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Three Families of Functions of Complexity One

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Three rare families of functions of analytic complexity one were studied. Main results are the description of linear differential equations with solutions of complexity one (Theorem 2), the description of L-pairs of complexity one (Theorem 5), the description of $O(2)$-simple functions (Theorem 7).

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Introduction

The complexity of analytic functions of several variables has been studied in [1–5]. A method of measuring the complexity of an analytic function in two variables, possibly multivalued, is proposed in [3]. For any analytic function of two variables $z(x, y)$ one can define its complexity $N(z)$. It attains values $0, 1, \dots, \infty$ and is preserved under any analytic continuation. Functions of one variable have complexity $N(z) = 0$. Complexity one have functions $z(x, y)$ of two variables if they have the form $z = c(a(x) + b(y))$, where a, b, c are nonconstant functions of one variable, and so on. In other words, for a function z of two variables we write $N(z) = n$ if z can be represented in the form $C(A(x, y) + B(x, y))$, where C is a function of one variable, and the complexity of A and B is less than n, and there is no such representation with a smaller value of n. This produces an increasing system of classes of functions

$$Cl_0 \subset Cl_1 \subset Cl_2 \dots .$$

If a function does not belong to any of these classes we write $N(z) = \infty$. Each of the above classes is defined by differential-algebraic relations. For example, Cl_0 is defined by the condition $z'_x\, z'_y = 0$, and Cl_1 by the condition

$$\delta(z) = z'_x z'_y (z'''_{xxy} z''_y - z'''_{xyy} z'_x) + z''_{xy}((z'_x)^2 z''_{yy} - (z'_y)^2 z''_{xx}) = 0. \tag{1}$$

The differential polynomial $\delta(z)$ is the numerator of the expression $(\ln(z'_y/z'_x))''_{xy}$.

1. Linear equations with constant coefficients

Consider the pair of functions $(z_1 = \mathrm{e}^{ax+by},\ z_2 = \mathrm{e}^{px+qy})$. If $ab = pq = 0$ then we have $\max(N(z_1), N(z_2)) = 0$. If it is not so, then $\max(N(z_1), N(z_2)) = 1$. What condition on $(a,\, b,\, p,\, q)$ provides that the complexity of all linear combinations of z_1 and z_2 does not exceed one? The answer gives

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Lemma 1. *Let $(ab, pq) \neq 0$. The complexity of all linear combinations of z_1 and z_2 does not exceed 1 only in three cases (1) $p = a$, (2) $q = b$, (3) $aq = bp$.*

Proof. The condition (1) for $z = t_1 z_1 + t_2 z_2$ has the form

$$(b-q)\,(a-p)\,(qa-bp)\left(\left(e^{ax+by}\right)^2 ab{t_2}^2 - \left(e^{px+qy}\right)^2 pq{t_1}^2\right) e^{ax+by} e^{px+qy}\, t_1\, t_2 = 0.$$

So the lemma is proved. □

There is a curious corollary from this lemma. Consider a homogeneous linear equation with constant coefficients $P(D)(z(x,y)) = 0$ and let $\mathcal{L}$ be the space of its analytic solutions. The complexity $N(\mathcal{L})$ of the space of solutions $\mathcal{L}$ is the maximum (finite or infinite) of the solutions' complexities.

Theorem 2. *If $N(\mathcal{L}) \leqslant 1$, then the equation $P(D)(z(x,y)) = 0$ has one of the forms:*
(1) $z'_x - Az = 0$, solutions have the form $z = e^{Ax}\, b(y)$,
(2) $z'_y - Bz = 0$, solutions have the form $z = e^{By}\, a(x)$,
(3) $kz'_x + lz'_y = 0$, solutions have the form $z = c(lx - ky)$,
(4) $z''_{xy} = 0$, solutions have the form $z = a(x) + b(y)$.

Proof. Let $\chi = \{P(\lambda_1, \lambda_2) = 0\}$ be the characteristic set of this equation and let ($z_1 = e^{ax+by}$, $z_2 = e^{px+qy}$) be two solutions, i.e. (a, b), $(p, q) \in \chi$. It follows from Lmma 1 that χ belongs to a vertical line (case (1)) or to a horizontal line (case (2)), or to a line passing through the origin (case (3)). There is another case (case (4)) outside Lemma 1. In this case χ is the coordinate cross and $N(z_1) = N(z_2) = 0$. The characteristic polynomials have one of the forms: in case (1) $P(\lambda_1, \lambda_2) = (\lambda_1 - A)^{n_1}$, in case (2) $P(\lambda_1, \lambda_2) = (\lambda_2 - B)^{n_2}$, in case (3) $P(\lambda_1, \lambda_2) = (k\lambda_1 + l\lambda_2)^{n_3}$, in case (4) $P(\lambda_1, \lambda_2) = (\lambda_1\lambda_2)^{n_4}$. In all cases it is not difficult to solve these differential equations. The condition $N(\mathcal{L}) \leqslant 1$ is true only for $n_1 = n_2 = n_3 = n_4 = 1$. The theorem is proved. □

Note that if the multiplicities (n_1, $n_2\, n_3$, n_4) are arbitrary, then the complexities of the space of solutions are finite but greater than one.

2. L-pairs

A collection of functions forms a linear space if this collection is closed under addition and multiplication by a constant (complex numbers). Multiplication by a nonzero constant does not change the complexity of a function: $N(\lambda z(x,\, y)) = N(z(x,\, y))$. This means that a nonzero function of complexity 1 generates a linear space lying in Cl_1. As for a sum of two functions, if $N(z_1(x, y))$ and $N(z_2(x, y))$ do not exceed n then $N(z_1(x, y) + z_2(x, y)) \leqslant (n + 1)$. It can be shown that in 'general position' this inequality becomes the equality. There is a simple example: $N(xy) = 1$, $N(x^2) = 0$, then $N(xy + x^2) = 2$. But there exist exceptional pairs. For example $N(xy) = 1$, $N(x + y) = 1$ and $N(t_1(xy) + t_2(x + y)) = 1$ for any (t_1, t_2).

Definition. *We call a pair of functions $(z_1(x, y),\ z_2(x, y)$ an L-pair of complexity n if*

$$N(t_1(z_1(x,\, y) + t_2 z_2(x,\, y)) \leqslant \max(N(z_1),\, N(z_2)) = n \text{ for any } (t_1,\, t_2).$$

Here we assume that z_1 and z_2 have analytic germs at the same point. Lemma 1 then becomes a classification of L-pairs of a special form.

Let us formulate several obvious statements.

Statement 3. *Two functions (z_1, z_2) is an L-pair of complexity zero if and only if they are functions of the same argument x or y.*

Statement 4. *The property of being an L-pair is invariant under the action of*
(1) the pseudo-group of transformations $\{(x \to p(x),\ y \to q(y))\}$,
(2) the change $\{(x \to y,\ y \to x)\}$,
(3) the affine group of transformations of $(z_1,\ z_2)$*-plane.*

The pseudo-group generated by the transformations (1), (2) and (3) we denote by $\mathcal{G}$. The description of L-pairs is natural to give up to the $\mathcal{G}$-action.

Now let us turn back to Lemma 1. If we assume only that $N(z_1 + z_2) \leqslant 1$, we have the same description. Indeed, the condition (1) for $z = z_1 + z_2$ has the form

$$(b-q)\,(a-p)\,(qa-bp)\left(\left(\mathrm{e}^{ax+by}\right)^2 ab - \left(\mathrm{e}^{px+qy}\right)^2 pq\right)\mathrm{e}^{ax+by}\mathrm{e}^{px+qy} = 0,$$

and it is enough to reach the conclusion of Lemma 1. Taking this into account we modify the definition.

Definition. *We call a pair* $(z_1(x,y),\ z_2(x,y))$ *a pair of complexity* n*, if* $N(z_1(x,y) + z_2(x,y)) \leqslant \max(N(z_1), N(z_2)) = n$.

We can strengthen Lemma 1 as follows.

Lemma 1'. *Let* $(ab, pq) \neq 0$*. The pair* $(z_1 = \mathrm{e}^{ax+by},\ z_2 = \mathrm{e}^{px+qy})$ *is a pair of complexity one only in three cases (1)* $p = a$*, (2)* $q = b$*, (3)* $aq = bp$.

Now we turn to the construction of an arbitrary L-pair of complexity one. Their description is given in the form of a list of cases that are specified and denoted in the course of exposition.

Let z_1 and z_2 be two functions of complexity not exceeding 1, that is $z_1 = c_1(a_1(x) + b_1(y))$, $z_2 = c_2(a_2(x) + b_2(y))$. Assume also that $\max(N(z_1), N(z_2)) = 1$, i.e one of the functions has complexity one, let it be z_2. Then a_2, b_2, and r are non constant and locally invertible at a general point. Replace x by $a_2^{-1}(x)$ and y by $b_2^{-1}(y)$. The condition takes the form

$$c(a(x) + b(y)) + t \cdot r(x+y) \in Cl_1 \quad \forall t, \quad r' \neq 0. \tag{2}$$

Let the first term have complexity zero, this is **Case (01)**. Then the first term is a function of one variable, denote it by $a(x)$. From (1) for $a(x) + t \cdot r(x+y)$ we get

$$\begin{aligned} a_1 r_1 r_3 &= 2\, a_1 {r_2}^2 - a_2 r_1 r_2, \\ r_1 r_3 &= {r_2}^2. \end{aligned}$$

By lower indices we denote orders of derivatives. If $r_2 = 0$ then $r(x+y) = k \cdot (x+y) + l$ and $a(x)$ is arbitrary. This is **Case (01.1)**. This pair is equivalent to $(a(x),\ (x+y))$.
If r_2 is not zero then from the second equation we have $r(t) = \rho \cdot e^{mt} + \tilde{\rho}$. And from the first equation we have $a(x) = \alpha \cdot e^{mt} + \tilde{\alpha}$. This pair is equivalent to $(kx,\ xy)$. We call this **Case (01.2)**

Consider now **Case (11)** when both terms have complexity one. This means that a', b', c', r' are nonconstant functions. From (1) for $c(a(x) + b(y)) + t \cdot r(x+y)$ we get

$$\begin{gathered}
{a_1}^2 b_1 c_3 {r_1}^2 - a_1 {b_1}^2 c_3 {r_1}^2 - {a_1}^2 c_2 r_1 r_2 - a_1 b_2 c_2 {r_1}^2 + a_2 b_1 c_2 {r_1}^2 + \\
+ {b_1}^2 c_2 r_1 r_2 - a_1 c_1 r_1 r_3 + 2\, a_1 c_1 {r_2}^2 - a_2 c_1 r_1 r_2 + b_1 c_1 r_1 r_3 - 2\, b_1 c_1 {r_2}^2 + b_2 c_1 r_1 r_2 = 0, \\
-a_1^3 b_1 c_1 c_3 {r_1}^2 - {a_1}^3 b_1 {c_2}^2 {r_1}^2 - a_1 {b_1}^3 c_1 c_3 {r_1}^2 + a_1 {b_1}^3 {c_2}^2 {r_1}^2 - 2\, {a_1}^2 b_2 c_1 c_2 {r_1}^2 + 2\, a_2 {b_1}^2 c_1 c_2 {r_1}^2 - \\
- {a_1}^2 {c_1}^2 r_1 r_3 + {a_1}^2 {c_1}^2 {r_2}^2 + 2\, a_1 b_2 {c_1}^2 r_1 r_2 - 2\, a_2 b_1 {c_1}^2 r_1 r_2 + {b_1}^2 {c_1}^2 r_1 r_3 - {b_1}^2 {c_1}^2 {r_2}^2 = 0, \\
{a_1}^3 {b_1}^2 c_1 c_3 r_1 - 2\, {a_1}^3 {b_1}^2 {c_2}^2 r_1 - {a_1}^2 {b_1}^3 c_1 c_3 r_1 + 2\, {a_1}^2 {b_1}^3 {c_2}^2 r_1 + {a_1}^3 b_1 c_1 c_2 r_2 - {a_1}^3 b_2 c_1 c_2 r_1 - \\
- a_1 {b_1}^3 c_1 c_2 r_2 + a_2 {b_1}^3 c_1 c_2 r_1 - {a_1}^2 b_1 {c_1}^2 r_3 + {a_1}^2 b_2 {c_1}^2 r_2 + a_1 {b_1}^2 {c_1}^2 r_3 - a_2 {b_1}^2 {c_1}^2 r_2 = 0.
\end{gathered} \tag{3}$$

Eliminating c_3 from the first and second equations and then from the first and third equations, we get two equations. Each of them is a quadratic form in (c_1, c_2) with a common factor $a_1 b_1 r_1 (a_1 - b_1)^2$. In our case this factor can be equal to zero only if $a_1 - b_1 = 0$ (**Case (11.1)**). This pair has the form $(c(x+y),\, r(x+y))$.
Assume now $a_1 - b_1 \neq 0$. After dividing by the common factor we get

$$\begin{gathered} {a_1}^2 b_1 {c_2}^2 {r_1}^2 + a_1 {b_1}^2 {c_2}^2 {r_1}^2 - {a_1}^2 c_1 c_2 r_1 r_2 - 2\, a_1 b_1 c_1 c_2 r_1 r_2 + a_1 b_2 c_1 c_2 {r_1}^2 + \\ + a_2 b_1 c_1 c_2 {r_1}^2 - {b_1}^2 c_1 c_2 r_1 r_2 + a_1 {c_1}^2 {r_2}^2 - a_2 {c_1}^2 r_1 r_2 + b_1 {c_1}^2 {r_2}^2 - b_2 {c_1}^2 r_1 r_2 = 0, \\ 2 {a_1}^2 {b_1}^2 {c_2}^2 {r_1}^2 - 2 {a_1}^2 b_1 c_1 c_2 r_1 r_2 + {a_1}^2 b_2 c_1 c_2 {r_1}^2 - 2 a_1 {b_1}^2 c_1 c_2 r_1 r_2 + \\ + a_2 2 {b_1}^2 c_1 c_2 {r_1}^2 + 4 a_1 b_1 {c_1}^2 {r_2}^2 - 2 a_1 b_2 {c_1}^2 r_1 r_2 - 2 a_2 b_1 {c_1}^2 r_1 r_2 = 0. \end{gathered} \tag{4}$$

After elimination of c_2/c_1 we have

$$(a_1 - b_1)^3\, a_1 b_1 {r_1}^6 a_2 b_2 r_2 \left({a_1}^2 b_1 r_2 - {a_1}^2 b_2 r_1 - a_1 {b_1}^2 r_2 + a_2 {b_1}^2 r_1\right) = 0. \tag{5}$$

Consider all the possibilities separately.

Case (11.2). One of the functions $a'' = 0$ and $b'' = 0$ is linear, let it be b, then $b(y) = k \cdot y + l$, where $k \neq 0$. Replace $k \cdot y + l$ by y and $k \cdot x - l$ by x, then $r(t)$ becomes $r(t/k)$. The condition (1) for $c(a(x)+y) + t \cdot r(x+y)$ takes the form

$$\begin{gathered} {a_1}^3 c_1 c_2 r_2 + {a_1}^3 c_1 c_3 r_1 - 2\, {a_1}^3 {c_2}^2 r_1 - {a_1}^2 {c_1}^2 r_3 - {a_1}^2 c_1 c_3 r_1 + \\ 2\, {a_1}^2 {c_2}^2 r_1 + a_1 {c_1}^2 r_3 - a_1 c_1 c_2 r_2 - a_2 {c_1}^2 r_2 + a_2 c_1 c_2 r_1 = 0, \\ {a_1}^3 c_1 c_3 {r_1}^2 - {a_1}^3 {c_2}^2 {r_1}^2 - {a_1}^2 {c_1}^2 r_1 r_3 + {a_1}^2 {c_1}^2 {r_2}^2 - a_1 c_1 c_3 {r_1}^2 + \\ a_1 {c_2}^2 {r_1}^2 - 2\, a_2 {c_1}^2 r_1 r_2 + 2\, a_2 c_1 c_2 {r_1}^2 + {c_1}^2 r_1 r_3 - {c_1}^2 {r_2}^2 = 0, \\ -{a_1}^2 c_2 r_1 r_2 + {a_1}^2 c_3 {r_1}^2 - a_1 c_1 r_1 r_3 + 2\, a_1 c_1 {r_2}^2 - a_1 c_3 {r_1}^2 - \\ a_2 c_1 r_1 r_2 + a_2 c_2 {r_1}^2 + c_1 r_1 r_3 - 2\, c_1 {r_2}^2 + c_2 r_1 r_2 = 0. \end{gathered}$$

The expressions for c_3 from each of these equations are fractions with the denominators

$${a_1}^2 c_1 r_1 (a_1 - 1)\,, \quad a_1 c_1 {r_1}^2 \left({a_1}^2 - 1\right), \quad a_1 {r_1}^2 (a_1 - 1)\,.$$

There are two possibilities for vanishing of one of the denominators: $a_1 = 1$ or $a_1 = -1$. In our case $a_1 \neq b_1$, hence we have only the second possibility $a_1 = -1$, $a(x) = -x + \alpha$. The condition (1) yields

$$\begin{gathered} -{c_1}^2 r_3 - c_1 c_3 r_1 + 2\, {c_2}^2 r_1 = 0, \\ c_1 r_1 r_3 - 2\, c_1 {r_2}^2 + c_3 {r_1}^2 = 0, \end{gathered}$$

where c and r are functions of two independent variables $x - y$ and $x + y$.
Separating the variables and solving the differential equations we arrive at **Case (11.2.1)** : $c(-x+y) = \gamma e^{m(-x+y)} + \tilde{\gamma}, \quad r(x+y) = \rho e^{\pm m(x+y)} + \tilde{\rho}$. The pair then has the form $(y/x, xy)$.
If $a_1 \neq \pm 1$, we can eliminate c_3 from (5) to get two quadratic form in (c_1, c_2):

$$\begin{gathered} (c_2 r_1 - c_1 r_2)\left(c_2 {a_1}^3 r_1 + c_2 {a_1}^2 r_1 - c_1 {a_1}^2 r_2 - c_1 a_1 r_2 + a_2 r_1\right) = 0, \\ (c_2 r_1 - c_1 r_2)\left(2\, c_2 {a_1}^2 r_1 - 2\, c_1 a_1 r_2 + c_1 a_2 r_1\right) = 0 \end{gathered}$$

with the common factor $(c_2 r_1 - r_2 c_1)$. If this factor is equal to zero (**Case (11.2.2)**), then we can separate the variables and, taking into account that the Jacobian of the change ($t = a(x) + y$; $s = x + y$) does not vanish, we see that both logarithmic derivatives are equal to the same constant m. From this we get $z_1 = \gamma e^{m(a(x)+y)} + \tilde{\gamma}, \quad z_2 = \rho e^{m(x+y)} + \tilde{\rho}$. The pair has the form $(a(x)y, xy)$.

Otherwise, (**Case (11.2.3)**), dividing out the common factor and eliminating c_2/c_1 from two linear forms, we get ${a_1}^2 a_2 {r_1}^2 (a_1 - 1) = 0$. It vanishes only if $a_2 = 0$, a_1 is then the constant A. In this case $Ac_2/c_1 = r_2/r_1$, and $z_1 = c(Ax+y) = \gamma e^{\frac{m}{A}(Ax+y)}, \quad z_2 = r(x+y) = \rho e^{m(x+y)}$. The pair has the form $(x^k y, xy)$
We see that Cases (11.2.1) and (11.2.3) are subcases of Case (11.2.2). Thus, in **Case (11.2)** the pair has the form $(a(x)y, xy)$.

In **Case (11.3)** $r_2 = 0$, i.e. $r(x+y) = \rho(x+y) + \tilde{\rho}$, where $\rho \neq 0$. By replacing x with $\rho x + \tilde{\rho}$ and y with ρy we obtain $r(x+y) = x+y$. The condition (1) for $c(a(x)+b(y)) + (x+y)$ has the form

$$
\begin{aligned}
{a_1}^3{b_1}^2c_1c_3 - 2\,{a_1}^3{b_1}^2{c_2}^2 - {a_1}^2{b_1}^3c_1c_3 + 2\,{a_1}^2{b_1}^3{c_2}^2 - {a_1}^3b_2c_1c_2 + a_2{b_1}^3c_1c_2 = 0,\\
{a_1}^3b_1c_1c_3 - {a_1}^3b_1{c_2}^2 - a_1{b_1}^3c_1c_3 + a_1{b_1}^3{c_2}^2 - 2\,{a_1}^2b_2c_1c_2 + 2\,a_2{b_1}^2c_1c_2 = 0,\\
{a_1}^2b_1c_3 - a_1{b_1}^2c_3 - a_1b_2c_2 + a_2b_1c_2 = 0.
\end{aligned}
$$

By eliminating c_3 and c_2/c_1, we get

$$(a_1 - b_1)\left({a_1}^2b_2 - a_2{b_1}^2\right) = 0.$$

It may vanish only because of the second factor, therefore, separating the variables we get $a_2/a_1^2 = b_2/b_1^2 = -m$ where m is a constant. Then

$$a(x) + b(y) = \frac{1}{m}(\ln(mx+\alpha) + \ln(my+\beta) + \ln(n)),$$

and three equations for $c(t)$ are

$$c_3 = mc_2^2, \quad c_3c_1 = c_2^2, \quad mc_1c_2 + c_1c_3 - 2c_2^2 = 0.$$

Consequently, $c(t) = \gamma e^{mt} + \tilde{\gamma}$, and the pair has the form $(xy, x+y)$.

Case (11.4)

$${a_1}^2b_1r_2 - {a_1}^2b_2r_1 - a_1{b_1}^2r_2 + a_2{b_1}^2r_1 = 0. \tag{6}$$

From this we get

$$\frac{r_2}{r_1} = \frac{{a_1}^2b_2 - a_2{b_1}^2}{a_1b_1(a_1 - b_1)} \tag{7}$$

(the denominator is not zero). The condition that $\frac{r_2}{r_1}$ is a function of $x+y$, namely the equality of its derivatives with respect to x and y, is

$$-{a_1}^4b_1b_3 + {a_1}^4{b_2}^2 + {a_1}^3{b_1}^2b_3 - 2\,{a_1}^3b_1{b_2}^2 - {a_1}^2a_3{b_1}^3 + 2\,a_1{a_2}^2{b_1}^3 + a_1a_3{b_1}^4 - {a_2}^2{b_1}^4 = 0 \tag{8}$$

We can decrease the order of equation (8) twice. First, putting $a_1 = a'(x) = A,\ b_1 = b'(y) = B$. Second, introducing $a_2 = a''(x) = P(A),\ b_2 = b''(y) = Q(B)$. In this notation we have $a_3 = a'''(x) = P'(A)\,P(A),\ b_3 = b'''(y) = Q'(B)\,Q(B)$ and we can write (8) as

$$
\begin{aligned}
-A^4B\left(\frac{\mathrm{d}}{\mathrm{d}B}G(B)\right)G(B) + A^4(G(B))^2 + A^3B^2\left(\frac{\mathrm{d}}{\mathrm{d}B}G(B)\right)G(B) -\\
-2\,A^3B(G(B))^2 - A^2\left(\frac{\mathrm{d}}{\mathrm{d}A}F(A)\right)F(A)B^3 + 2\,A(F(A))^2B^3 +\\
+A\left(\frac{\mathrm{d}}{\mathrm{d}A}F(A)\right)F(A)B^4 - (F(A))^2B^4 = 0.
\end{aligned}
$$

After the substitution $f(A)=\sqrt{F(A)}$, $\quad g(B)=\sqrt{G(B)}$ we previous equation becomes linear

$$-A^4B\frac{\mathrm{d}}{\mathrm{d}B}g\left(B\right)+2\,A^4g\left(B\right)+A^3B^2\frac{\mathrm{d}}{\mathrm{d}B}g\left(B\right)-4\,A^3Bg\left(B\right)-$$
$$A^2B^3\frac{\mathrm{d}}{\mathrm{d}A}f\left(A\right)+4\,Af\left(A\right)B^3+AB^4\frac{\mathrm{d}}{\mathrm{d}A}f\left(A\right)-2\,f\left(A\right)B^4=0.$$

From this we find $\dfrac{\mathrm{d}}{\mathrm{d}B}g$ and write the condition of its independence from A:

$$-A^4B^2\frac{\mathrm{d}^2}{\mathrm{d}A^2}f\left(A\right)+2\,A^3B^3\frac{\mathrm{d}^2}{\mathrm{d}A^2}f\left(A\right)-A^2B^4\frac{\mathrm{d}^2}{\mathrm{d}A^2}f\left(A\right)+6\,A^3B^2\frac{\mathrm{d}}{\mathrm{d}A}f\left(A\right)-10\,A^2B^3\frac{\mathrm{d}}{\mathrm{d}A}f\left(A\right)+$$
$$+4\,AB^4\frac{\mathrm{d}}{\mathrm{d}A}f\left(A\right)+2\,A^4g\left(B\right)-12\,A^2B^2f\left(A\right)+16\,Af\left(A\right)B^3-6\,f\left(A\right)B^4=0.$$

Now we express $g(B)$ and write the condition of its independence from A:

$$A^3\frac{\mathrm{d}^3}{\mathrm{d}A^3}f\left(A\right)-6\,A^2\frac{\mathrm{d}^2}{\mathrm{d}A^2}f\left(A\right)+18\,A\frac{\mathrm{d}}{\mathrm{d}A}f\left(A\right)-24\,f\left(A\right)=0.$$

By looking for solutions of the form $f(A)=A^m$, we get the equation

$$m(m-1)(m-2)-6m(m-1)+18m-24=(m-2)(m-3)(m-4).$$

Hence, a general solution to (9) is $f(A)=l_1A^4+m_1A^3+n_1A^2$. By eliminating $f(A)$ from (9), we obtain $g(B)=l_2B^4+m_2B^3+n_2B^2$. Substituting these $f(A)$ and $g(B)$ in (9), we get $l_1=l_2,\ m_1=m_2,\ n_1=n_2$. Finally, $f(A)=lA^4+mA^3+nA^2,\quad g(B)=lB^4+mB^3+nB^2$. We see that $\alpha(x)=a'(x)$ and $\beta(y)=b'(y)$ satisfy the same differential equation

$$\frac{\mathrm{d}\alpha}{\mathrm{d}x}=\sqrt{l\alpha^4+m\alpha^3+n\alpha^2},\qquad \frac{\mathrm{d}\beta}{\mathrm{d}y}=\sqrt{l\beta^4+m\beta^3+n\beta^2}. \tag{9}$$

Since a and b are not linear, we may assume that the constants l, m, and n are not zeros simultaneously. Thus, if $l=n=0$ and $m\neq 0$ (**Case (11.4.1)**), then

$$\int\frac{dt}{t\sqrt{mt}}=\frac{-2}{\sqrt{t}}.$$

Therefore

$$a'(x)=\alpha(x)=\frac{4}{m(x+C)^2},\quad a(x)=-\frac{4}{m(x+C)}+\tilde{C},\quad a''(x)=\frac{-8}{m(x+C)^3}.$$

Analogously,

$$b'(x)=\beta(x)=\frac{4}{m(y+D)^2},\quad b(y)=-\frac{4}{m(y+D)}+\tilde{D},\quad b''(y)=\frac{-8}{m(y+D)^3}.$$

Now, from (7) we get

$$\frac{r_2}{r_1}=\frac{a_1^2b_2-a_2b_1^2}{a_1b_1(a_1-b_1)},$$

and then we have $r(t)=-\dfrac{\rho}{t+C+D}$. Computing c_2/c_1 from any of (4) and substituting the expression for r_2/r_1, we get

$$\frac{c_2}{c_1}=\frac{a_1b_2-a_2b_1}{a_1b_1(a_1-b_1)}$$

and

$$c(a(x)+b(y)) = \frac{1}{\frac{1}{x+C}+\frac{1}{y+D}}.$$

Thus, the pair has the form

$$\Big(z_1 = \frac{xy}{x+y}, \quad z_2 = \frac{1}{x+y}\Big).$$

If $l \neq 0$ or $n \neq 0$ (**Case (11.4.2)**), then

$$\int \frac{dt}{t\sqrt{lt^2+mt+n}} = \frac{-2}{\sqrt{lt^2+n}}\text{arctgh}\left(\frac{\sqrt{lt^2+mt+n}}{\sqrt{lt^2+n}}\right),$$

and we get $a'(x) = \alpha(x)$ and $b'(y) = \beta(y)$ as inversion of the integrals, and $a(x)$ and $b(y)$ by one more integration. As in the previous case, from (7) we get $r(t)$ and $c(t)$ from any relation of (4).

Finally, we have the theorem.

Theorem 5. *Let $z_1(x,y)$ and $z_2(x,y)$ is an L-pair of complexity one, then this pair up $\mathcal{G}$-action has the form*
For $N(z_1)=0, \quad N(z_2)=1$
(01.1) $z_1 = a(x), \quad z_2 = x+y, \quad a$ is arbitrary,
(01.2) $z_1 = x, \quad z_2 = xy,$
For $N(z_1)=N(z_2)=1$
(11.1) $z_1 = c(x+y), \quad z_2 = r(x+y)$, where c and r are arbitrary,
(11.2) $z_1 = a(x)y, \quad z_2 = xy, \quad a$ is arbitrary,
(11.3) $z_1 = xy, \quad z_2 = x+y,$
(11.4.1) $z_1 = \dfrac{xy}{x+y}, \quad z_2 = \dfrac{1}{x+y},$
(11.4.2) In this case there are no explicit expressions for the pair $z_1 = c(a(x)+b(y))$, $z_2 = r(x+y)$. The four functions (a,b,c,r) are constructed as described above. In particular, they can be expressed by quadratures.

As shown above, all pairs in this list are L-pairs. In Cases (01.1), (01.2), (11.1), (11.2) it is obvious. In Case (11.3) we can also see it easily: $z = xy + t(x+y) = (x+t)(y+t) - t^2$. In Case (11.4.1) it is not that clear. We need to check that

$$z = \frac{xy}{x+y} + t\frac{1}{x+y} = \frac{t+xy}{x+y} \in Cl_1 \text{ for all } t.$$

After the change t by t^2 we get

$$z = \frac{t^2+xy}{x+y}.$$

By replacing x with tx, y with ty, and z with t/z, we get

$$z = \frac{x+y}{1+xy}.$$

Now, we replace x with $\text{th}(x)$, y with $\text{th}(y)$, and z with $\text{th}(z)$ and use the addition formula

$$\text{th}(x+y) = \frac{\text{th}(x)+\text{th}(y)}{1+\text{th}(x)\,\text{th}(y)}$$

to get $z = x+y$. Since all the transformations here do not change complexity, this proves that the complexity of the original function is 1.

For Case (11.4.2) the author does not know a similar reasoning. The open question is what mysterious relations are behind that fact.

The set of pairs of complexity one is certainly wider than the set of L-pairs of complexity one. This is another open problem: to describe all pairs of complexity one.

3. O(2)-simplicity

The standard action of the $O(2)$ on the (x, y)-plane is

$$g_\phi = (\ x \to \cos(\phi)x - \sin(\phi)y, \quad y \to \sin(\phi)x + \cos(\phi)y\ )$$

where $\phi \in \mathbf{C}$. This action induces an action on functions

$$z(x, y) \to g_\phi(z)(x, y) = z(\cos(\phi)x - \sin(\phi)y,\ \sin(\phi)x + \cos(\phi)y).$$

Denote $t = \mathrm{tg}(\phi/2)$, then we have another form for this action

$$g_t = \left(x\ \to \frac{1-t^2}{1+t^2}\,x - 2\,\frac{t}{t^2+1}\,y,\ y\ \to \frac{1-t^2}{t^2+1}\,y + 2\,\frac{t}{1+t^2}x\right).$$

If $N(z(x, y)) = n$, then $N(z(\lambda x, \lambda y)) = n$ also, therefore we can replace $g_t(x, y)$ with $h_t(x, y) = (1 + t^2)\, g_t(x, y)$.

If $N(z) \leqslant n$, then $N(g_\phi(z)) \leqslant n + 1$, and for arbitrary z and ϕ there is no reason to expect that $N(g_\phi(z)) \leqslant n$. For example, let $z = xy$, then $N(z) = 1$. For $\delta(h_t(z))$ we have

$$4\,t\left(x^2+y^2\right)(t-1)\,(t+1)\left(t^2+2\,t-1\right)\left(t^2-2\,t-1\right)\left(t^2+1\right)^4.$$

We see that $N(h_t(xy)) = 1$ only for 9 values of t, namely $t = 0, \pm 1, \pm i, \pm 1 \pm \sqrt{2}$. The corresponding functions are proportional to

$$xy, \quad x^2 - y^2, \quad (x \pm iy)^2.$$

For another values t the complexity $N(h_t(xy))$ is equal to two.

Definition. *A function $z(x, y)$ is called $O(2)$-simple if $N(g_t(z)) \leqslant 1$ for all t.*

All linear functions are, of course, $O(2)$-simple. Now, we want to describe all $O(2)$-simple functions. It is clear that for such functions $N(z) \leqslant 1$, then $z = c(a(x) + b(y))$. If one of the functions (a, b, c) is constant, then $N(z) = 0$, and z depends on only one variable or a constant. Any such function is $O(2)$-simple (**Case 0**). Assume that $N(z) = 1$, i.e. a, b, c are not constant.

Statement 6. *(1) z is $O(2)$-simple if and only if $\delta(g_t(z)) = 0$ for all (x, y, t).*
(2) $c(a(x) + b(y))$ is $O(2)$-simple if and only if $a(x) + b(y)$ is $O(2)$-simple. (3) $z(x, y)$ is $O(2)$-simple if and only if $z(y, x)$ is $O(2)$-simple.

The proof is obvious.

Let $a(x) + b(y)$ is $O(2)$-simple, then, in particular,

$$\frac{d}{dt}\delta(g_t(a(x) + b(y)))|_{t=0} = 0, \tag{10}$$

in index notation for derivatives we have

$$-{a_1}^2 a_2 b_2 - {a_1}^2 b_1 b_3 + {a_1}^2 {b_2}^2 - a_1 a_3 {b_1}^2 + {a_2}^2 {b_1}^2 - a_2 {b_1}^2 b_2 = 0. \tag{11}$$

We can decrease the order of equation (11) twice. First, putting $a_1 = a'(x) = A,\ b_1 = b'(y) = B$. Second, introducing $P(A) = a_2 = a''(x),\ Q(B) = b_2 = b''(y)$. In this notation we have $a_3 = a'''(x) = P'(A)\,P(A),\ b_3 = b'''(y) = Q'(B)\,Q(B)$ and we can write (11) as

$$-QA^2P - BQ_1QA^2 + Q^2A^2 - B^2AP_1P + B^2P^2 - B^2QP = 0 \tag{12}$$

By differentiating (12) wit respect to A, we get

$$-2\,QAP - QA^2P_1 - 2\,ABQ_1Q + 2\,Q^2A + B^2P_1P - B^2AP_2P - B^2AP_1^2 - B^2QP = 0. \quad (13)$$

The relations (12) и (13) are a system of linear equations in $Q(B)$ and $Q'(B)$, its determinant is equal to

$$-B\,P\,A\left(A^3P_1 + B^2AP_1 - 2\,B^2P\right).$$

This determinant is identically equal to zero only if $P(A) = 0$ (**Case 1**). The solution to the system for $Q(B)$ is

$$Q(B) = -\frac{B^2(A^2P_2P + A^2P_1^2 - 3\,APP_1 + 2\,P^2)}{A^3P_1 + B^2AP_1 - 2\,B^2P}.$$

The condition of independence Q from A is

$$\begin{gathered}-A^3P_3\,PP_1 + A^3{P_2}^2P - 2\,A^3P_2\,{P_1}^2 - AB^2P_3\,PP_1 + \\ +AB^2{P_2}^2P - 2\,AB^2P_2\,{P_1}^2 + A^2P_2\,PP_1 + 4\,A^2{P_1}^3 + \\ +2\,B^2P_3\,P^2 + 3\,B^2P_2\,PP_1 + 2\,AP_2\,P^2 - 10\,AP{P_1}^2 + 6\,P^2P_1 = 0,\end{gathered} \quad (14)$$

which splits into two relations: terms free of B and terms with the factor B^2. Eliminating $P'''(A)$ from them, we get

$$P\,\left(AP_1 - 2\,P\right)\left(AP_2\,P - 2\,A{P_1}^2 + 3\,P_1\,P\right)\left(A^2PP_2 + A^2{P_1}^2 - 3\,APP_1 + 2\,P^2\right) = 0.$$

The case $P = 0$ (Case 1) has been considered above. Now we turn to the remaining cases.

$$\begin{aligned}(AP_1 - 2\,P) &= 0 \text{ (Case 2)},\\ \left(AP_2\,P - 2\,A{P_1}^2 + 3\,P_1\,P\right) &= 0 \text{ (Case 3)},\\ \left(A^2PP_2 + A^2{P_1}^2 - 3\,APP_1 + 2\,P^2\right) &= 0 \text{ (Case 4)}.\end{aligned}$$

The solutions to the corresponding differential equations are

$$\begin{aligned}P(A) &= 0 \quad \text{(Case 1)},\\ P(A) &= CA^2 \text{ (Case 2)},\\ P\,(A) &= \frac{A^2}{A^2C_1 + C_2} \text{ (Case 3)},\\ P(A) &= A\,\sqrt{C_1\,\ln(A) + C_2} \text{ (Case 4)}.\end{aligned}$$

To find $Q(B)$ corresponding to $P(A)$, we substitute these solutions in (13).
In Case 1 $P(A) = 0,\ Q(B) = CB$.
In Case 2 $P(A) = CA^2,\ Q(B) = -CB^2$.
In Case 3 $P(A) = A^2/(cA^2 + d)$ and for $Q(B)$ we have

$$\begin{gathered}-A^6BQQ_1\,c^3 + A^6Q^2c^3 - 3\,A^4BQQ_1\,c^2d - A^6Qc^2 - A^4B^2Qc^2 + 3\,A^4Q^2c^2d - \\ -3\,A^2BQQ_1\,cd^2 + B^2A^4c - 2\,A^4Qcd - 2\,A^2B^2Qcd + 3\,A^2Q^2cd^2 - \\ -BQQ_1\,d^3 - A^2B^2d - A^2Qd^2 - B^2Qd^2 + Q^2d^3 = 0,\end{gathered} \quad (15)$$

which is a polynomial in A^2 and splits into four differential equations of first order on $Q(B)$ (the coefficients at 1, A^2, A^4, A^6). These equations yield $d = 0$, and $P(A) = Q(B) = C = \text{const}$.
In Case 4 we have $P(A) = A\,\sqrt{c\,\ln(A) + d}$ and

$$-2\,QA^2\sqrt{c\ln(A) + d} - B^2Ac - 2\,ABQQ_1 - 2\,B^2Q\sqrt{c\ln(A) + d} + 2\,AQ^2 = 0.$$

The functions

$$\sqrt{c\ln(A)+d}, \quad A, \quad A^2\sqrt{c\ln(A)+d}$$

are linearly independent, hence $Q(B)=0$ and $c=0$. So the answer in Case 4 coincides with the answer in Case 1 after replacing $A\to B$.

Now we can return to equations in $a(x)$ and $b(y)$ and find the answers:

In Case 1: $P(A)=0$ means $a''(x)=0$ and $a(x)=\alpha_1 x+\alpha_0$, then $Q(B)=CB$ means $b''(y)=Cb'(y)$ and $b(y)=\beta_1 e^{Cy}+\beta_0$. Then we write the $O(2)$-simplicity condition $\delta(g_t(z))=0$ for $a+b$ and see that it holds only for $\alpha_1\beta_1=0$. The same goes in Case 4.

In Case 2: $P(A)=CA^2$ means $a''(x)=C(a'(x))^2$ and $a(x)=-\ln(\alpha_1 x+\alpha_0)/C$, then from $Q(B)=CB^2$ we get $b(y)=\ln(\beta_1 y+\beta_0)/C$. Since

$$a(x)+b(y)=\frac{1}{C}\ln\left(\frac{\beta_1 y+\beta_0}{\alpha_1 x+\alpha_0}\right),$$

it is enough to check the $O(2)$-simplicity condition only for

$$z=\frac{\beta_1 y+\beta_0}{\alpha_1 x+\alpha_0}.$$

It is easy to see that the condition $\delta(g_t(z))=0$ holds.

In Case 3: $P(A)=C$ means $a''(x)=C$ and $a(x)=Cx^2+\alpha_1 x+\alpha_0$, then from $Q(B)=C$ we get $b(y)=Cy^2+\beta_1 y+\beta_0$. We see that the $O(2)$-simplicity condition for $a+b$ holds.

Thus, we have the theorem.

Theorem 7. *The complete list of $O(2)$-simple functions up to transformations ($z(x,y)\to f(z(x,y))$ and ($z(x,y)\to z(y,x)$) is*

$$z=\frac{\beta_1\,y+\beta_0}{\alpha_1\,x+\alpha_0},$$
$$z=(x^2+y^2)+\alpha\,x+\beta\,y,$$
$$z=\alpha\,x+\beta\,y.$$

Corollary 8. *Any $O(2)$-simple function is a rational function, up to a transformation ($z(x,y)\to f(z(x,y))$).*

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Три семейства функций сложности один

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В работе описаны некоторые семейства функций двух переменных аналитической сложности единица, обладающие некоторыми редкими свойствами. Во-первых, классифицированы линейные уравнения с постоянными коэффициентами,т.ч. все их аналитические решения имеют сложность не выше единицы (теорема 2). Во-вторых, классифицированы пары аналитических функций, таких что любая их линейная комбинация имеет сложность не выше единицы (теорема 5). В-третьих, дано явное описание функций, т.ч. их орбиты под действием группы $O(2)$ состоят из функций сложности не выше единицы (теорема 7).

Ключевые слова: редкие семейства, аналитическая сложность.