow 0,$$

for all $0 \leqslant s < 1$, where

$$F(t; s) = \sum_{j \in \mathbb{N}_0} P_{1j}(t) s^j \quad \text{and} \quad f(s) = \sum_{j \in \mathbb{N}_0} a_j s^j.$$

Owing to Markovian property the GF

$$\sum_{j \in \mathbb{N}_0} P_{ij}(t) s^j = [F(t; s)]^i, \quad \text{for all} \quad i \in \mathbb{N}. \tag{1.3}$$

Assuming $a := f'(1)$ is finite and using equation (1.4) we have $\mathbb{E}_i Z(t) = \sum_{j \in \mathbb{N}_0} j P_{ij}(t) = ie^{at}$. The last formula shows that long-term properties of MBP seem to be variously depending on the parameter a. Hence, the MBP is classified as critical if $a = 0$ and sub-critical or super-critical if $a < 0$ or $a > 0$ respectively. Monographs [2, 5, 19] are general references for mentioned and other classical facts on theory of MBP.

Throughout this paper we write $\mathbb{P}\{*\}$ and $\mathbb{E}[*]$ instead of $\mathbb{P}_1\{*\}$ and $\mathbb{E}_1[*]$ respectively.

Let the random variable $\mathcal{H} := \inf\{t \in \mathcal{T} : Z(t) = 0\}$ be a hitting time of the zero state of MBP, i.e. the time of extinction of the process $Z(t)$. By the extinction theorem $\mathbb{P}_i\{\mathcal{H} < \infty\} = q^i$, where $q = \lim_{t\to\infty} P_{10}(t)$ is the extinction probability of MBP, which is the least non-negative root of $f(s) = 0$. Moreover $\lim_{t\to\infty} F(t; s) = q$ uniformly for $0 \leqslant s \leqslant r < 1$. Let us consider the conditioned distribution function $\mathbb{P}_i^{\mathcal{H}(t)}\{*\} := \mathbb{P}_i\{* | t < \mathcal{H} < \infty\}$. It is known that if $a \leqslant 0$ then $q = 1$. Therefore in this case $\mathbb{P}_i^{\mathcal{H}(t)}\{*\} = \mathbb{P}_i\{* | \mathcal{H} > t\}$ and $\mathbb{P}\{t < \mathcal{H} < \infty\} \equiv \mathbb{P}\{Z(t) > 0\}$. On the other hand in this case $0 \leqslant P_{1j}(t) \leqslant \mathbb{P}\{\mathcal{H} > t\} \to 0$ as $t \to \infty$. But the ratio $P_{1j}(t)/\mathbb{P}\{\mathcal{H} > t\}$ has a limiting finite law. So long-term properties of non-supercritical MBP are traditionally investigated on non-zero trajectories, that is under the condition of event $\{\mathcal{H} > t\}$. Sevastyanov [18] proved that in the sub-critical case there is a limiting distribution law $\lim_{t\to\infty} \mathbb{P}^{\mathcal{H}(t)}\{Z(t) = j\}$ if and only if $\sum_{j\in\mathbb{N}} a_j j \ln j < \infty$. In the critical situation he also proved that if $2b := f''(1) < \infty$, then $Z(t)/bt$ has a limiting exponential law. In this case Chistyakov [3] proved that if $f^{(4)}(1) < \infty$ and j/bt is bounded, then $t \cdot \mathbb{P}^{\mathcal{H}(t)}\{Z(t) = j\} = 1/b + \mathcal{O}\left(\sqrt{\ln t/t}\right)$ as $t \to \infty$. The author [6] restated this result without error term being on the condition of $b < \infty$ only.

More interesting phenomenon arises if we observe the limit of conditioned distribution $\mathbb{P}_i^{\mathcal{H}(t+\tau)}\{*\}$ letting $\tau \to \infty$. In the discrete-time situation this limit represents a distribution measure, which defines a homogeneous Markov chain called the Q-process (see [2, pp. 56–60]) The Q-process was considered first by Lamperti and Ney [14]. Some properties of it were discussed by Pakes [15–17], Imomov [7,9–11], Formanov and Imomov [4]. A considerable part of the paper [12] is devoted to the discussion of this process from the viewpoint of branching transformation called the Lamperti-Ney transformation. A closer look shows that in the MBP case the limit $\lim_{\tau\to\infty} \mathbb{P}_i^{\mathcal{H}(t+\tau)}\{Z(t) = j\}$ is an honest probability measure $\mathbb{Q}(t) = \{\mathcal{Q}_{ij}(t)\}$ which defines a homogeneous continuous-time Markov chain with state space on the set $\mathbb{N}$ called the Markov Q-Process (see [8]). Let $W(t)$ be the state size at the moment $t \in \mathcal{T}$ in the Markov Q-Process. Then $W(0) \overset{d}{=} Z(0)$ and

$$\mathbb{P}_i\{W(t) = j\} = \mathcal{Q}_{ij}(t).$$

In the mentioned above paper [8] some asymptotic properties of distribution of $W(t)$ were established. Namely, it was proved that if the corresponding MBP is critical, then $W(t)/\mathbb{E}W(t)$ has a limiting Erlang law. In this case there is an invariant measure if the second moment of the GF $f(s)$ is finite. In the non-critical situation under some moment condition there exists an invariant distribution for the process $W(t)$.

In Sec. 2 we define the Markov Q-Process and discuss properties concerning its construction and its transition function $\mathbb{Q}(t)$. In the Sec. 3 an ergodic property of $\mathbb{Q}(t)$ is established.

2. The Markov Q-process

In this section we are interested in the limiting interpretation of the conditioned transition function $\mathbb{P}_i^{\mathcal{H}(t+\tau)}\{Z(t)=j\}$ letting $\tau\to\infty$ and for all fixed $t\in\mathcal{T}$. First, by the law of total probability we write

$$\mathbb{P}_i\{t<\mathcal{H}<\infty,\ Z(t)=j\}=\mathbb{P}\{t<\mathcal{H}<\infty\,|Z(t)=j\}\cdot P_{ij}(t).$$

Since the probability of extinction of j particles is q^j, it follows that

$$\mathbb{P}_i\{t<\mathcal{H}<\infty,\ Z(t)=j\}=P_{ij}(t)\cdot q^j. \tag{2.1}$$

Using formula (3.1), from the last relation we obtain that

$$\mathbb{P}_i\{t<\mathcal{H}<\infty\}=\sum_{j\in\mathbb{N}}\mathbb{P}_i\{Z(t)=j,\ t<\mathcal{H}<\infty\}=\sum_{j\in\mathbb{N}}P_{ij}(t)q^j. \tag{2.2}$$

The relation (2.1) implies

$$\begin{aligned}\mathbb{P}_i\{Z(t)=j,\ t+\tau<\mathcal{H}<\infty\} &= P_{ij}(t)\cdot\sum_{k\in\mathbb{N}}\mathbb{P}_j\{\tau<\mathcal{H}<\infty,\ Z(\tau)=k\}=\\ &= P_{ij}(t)\cdot\sum_{k\in\mathbb{N}}P_{jk}(\tau)q^k.\end{aligned}$$

Therefore, considering the identity (2.2) we have

$$\mathbb{P}_i^{\mathcal{H}(t+\tau)}\{Z(t)=j\}=P_{ij}(t)\cdot\frac{\sum_{k\in\mathbb{N}}\frac{P_{jk}(\tau)}{P_{11}(\tau)}q^k}{\sum_{j\in\mathbb{N}}\frac{P_{ij}(t+\tau)}{P_{11}(t+\tau)}q^j}\cdot\frac{P_{11}(\tau)}{P_{11}(t+\tau)}.$$

Using the ratio limit property [6, Lemma 7], after short calculation it follows that

$$\lim_{\tau\to\infty}\mathbb{P}_i^{\mathcal{H}(t+\tau)}\{Z(t)=j\}=\frac{jq^{j-i}}{i\beta^t}P_{ij}(t)=:\mathcal{Q}_{ij}(t),$$

where $\beta=\exp\{f'(q)\}$. It is easily seen that $0<\beta\leqslant 1$. Namely, $\beta=1$ if $a=0$, and $\beta<1$ otherwise. Since $F'(t;q)=\beta^t$

$$\sum_{j\in\mathbb{N}}\mathcal{Q}_{ij}(t)=\sum_{j\in\mathbb{N}}\frac{jq^{j-i}}{i\beta^t}P_{ij}(t)=\frac{F_i'(t;q)}{iq^{i-1}\beta^t}=1,$$

thus, we have an honest probability measure $\mathbb{Q}(t)=\{\mathcal{Q}_{ij}(t)\}$. This measure defines a new stochastic process $W(t)$, $t\in\mathcal{T}$, called a Markov Q-Process (MQP) is a homogeneous continuous-time Markov chain with the state space $\mathcal{E}\subset\mathbb{N}$ (see [8]). In view of Markovian nature of this process, the transition functions $\mathcal{Q}_{ij}(t)$ satisfy the Kolmogorov-Chapman equations:

$$\mathcal{Q}_{ij}(t+\varepsilon)=\sum_{k\in\mathcal{E}}\mathcal{Q}_{ik}(\varepsilon)\mathcal{Q}_{kj}(t). \tag{2.3}$$

Thus, the random function $W(t)$ denotes the state size at the moment $t \in \mathcal{T}$ in MQP, so

$$\mathcal{Q}_{ij}(t) = \mathbb{P}_i\left\{W(t) = j\right\} = \frac{jq^{j-i}}{i\beta^t} P_{ij}(t). \tag{2.4}$$

Considered together, the equalities (1.2) and (2.4) entail the following important representation for the transition functions $\mathcal{Q}_{1j}(\varepsilon)$:

$$\mathcal{Q}_{1j}(\varepsilon) = \delta_{1j} + p_j\varepsilon + o(\varepsilon), \quad \text{as} \;\; \varepsilon \downarrow 0, \tag{2.5}$$

with the probability densities

$$p_0 = 0, \quad p_1 = a_1 - \ln\beta, \quad \text{and} \quad p_j = jq^{j-1}a_j \geqslant 0 \quad \text{for} \;\; j \in \mathcal{E}\backslash\{1\},$$

where $\{a_j\}$ are the evolution intensities of MBP $Z(t)$. It follows from (2.5) that the GF of intensities $\{p_j\}$ has the form

$$g(s) := \sum_{j\in\mathcal{E}} p_j s^j = s\left[f'(qs) - f'(q)\right]. \tag{2.6}$$

We see that $g(1) = 0$, so the infinitesimal GF $g(s)$ completely defines the process $W(t)$ and

$$0 < -p_1 = \sum_{j\in\mathcal{E}\backslash\{1\}} p_j < \infty.$$

In the following theorem we discuss basic properties of the transition matrix $\mathbb{Q}(t) = \{\mathcal{Q}_{ij}(t)\}$. Herewith we will follow methods and facts from the monograph by Anderson [1].

Theorem 1. *The transition matrix $\mathbb{Q}(t)$ of the MQP is standard. Its components $\mathcal{Q}_{ij}(t)$ are positive and uniformly continuous functions of $t \in \mathcal{T}$ for all $i, j \in \mathcal{E}$.*

Proof. According to the branching property (1.1) for the chain $Z(t)$, we see

$$P_{ij}(\varepsilon) = \delta_{ij} + ia_{j-i+1}\varepsilon + o(\varepsilon), \quad \text{as} \;\; \varepsilon \downarrow 0.$$

Hence, seeing representation (2.4)

$$\begin{cases} \mathcal{Q}_{ii}(\varepsilon) = 1 + (ia_1 - \ln\beta)\,\varepsilon + o(\varepsilon), \\ \mathcal{Q}_{ij}(\varepsilon) = jq^{j-i}a_{j-i+1}\varepsilon + o(\varepsilon), \end{cases} \quad \text{as} \;\; \varepsilon \downarrow 0, \tag{2.7}$$

for all $i, j \in \mathcal{E}$. It follows from (2.7) that

$$\begin{aligned} \sum_{j\in\mathcal{E}} \left|\mathcal{Q}_{ij}(\varepsilon) - \delta_{ij}\right| \;\; &= \sum_{j\in\mathcal{E}\backslash\{i\}} \mathcal{Q}_{ij}(\varepsilon) + \left|\mathcal{Q}_{ii}(\varepsilon) - 1\right| = \sum_{j\in\mathcal{E}\backslash\{i\}} \mathcal{Q}_{ij}(\varepsilon) + 1 - \mathcal{Q}_{ii}(\varepsilon) \leqslant \quad (2..1) \\ &\leqslant \;\; 2\left[1 - \mathcal{Q}_{ii}(\varepsilon)\right] \longrightarrow 0, \qquad \text{as} \;\; \varepsilon \downarrow 0. \end{aligned}$$

So the probability measure $\mathbb{Q}(t) = \{\mathcal{Q}_{ij}(t)\}$ is standard.

Positiveness of the functions $\mathcal{Q}_{ij}(t)$ is obvious owing to (2.7). Supposing $\varepsilon > 0$, it follows from equation (2.3) that

$$\mathcal{Q}_{ij}(t+\varepsilon) - \mathcal{Q}_{ij}(t) = \sum_{k\in\mathcal{E}} \mathcal{Q}_{ik}(\varepsilon)\mathcal{Q}_{kj}(t) - \mathcal{Q}_{ij}(t) = \sum_{k\in\mathcal{E}\backslash\{i\}} \mathcal{Q}_{ik}(\varepsilon)\mathcal{Q}_{kj}(t) - \mathcal{Q}_{ij}(t)\cdot\left[1 - \mathcal{Q}_{ii}(\varepsilon)\right].$$

The last relation gives

$$-\left[1-\mathcal{Q}_{ii}(\varepsilon)\right] \leqslant -\mathcal{Q}_{ij}(t)\cdot\left[1-\mathcal{Q}_{ii}(\varepsilon)\right] \leqslant \mathcal{Q}_{ij}(t+\varepsilon)-\mathcal{Q}_{ij}(t) \leqslant \sum_{k\in\mathcal{E}\setminus\{i\}} \mathcal{Q}_{ik}(t)\mathcal{Q}_{kj}(\varepsilon) \leqslant \leqslant \sum_{k\in\mathcal{E}\setminus\{i\}} \mathcal{Q}_{kj}(\varepsilon) = 1-\mathcal{Q}_{ii}(\varepsilon),$$

so $|\mathcal{Q}_{ij}(t+\varepsilon)-\mathcal{Q}_{ij}(t)| \leqslant 1-\mathcal{Q}_{ii}(\varepsilon)$. Similarly

$$|\mathcal{Q}_{ij}(t-\varepsilon)-\mathcal{Q}_{ij}(t)| = |\mathcal{Q}_{ij}(t)-\mathcal{Q}_{ij}(t-\varepsilon)| \leqslant 1-\mathcal{Q}_{ii}\left(t-(t-\varepsilon)\right) = 1-\mathcal{Q}_{ii}(\varepsilon).$$

Therefore we obtain $|\mathcal{Q}_{ij}(t+\varepsilon)-\mathcal{Q}_{ij}(t)| \leqslant 1-\mathcal{Q}_{ii}\left(|\varepsilon|\right)$ for any $\varepsilon \neq 0$ and for all $i,j \in \mathcal{E}$. The obtained relation implies that $\mathcal{Q}_{ij}(t)$ is a uniformly continuous function of $t \in \mathcal{T}$ because $\lim_{\varepsilon\downarrow 0} \mathcal{Q}_{ii}(\varepsilon) = 1$ for all $i \in \mathcal{E}$. □

Considering the property (1.3) it is easily seen that a GF version of (2.4) is

$$G_i(t;s) := \mathbb{E}_i s^{W(t)} = \sum_{j\in\mathcal{E}} \mathcal{Q}_{ij}(t)s^j = \frac{qs}{i\beta^t}\left[\frac{\partial}{\partial x}\left(\frac{F(t;x)}{q}\right)^i\right]_{x=qs},$$

or, more obviously, that

$$G_i(t;s) = \left[\frac{F(t;qs)}{q}\right]^{i-1} G(t;s), \tag{2.8}$$

where

$$G(t;s) := G_1(t;s) = \frac{s}{\beta^t}\left.\frac{\partial F(t;x)}{\partial x}\right|_{x=qs}.$$

It is known that $F\left(t;q\right) = q$ and $F'\left(t;q\right) = \beta^t$ (see [19, pp. 52–53]). Therefore seeing (2.8) we obtain once again that $\sum_{j\in\mathcal{E}} \mathcal{Q}_{ij}(t) = G_i(t;1) = 1$.

It can be easily seen that a GF version of the relation (2.5) is

$$G(\varepsilon;s) = s + g(s)\cdot\varepsilon + o(\varepsilon), \quad \text{as } \varepsilon\downarrow 0 \text{ and for all } 0 \leqslant s < 1. \tag{2.9}$$

According to formulas (1.3) and (2.8), it follows that the GF $G(t;s)$ satisfies the following functional equation:

$$G(t+\tau;s) = \frac{G\left(t;\widehat{F}(\tau;s)\right)}{G\left(0;\widehat{F}(\tau;s)\right)} G(\tau;s), \tag{2.10}$$

where $\widehat{F}(t;s) = F(t;qs)/q$ is the probability GF of sub-critical MBP. Using (2.9) and (2.10) for the difference $\Delta_\varepsilon G(t;s) = G(t-\varepsilon;s) - G(t+\varepsilon;s)$ we get

$$\Delta_\varepsilon G(t;s) = [\text{some function}]\cdot\varepsilon + o(\varepsilon), \quad \text{as } \varepsilon\downarrow 0,$$

for any $t \in \mathcal{T}$ and all $0 \leqslant s < 1$, which implies that $G(t;s)$ is differentiable.

3. Classification and ergodic behavior of the transition functions $\{\mathcal{Q}_{ij}(t)\}$

Note that evolution of MQP is ruled, in essence, by the structural parameter β. Afterwards we will see that there are two types of processes depending on a value of this parameter. It has been shown in [8] that

$$G(t;s) = s \,\exp\left\{\int_0^t h\left(\widehat{F}(\tau;s)\right) d\tau\right\},$$

where $h(s) = g(s)/s$. Putting together this formula and (2.8) we write

$$G_i(t;s) = s\left[\widehat{F}(t;s)\right]^{i-1} \exp\left\{\int_0^t h\left(\widehat{F}(\tau;s)\right) d\tau\right\}. \tag{3.1}$$

Let $\alpha := g'(1)$ be finite. Differentiating at the point $s = 1$, from (3.1) we get

$$\mathbb{E}_i W(t) = (i-1)\,\beta^t + \mathbb{E}W(t)$$

and

$$\mathbb{E}W(t) = \begin{cases} 1 + \gamma\,(1-\beta^t)\,, & \text{when } \beta < 1, \\ \alpha t + 1\,, & \text{when } \beta = 1. \end{cases} \tag{3.2}$$

Moreover we obtain the variance structure

$$\mathbb{D}_i W(t) = \begin{cases} \left[\gamma + (i-1)\,(1+\gamma)\,\beta^t\right](1-\beta^t)\,, & \text{when } \beta < 1, \\ \alpha\, it\,, & \text{when } \beta = 1. \end{cases} \tag{3.3}$$

Where $\gamma = \alpha/|\ln\beta|$ and $\mathbb{D}_i W(t) = \mathbb{D}\left[W(t)\,|W(0) = i\right]$ in (3.3).

Formula (3.2) implies that when $\beta = 1$

$$\mathbb{E}_i W(t) \sim \alpha t, \quad \text{as } t \to \infty,$$

and if $0 < \beta < 1$

$$\mathbb{E}_i W(t) \longrightarrow 1 + \gamma, \quad \text{as } t \to \infty.$$

So in the case of $\beta = 1$ the MQP has the transience property.

Definition 1. *We classify the MQP as* restrictive *if* $\beta < 1$ *and* explosive *if* $\beta = 1$.

Theorem 2. *The MQP is*

(i) positive if it is restrictive and $\alpha := g'(1)$ *is finite;*

(ii) null if it is explosive.

Proof. To prove assertion (i) from (2.11) we get

$$\ln \mathcal{Q}_{11}(t) = \int_0^t h\left(\widehat{F}(\tau;0)\right) d\tau = \int_0^{\widehat{F}(t;0)} \frac{h(x)}{\widehat{f}(x)}\,dx \longrightarrow \int_0^1 \frac{h(x)}{\widehat{f}(x)}\,dx,$$

since $\widehat{F}(t;0) \uparrow 1$ as $t \to \infty$, where $\widehat{f}(s) = f(qs)/q$ is an infinitesimal GF of a subcritical process. Herein we used the fact that $\lim_{s\downarrow 0}\left[G(t;s)/s\right] = \mathcal{Q}_{11}(t)$. The condition $\alpha < \infty$ implies that the integral in the right hand side converges. Hence $\lim_{t\to\infty} \mathcal{Q}_{11}(t) > 0$. For part (ii) we recall that in this case $q = 1$ and $h(s) = f'(s)$ if $\beta = 1$. Similarly

$$\ln \mathcal{Q}_{11}(t) = \int_0^t h\left(F(\tau;0)\right) d\tau = \int_0^{F(t;0)} \frac{h(x)}{f(x)}\,dx \longrightarrow \int_0^1 \frac{f'(x)}{f(x)}\,dx = -\infty.$$

So that $\lim_{t\to\infty} \mathcal{Q}_{11}(t) = 0$. □

Now let us recall the following statement.

Lemma 1 ([6]). *The following assertions are valid.*

- *Let* $a \neq 0$. *Then*

$$\frac{\partial F(t;s)}{\partial s} = \frac{|\ln \beta|}{f(s)} \mathcal{A}(s) \cdot \beta^t (1 + o(1)), \quad as \ \ t \to \infty, \tag{3.4}$$

where

$$\mathcal{A}(s) = (q - s) \exp\left\{ \int_s^q \left[\frac{1}{u - q} + \frac{|\ln \beta|}{f(u)} \right] du \right\}. \tag{3.5}$$

- *Let* $a = 0$. *If the second moment* $f''(1) =: 2b$ *is finite, then*

$$\frac{\partial F(t;s)}{\partial s} = \frac{b(1-s)^2}{f(s) \left[bt(1-s) + 1 \right]^2} (1 + o(1)), \quad as \ \ t \to \infty. \tag{3.6}$$

Putting together (3.1) and (3.4)–(3.6) and considering that $\lim_{t\to\infty} \widehat{F}(t;s) = 1$ uniformly for all $0 \leqslant s \leqslant r < 1$, we obtain the following theorem.

Theorem 3. *Let* $\alpha := g'(1)$ *is finite.*

1. *If MQP is restrictive, then*

$$G_i(t;s) = s \frac{|\ln \beta|}{f(qs)} \mathcal{A}(qs) (1 + o(1)), \quad as \ \ t \to \infty, \tag{3.7}$$

where the function $\mathcal{A}(s)$ *has the form (3.5).*

2. *If MQP is explosive, then*

$$G_i(t;s) = s \frac{2\alpha}{f(s)} \left[\frac{(1-s)}{(1-s)\alpha t + 2} \right]^2 (1 + o(1)), \quad as \ \ t \to \infty. \tag{3.8}$$

Since $\mathcal{Q}_{11}(t) = \lim_{s\downarrow 0} \left[G(t;s)/s \right]$, the relations (3.7) and (3.8) give the following local limit theorem.

Theorem 4. *Let* $\alpha := g'(1)$ *is finite.*

1. *In the restrictive case*

$$\mathcal{Q}_{11}(t) = \frac{|\ln \beta|}{a_0} \mathcal{A}(0) (1 + o(1)), \quad as \ \ t \to \infty,$$

if only the following condition is satisfied:

$$\int_0^q \frac{|\ln \beta| \cdot u - f(q-u)}{u f(q-u)} du = \ln \frac{q}{\mathcal{A}(0)} < \infty. \tag{$\mathcal{A}$}$$

2. *If MQP is explosive, then*

$$t^2 \mathcal{Q}_{11}(t) = \frac{2}{a_0 \alpha} \left(1 + \mathcal{O}\left(\frac{1}{t} \right) \right), \quad as \ \ t \to \infty.$$

Further on we establish limit properties of $\{\mathcal{Q}_{ij}(t)\}$ for all $i, j \in \mathcal{E}$. For the general MQP the following ratio limit property holds.

Theorem 5. *The limits*

$$\lim_{t\to\infty}\frac{\mathcal{Q}_{ij}(t)}{\mathcal{Q}_{11}(t)}=\omega_j \tag{3.9}$$

exist for all $i,j\in\mathcal{E}$, *and these determined by the GF*

$$\mathcal{W}(s)=\sum_{j\in\mathcal{E}}\omega_j s^j = s\,\exp\left\{\int_0^s\frac{|h(x)|}{\widehat{f}(x)}dx\right\}, \tag{3.10}$$

where $h(s)=g(s)/s$ *and* $\widehat{f}(s)=f(qs)/q$. *The limiting GF* $\mathcal{W}(s)$ *converges for all* $0\leqslant s<1$.

Proof. Let us consider the GF

$$\mathcal{W}_i(t;s)=\sum_{j\in\mathcal{E}}\frac{\mathcal{Q}_{ij}(t)}{\mathcal{Q}_{11}(t)}s^j=\frac{1}{\mathcal{Q}_{11}(t)}G_i(t;s)=\left[\widehat{F}(t;s)\right]^{i-1}\mathcal{W}(t;s), \tag{3.11}$$

where

$$\mathcal{W}(t;s)=\sum_{j\in\mathcal{E}}\frac{\mathcal{Q}_{1j}(t)}{\mathcal{Q}_{11}(t)}s^j.$$

It follows from (3.11) that it suffices to consider the case $i=1$ because $\widehat{F}(t;s)\uparrow 1$ as $t\to\infty$ uniformly for all $0\leqslant s\leqslant r<1$. So write

$$\mathcal{W}(t;s)=s\exp\left\{\int_0^t\left[h\left(\widehat{F}(u;s)\right)-h\left(\widehat{F}(u;0)\right)\right]du\right\}.$$

One can choose $\tau\in\mathcal{T}$ for any $0\leqslant s<1$ so that $s=\widehat{F}(\tau;0)$. On the other hand we know that $\widehat{F}\left(t;\widehat{F}(\tau;0)\right)=\widehat{F}(t+\tau;0)$ ([19, p. 24]). Therefore we obtain equalities

$$\mathcal{W}(t;s)=s\exp\left\{\int_\tau^{t+\tau}h\left(\widehat{F}(u;0)\right)du-\int_0^t h\left(\widehat{F}(u;0)\right)du\right\}=$$
$$=s\exp\left\{\int_0^\tau\left[h\left(\widehat{F}(t;\widehat{F}(u;0))\right)-h\left(\widehat{F}(u;0)\right)\right]du\right\}=s\exp\left\{\int_0^s\frac{h\left(\widehat{F}(t;x)\right)-h(x)}{\widehat{f}(x)}dx\right\},$$

where $\widehat{f}(s):=f(qs)/q$. In the last step we have used the Kolmogorov backward equation

$$\frac{\partial F(t;s)}{\partial t}=f\left(F(t;s)\right),\quad \text{for all}\ \ 0<s<1,$$

(see [19, pp. 27–30]). To get to (3.10) it suffices to take limit as $t\to\infty$ in the obtained relation for $\mathcal{W}(t;s)$ being that $\widehat{F}(t;s)\to 1$ and $h(1)=0$. The assertion (3.9) follows now from the continuity theorem for a GF. Lastly it is easy to see that $\mathcal{W}(s)<\infty$ for all $0\leqslant s<1$. □

Aggregating Theorems 4 and 5 yields the following

Theorem 6. *Let* $\alpha:=g'(1)$ *is finite.*

1. *If in the restrictive case the condition* $[\mathcal{A}]$ *is satisfied, then*

$$\mathcal{Q}_{ij}(t)=\omega_j\frac{|\ln\beta|}{a_0}\mathcal{A}(0)\left(1+o(1)\right),\quad \text{as}\ \ t\to\infty.$$

2. If MQP is explosive, then

$$t^2\mathcal{Q}_{ij}(t) = \omega_j \frac{2}{a_0\alpha}\left(1 + \mathcal{O}\left(\frac{1}{t}\right)\right), \quad as \;\; t \to \infty.$$

Now using the Kolmogorov-Chapman equation (2.3) we obtain

$$\frac{\mathcal{Q}_{ij}(t+\tau)}{\mathcal{Q}_{11}(t+\tau)} \cdot \frac{\mathcal{Q}_{11}(t+\tau)}{\mathcal{Q}_{11}(t)} = \sum_{k\in\mathcal{E}} \frac{\mathcal{Q}_{ik}(t)}{\mathcal{Q}_{11}(t)}\mathcal{Q}_{kj}(\tau).$$

On the other hand setting $s = 0$ in (2.10) we see that $\mathcal{Q}_{11}(t+\tau)/\mathcal{Q}_{11}(t) \to 1$ as $t \to \infty$. Hence we get the following invariance equation for $\{\omega_j\}$:

$$\omega_j = \sum_{k\in\mathcal{E}} \omega_k \mathcal{Q}_{kj}(t), \quad \text{for all} \;\; t \in \mathcal{T}. \tag{3.12}$$

The GF version of (3.12) is

$$\mathcal{W}\left(\widehat{F}(t;s)\right) = \frac{\widehat{F}(t;s)}{G(t;s)}\mathcal{W}(s), \quad \text{for} \;\; 0 \leqslant s < 1,$$

the functional equation of generalized Schroeder form. So the set $\{\omega_j\}$ is the ergodic invariant measure for MQP.

We conclude the paper stating the following limit theorem.

Theorem 7. *Let* $\alpha := g'(1)$ *is finite.*

1. If MQP is restrictive, then the variable $W(t)$ tends in mean square and with probability one to the random variable $\boldsymbol{w}$ having the finite mean and variance:

$$\mathbb{E}\boldsymbol{w} = 1 + \gamma \quad and \quad \mathbb{D}\boldsymbol{w} = \gamma.$$

2. If MQP is explosive, then for any $x > 0$

$$\lim_{t\to\infty} \mathbb{P}\left\{\frac{W(t)}{\mathbb{E}W(t)} \leqslant x\right\} = 1 - e^{-2x} - 2xe^{-2x}.$$

For proof see [8].

4. Conclusion remarks

The paper is devoted to the study of a population process which is defined as the long-living continuous-time Markov branching process. This is a homogeneous Markov chain called Markov Q-process (MQP). In the discrete-time situation the same process was defined in [2]. We see that the structural parameter $\beta = f'(q)$ plays the regulating role. In fact the long-time behavior of MQP depends on this parameter and unlike the branching process it is classified only in two types. In the study of transition functions $\mathcal{Q}_{ij}(t)$ we essentially use asymptotic properties of the first derivative of the probability GF of Markov Branching process. As infinitesimal GF $\widehat{f}(s) = f(qs)/q$ generates a sub-critical branching process, referring to [19, pp. 54–57] we see that the condition $[\mathcal{A}]$ is equivalent to the convergence of the series $\sum_{j\in\mathbb{N}} \widehat{a}_j j \ln j$, where $\widehat{a}_j = a_j q^{j-1}$. Therefore by definition of Q-process the first assertions of Theorems 4 and 6 hold under the condition $\sum_{j\in\mathcal{E}} p_j \ln j < \infty$. The Ratio limit property (Theorem 5) for transition functions states the existence of invariant measure for MQP without any moment assumptions. Theorem 7 shows the limit properties of states of process. The considered model will be extended to the age-dependent Bellman-Harris process case in our future studies.

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О предельной структуре марковского ветвящегося процесса с непрерывным временем

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В статье мы исследуем предельные свойства марковского ветвящегося процесса при условии не вырождения в далеком будущем. Предельная вероятностная мера определяет случайный процесс, называемый марковский Q-процесс. Исследуем структурные и асимптотические свойства марковского Q-процесса. Изучаем асимптотические свойства переходных вероятностей и их сходимость к инвариантным мерам.

Ключевые слова: марковский ветвящийся процесс, марковский Q-процесс, переходные вероятности, инвариантные меры.