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Sharp Theorems on Multipliers and Distances in Harmonic Function Spaces in Higher Dimension

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We present new sharp results concerning multipliers and distance estimates in various spaces of harmonic functions in the unit ball of $\mathbb{R}^n$.

Keywords: multipliers, harmonic functions, Bergman spaces, mixed norm spaces, distance estimates.

1. Introduction and Preliminaries

The aim of this paper is twofold. One is to describe spaces of multipliers between certain spaces of harmonic functions on the unit ball. We note that so far there are no results in this direction in the multidimensional case, where the use of spherical harmonics is a natural substitute for power series expansion. In fact, even the case of the unit disc has not been extensively studied in this context. We refer the reader to [1], where multipliers between harmonic Bergman type classes were considered, and to [2] and [3] for the case of harmonic Hardy classes. Most of our results are present in these papers in the special case of the unit disc.

The other topic we investigate is distance estimates in spaces of harmonic functions on the unit ball. This line of investigation can be considered as a continuation of papers [4–6].

Let $\mathbb{B}$ be the open unit ball in $\mathbb{R}^n$, $\mathbb{S} = \partial\mathbb{B}$ is the unit sphere in $\mathbb{R}^n$, for $x \in \mathbb{R}^n$ we have $x = rx'$, where $r = |x| = \sqrt{\sum_{j=1}^{n} x_j^2}$ and $x' \in \mathbb{S}$. Normalized Lebesgue measure on $\mathbb{B}$ is denoted by $dx = dx_1 \dots dx_n = r^{n-1} dr dx'$ so that $\int_{\mathbb{B}} dx = 1$. We denote the space of all harmonic functions in an open set Ω by $h(\Omega)$. In this paper letter C designates a positive constant which can change its value even in the same chain of inequalities.

For $0 < p < \infty$, $0 \leqslant r < 1$ and $f \in h(\mathbb{B})$ we set

$$M_p(f, r) = \left(\int_{\mathbb{S}} |f(rx')|^p dx' \right)^{1/p},$$

with the usual modification to cover the case $p = \infty$. Weighted Hardy spaces are defined, for $\alpha \geqslant 0$ and $0 < p \leqslant \infty$, by $H^p_\alpha(\mathbb{B}) = H^p_\alpha = \{f \in h(\mathbb{B}) : \|f\|_{p,\alpha} = \sup_{r<1} M_p(f, r)(1 - r)^\alpha < \infty\}$. For $\alpha = 0$ the space H^p_α is denoted simply by H^p.

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For $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ and $\alpha > 0$ and we consider mixed (quasi)-norms $\|f\|_{p,q;\alpha}$ defined by

$$\|f\|_{p,q;\alpha} = \left(\int_0^1 M_q(f,r)^p (1-r^2)^{\alpha p - 1} r^{n-1} dr \right)^{1/p}, \qquad f \in h(\mathbb{B}), \tag{1}$$

again with the usual interpretation for $p = \infty$, and the corresponding spaces $B^{p,q}_\alpha(\mathbb{B}) = B^{p,q}_\alpha = \{f \in h(\mathbb{B}) : \|f\|_{p,q;\alpha} < \infty\}$. It is not hard to show that these spaces are complete metric spaces and that for $\min(p,q) \geqslant 1$ they are Banach spaces. These spaces include weighted Bergman spaces $A^p_\beta(\mathbb{B}) = A^p_\beta = B^{p,p}_{\frac{\beta+1}{p}}$ where $\beta > -1$ and $0 < p < \infty$. We set $A^\infty_\beta = B^{\infty,\infty}_\beta$ for $\beta > 0$.

Note that $A^\infty_\alpha = H^\infty_\alpha$ for $\alpha \geqslant 0$ and $B^{\infty,q}_\alpha = H^q_\alpha$ for $0 < q \leqslant \infty$, $\alpha > 0$. We also have, for $0 < p_0 \leqslant p_1 \leqslant \infty$, $B^{p_0,1}_\alpha \subset B^{p_1,1}_\alpha$, see [7].

Next we need certain facts on spherical harmonics and Poisson kernel, see [1] for a detailed exposition. Let $Y^{(k)}_j$ be the spherical harmonics of order k, $j \leqslant 1 \leqslant d_k$, on $\mathbb{S}$. Next, $Z^{(k)}_{x'}(y') = \sum_{j=1}^{d_k} Y^{(k)}_j(x') \overline{Y^{(k)}_j(y')}$ are zonal harmonics of order k. Note that the spherical harmonics $Y^{(k)}_j$, $(k \geqslant 0,\ 1 \leqslant j \leqslant d_k)$ form an orthonormal basis of $L^2(\mathbb{S}, dx')$. Every $f \in h(\mathbb{B})$ has an expansion $f(x) = f(rx') = \sum_{k=0}^\infty r^k b_k \cdot Y^k(x')$, where $b_k = (b^1_k, \dots, b^{d_k}_k)$, $Y^k = (Y^{(k)}_1, \dots, Y^{(k)}_{d_k})$ and $b_k \cdot Y^k$ is interpreted in the scalar product sense: $b_k \cdot Y^k = \sum_{j=1}^{d_k} b^j_k Y^{(k)}_j$. We often write, to stress dependence on a function $f \in h(\mathbb{B})$, $b_k = b_k(f)$ and $b^j_k = b^j_k(f)$, in fact we have linear functionals b^j_k, $k \geqslant 0, 1 \leqslant j \leqslant d_k$ on the space $h(\mathbb{B})$.

We denote the Poisson kernel for the unit ball by $P(x,y')$, it is given by

$$P(x,y') = P_{y'}(x) = \sum_{k=0}^\infty r^k \sum_{j=1}^{d_k} Y^{(k)}_j(y') Y^{(k)}_j(x') = \frac{1}{n\omega_n} \frac{1-|x|^2}{|x-y'|^n}, \quad x = rx' \in \mathbb{B}, \quad y' \in \mathbb{S},$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. We are going to use also a Bergman kernel for A^p_β spaces, this is the following function

$$Q_\beta(x,y) = 2 \sum_{k=0}^\infty \frac{\Gamma(\beta+1+k+n/2)}{\Gamma(\beta+1)\Gamma(k+n/2)} r^k \rho^k Z^{(k)}_{x'}(y'), \quad x = rx',\ y = \rho y' \in \mathbb{B},\ \beta \geq 0. \tag{2}$$

For details on this kernel we refer to [7], where the following theorem can be found.

Theorem 1 ([7]). *Let $p \geqslant 1$ and $\beta \geqslant 0$. Then for every $f \in A^p_\beta$ and $x \in \mathbb{B}$ we have*

$$f(x) = \int_0^1 \int_{\mathbb{S}^{n-1}} Q_\beta(x,y) f(\rho y') (1-\rho^2)^\beta \rho^{n-1} d\rho dy', \qquad y = \rho y'.$$

This theorem is a cornerstone for our approach to distance problems in the case of the unit ball. The following lemma gives estimates for this kernel, see [7, 8]. Note the Bergman kernel can be also defined for all $\beta > -1$

Lemma 1. *1. Let $\beta > 0$. Then, for $x = rx', y = \rho y' \in \mathbb{B}$ we have $|Q_\beta(x,y)| \leqslant \dfrac{C}{|\rho x - y'|^{n+\beta}}$.*

2. Let $\beta > -1$. Then

$$\int_{\mathbb{S}^{n-1}} |Q_\beta(rx', y)| dx' \leqslant \frac{C}{(1-r\rho)^{1+\beta}}, \qquad |y| = \rho, \quad 0 \leqslant r < 1.$$

3. Let $\beta > n-1$, , $0 \leqslant r < 1$ and $y' \in \mathbb{S}^{n-1}$. Then

$$\int_{\mathbb{S}^{n-1}} \frac{dx'}{|rx'-y'|^\beta} \leqslant \frac{C}{(1-r)^{\beta-n+1}}.$$

Lemma 2 ([7]). *Let $\alpha > -1$ and $\lambda > \alpha+1$. Then*

$$\int_0^1 \frac{(1-r)^\alpha}{(1-r\rho)^\lambda} dr \leqslant C(1-\rho)^{\alpha+1-\lambda}, \qquad 0 \leqslant \rho < 1.$$

Lemma 3. *Let $G(r)$, $0 \leqslant r < 1$, be a positive increasing function. Then, for $\alpha > -1$, $\beta > -1$, $\gamma \geqslant 0$ and $0 < q \leqslant 1$ we have*

$$\left(\int_0^1 G(r) \frac{(1-r)^\beta}{(1-\rho r)^\gamma} r^\alpha dr \right)^q \leqslant C \int_0^1 G(r)^q \frac{(1-r)^{\beta q + q - 1}}{(1-\rho r)^{q\gamma}} r^\alpha dr, \quad 0 \leqslant \rho < 1. \tag{3}$$

A special case of the above lemma appears in [9], for reader's convenience we produce a proof.

Proof. We use a subdivision of $I = [0,1)$ into subintervals $I_k = [r_k, r_{k+1})$, $k \geqslant 0$, where $r_k = 1 - 2^{-k}$. Since $1 - \rho r_k \asymp 1 - \rho r_{k+1}$, $0 \leqslant \rho < 1$, we have

$$J = \left(\int_0^1 G(r) \frac{(1-r)^\beta}{(1-\rho r)^\gamma} r^\alpha dr \right)^q = \left(\sum_{k \geqslant 0} \int_{I_k} G(r) \frac{(1-r)^\beta}{(1-\rho r)^\gamma} r^\alpha dr \right)^q \leqslant$$
$$\leqslant \sum_{k \geqslant 0} \left(\int_{I_k} G(r) \frac{(1-r)^\beta}{(1-\rho r)^\gamma} r^\alpha dr \right)^q \leqslant \sum_{k \geqslant 0} 2^{-kq\beta} G^q(r_{k+1}) \left(\int_{I_k} \frac{r^\alpha dr}{(1-\rho r)^\gamma} \right)^q \leqslant$$
$$\leqslant C \sum_{k \geqslant 0} 2^{-kq\beta} G^q(r_{k+1}) 2^{-kq} (1-\rho r_{k+1})^{-q\gamma} \leqslant C \sum_{k \geqslant 0} 2^{-kq\beta} G^q(r_{k+1}) 2^{-kq} (1-\rho r_k)^{-q\gamma} \leqslant$$
$$\leqslant C \sum_{k \geqslant 0} G^q(r_{k+1}) \int_{I_{k+1}} \frac{(1-r)^{\beta q + q - 1} r^\alpha dr}{(1-\rho r)^{q\gamma}} \leqslant C \int_0^1 G(r)^q \frac{(1-r)^{\beta q + q - 1}}{(1-\rho r)^{q\gamma}} r^\alpha dr.$$

□

Lemma 4. *For $\delta > -1$, $\gamma > n + \delta$ and $\beta > 0$ we have*

$$\int_{\mathbb{B}} |Q_\beta(x,y)|^{\frac{\gamma}{n+\beta}} (1-|y|)^\delta dy \leqslant C(1-|x|)^{\delta-\gamma+n}, \qquad x \in \mathbb{B}.$$

Proof. Using Lemma 1 and Lemma 2 we obtain:

$$\int_{\mathbb{B}} |Q_\beta(x,y)|^{\frac{\gamma}{n+\beta}} (1-|y|)^\delta dy \leqslant C \int_{\mathbb{B}} \frac{(1-|y|)^\delta}{|\rho r x' - y'|^\gamma} dy \leqslant$$
$$\leqslant C \int_0^1 (1-\rho)^\delta \int_{\mathbb{S}} \frac{dy'}{|\rho r x' - y'|^\gamma} dy' d\rho \leqslant C \int_0^1 (1-\rho)^\delta (1-r\rho)^{n-\gamma-1} d\rho \leqslant C(1-r)^{n+\delta-\gamma}.$$

□

Lemma 5 ([7]). *For real s, t such that $s > -1$ and $2t + n > 0$ we have*

$$\int_0^1 (1-r^2)^s r^{2t+n-1} dr = \frac{1}{2} \frac{\Gamma(s+1)\Gamma(n/2+t)}{\Gamma(s+1+n/2+t)}.$$

We set $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, t > 0\} \subset \mathbb{R}^{n+1}$. We usually denote the points in $\mathbb{R}^{n+1}_+$ by $z = (x,t)$ or $w = (y,s)$ where $x, y \in \mathbb{R}^n$ and $s, t > 0$.

For $0 < p < \infty$ and $\alpha > -1$ we consider spaces

$$\tilde{A}^p_\alpha(\mathbb{R}^{n+1}_+) = \tilde{A}^p_\alpha = \left\{ f \in h(\mathbb{R}^{n+1}_+) : \int_{\mathbb{R}^{n+1}_+} |f(x,t)|^p t^\alpha dxdt < \infty \right\}.$$

Also, for $p = \infty$ and $\alpha > 0$, we set

$$\tilde{A}^\infty_\alpha(\mathbb{R}^{n+1}_+) = \tilde{A}^\infty_\alpha = \left\{ f \in h(\mathbb{R}^{n+1}_+) : \sup_{(x,t)\in\mathbb{R}^{n+1}_+} |f(x,t)| t^\alpha < \infty \right\}.$$

These spaces have natural (quasi)-norms, for $1 \leqslant p \leqslant \infty$ they are Banach spaces and for $0 < p \leqslant 1$ they are complete metric spaces.

We denote the Poisson kernel for $\mathbb{R}^{n+1}_+$ by $P(x,t)$, i.e.

$$P(x,t) = c_n \frac{t}{(|x|^2 + t^2)^{\frac{n+1}{2}}}, \qquad x \in \mathbb{R}^n, t > 0.$$

For an integer $m \geqslant 0$ we introduce a Bergman kernel $Q_m(z,w)$, where $z = (x,t) \in \mathbb{R}^{n+1}_+$ and $w = (y,s) \in \mathbb{R}^{n+1}_+$, by

$$Q_m(z,w) = \frac{(-2)^{m+1}}{m!} \frac{\partial^{m+1}}{\partial t^{m+1}} P(x-y, t+s).$$

The terminology is justified by the following result from [7].

Theorem 2. *Let $0 < p < \infty$ and $\alpha > -1$. If $0 < p \leqslant 1$ and $m \geqslant \frac{\alpha+n+1}{p} - (n+1)$ or $1 \leqslant p < \infty$ and $m > \frac{\alpha+1}{p} - 1$, then*

$$f(z) = \int_{\mathbb{R}^{n+1}_+} f(w) Q_m(z,w) s^m dyds, \qquad f \in \tilde{A}^p_\alpha, \quad z \in \mathbb{R}^{n+1}_+. \tag{4}$$

The following elementary estimate of this kernel is contained in [7]:

$$|Q_m(z,w)| \leqslant C \left[|x-y|^2 + (s+t)^2 \right]^{-\frac{n+m+1}{2}}, \quad z = (x,t), w = (y,s) \in \mathbb{R}^{n+1}_+. \tag{5}$$

2. Multipliers on Spaces of Harmonic Functions

In this section we present our results on multipliers between spaces of harmonic functions on the unit ball. The following definitions are needed to formulate these theorems.

Definition 1. *For a double indexed sequence of complex numbers $c = \{c^j_k : k \geqslant 0, 1 \leqslant j \leqslant d_k\}$ and a harmonic function $f(rx') = \sum_{k=0}^\infty \sum_{j=1}^{d_k} r^k b^j_k(f) Y^{(k)}_j(x')$ we define $(c * f)(rx') = \sum_{k=0}^\infty \sum_{j=1}^{d_k} r^k c^j_k b^j_k(f) Y^{(k)}_j(x')$, $rx' \in \mathbb{B}$, if the series converges in $\mathbb{B}$. Similarly we define convolution of $f, g \in h(\mathbb{B})$ by $(f * g)(rx') = \sum_{k=0}^\infty \sum_{j=1}^{d_k} r^k b^j_k(f) b^j_k(g) Y^{(k)}_j(x')$, $rx' \in \mathbb{B}$, it is easily seen that $f * g$ is defined and harmonic in $\mathbb{B}$.*

Definition 2. *For $t > 0$ and a harmonic function $f(x) = \sum_{k=0}^{\infty} b_k(f)Y^k(x')$ on the unit ball we define a fractional derivative of order t of f by the following formula:*

$$(\Lambda_t f)(x) = \sum_{k=0}^{\infty} r^k \frac{\Gamma(k+n/2+t)}{\Gamma(k+n/2)\Gamma(t)} b_k(f) \cdot Y^k(x'), \qquad x = rx' \in \mathbb{B}.$$

Clearly, for $f \in h(\mathbb{B})$ and $t > 0$ the function $\Lambda_t h$ is also harmonic in $\mathbb{B}$.

Definition 3. *Let X and Y be subspaces of $h(\mathbb{B})$. We say that a double indexed sequence c is a multiplier from X to Y if $c * f \in Y$ for every $f \in X$. The vector space of all multipliers from X to Y is denoted by $M_H(X,Y)$.*

Clearly every multiplier $c \in M_H(X,Y)$ induces a linear map $M_c : X \to Y$. If, in addition, X and Y are (quasi)-normed spaces such that all functionals b_k^j are continuous on both spaces X and Y, then the map $M_c : X \to Y$ is continuous, as is easily seen using the Closed Graph Theorem. We note that this holds for all spaces we consider in this paper : A^p_α, $B^{p,q}_\alpha$ and H^p_α.

Lemma 6. *Let $f, g \in h(\mathbb{B})$ have expansions*

$$f(rx') = \sum_{k=0}^{\infty} r^k \sum_{j=1}^{d_k} b_k^j Y_j^{(k)}(x'), \qquad g(rx') = \sum_{l=0}^{\infty} r^k \sum_{i=1}^{d_k} c_l^i Y_i^{(l)}(x').$$

Then we have

$$\int_{\mathbb{S}} (g * P_{y'})(rx') f(\rho x') dx' = \sum_{k=0}^{\infty} r^k \rho^k \sum_{j=1}^{d_k} b_k^j c_k^j Y_j^{(k)}(y'), \qquad y' \in \mathbb{S}, \quad 0 \leqslant r, \rho < 1.$$

Moreover, for every $m > -1$, $y' \in \mathbb{S}$ and $0 \leqslant r, \rho < 1$ we have

$$\int_{\mathbb{S}} (g * P_{y'})(rx') f(\rho x') dx' = 2 \int_0^1 \int_{\mathbb{S}} \Lambda_{m+1}(g * P_{y'})(rRx') f(\rho R x')(1-R^2)^m R^{n-1} dx' dR.$$

Proof. The first assertion of this lemma easily follows from the orthogonality relations for spherical harmonics $Y_j^{(k)}$. Using Lemma 5 and orthogonality relations we have

$$\begin{aligned} I &= 2\int_0^1 \int_{\mathbb{S}} \Lambda_{m+1}(g * P_{y'})(rRx') f(\rho R x')(1-R^2)^m R^{n-1} dx' dR = \\ &= 2\int_0^1 \sum_{k=0}^{\infty} r^k \rho^k R^{2k+n-1}(1-R^2)^m \frac{\Gamma(k+n/2+m+1)}{\Gamma(k+n/2)\Gamma(m+1)} \sum_{j=1}^{d_k} b_k^j c_k^j Y_j^{(k)} dR = \\ &= \sum_{k=0}^{\infty} r^k \rho^k \sum_{j=1}^{d_k} b_k^j c_k^j Y_j^{(k)}(y'), \end{aligned}$$

which proves the second assertion. □

We note that $(g * P_{y'})(rx') = (g * P_{x'})(ry')$ and $\Lambda_t(g * P_{y'})(x) = (\Lambda_t g * P_{y'})(x)$, these easy to prove formulae are often used in our proofs.

In this section $f_{m,y}$ stands for the harmonic function $f_{m,y}(x) = Q_m(x,y)$, $y \in \mathbb{B}$. We often write f_y instead of $f_{m,y}$. Let us collect some norm estimates of f_y.

Lemma 7. *For $0 < p \leqslant \infty$ and $m > 0$ we have*

$$M_\infty(f_{m,y}, r) \leqslant C(1-|y|r)^{-n-m}, \tag{6}$$

$$M_1(f_{m,y}, r) \leqslant C(1-|y|r)^{-1-m}, \tag{7}$$

$$\|f_{m,y}\|_{B^{p,1}_\alpha} \leqslant C(1-|y|)^{\alpha-1-m}, \qquad m > \alpha - 1, \quad \alpha > 0, \tag{8}$$

$$\|f_{m,y}\|_{B^{p,\infty}_\alpha} \leqslant C(1-|y|)^{\alpha-n-m}, \qquad m > \alpha - n, \quad \alpha > 0, \tag{9}$$

$$\|f_{m,y}\|_{A^1_\alpha} \leqslant C(1-|y|)^{\alpha-m}, \qquad m > \alpha > -1, \tag{10}$$

$$\|f_{m,y}\|_{H^1_\alpha} \leqslant C(1-|y|)^{\alpha-1-m}, \qquad m > \alpha - 1, \quad \alpha \geqslant 0. \tag{11}$$

Proof. Using Lemma 1 we obtain

$$M_\infty(f_{m,y}, r) = \max_{x' \in \mathbb{S}} |Q_m(y, rx')| \leqslant \max_{x' \in \mathbb{S}} \frac{C}{|\rho r x' - y'|^{n+m}} = C(1 - r|y|)^{-n-m},$$

which gives (6). The estimate (7) follows from Lemma 1. The estimates (8), for finite p, and (10) follow from Lemma 2 and (7). Similarly, for finite p (9) follows from (6) and Lemma 2. Next, using (7),

$$\|f_{m,y}\|_{H^1_\alpha} \leqslant C \sup_{0 \leqslant r < 1} (1-r)^\alpha (1 - r\rho)^{-m-1}, \qquad \rho = |y|.$$

The function $\phi(r) = (1-r)^\alpha(1-r\rho)^{-m-1}$ attains its maximum on $[0,1]$ at

$$r_0 = 1 - (1-\rho)\frac{\alpha}{\rho(1+m-\alpha)},$$

as is readily seen by a simple calculus, and this suffices to establish (11) and therefore (8) for $p = \infty$. Finally, (9) directly follows from Lemma 1. □

In this section we are looking for sufficient and/or necessary condition for a double indexed sequence c to be in $M_H(X, Y)$, for certain spaces X and Y of harmonic functions. We associate to such a sequence c a harmonic function

$$g_c(x) = g(x) = \sum_{k \geqslant 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x'), \qquad x = rx' \in \mathbb{B}, \tag{12}$$

and express our conditions in terms of g_c. Our main results give conditions in terms of fractional derivatives of g_c, however it is possible to obtain some results on the basis of the following formula, contained in Lemma 6:

$$(c * f)(r^2 x') = \int_{\mathbb{S}} (g * P_{y'})(rx') f(ry') dy'. \tag{13}$$

Using continuous form of Minkowski's inequality, or more generally Young's inequality, this formula immediately gives the following proposition.

Proposition 1. *Let $c = \{c_k^j : k \geqslant 0, 1 \leqslant j \leqslant d_k\}$ be a double indexed sequence and let $g(x) = \sum_{k \geqslant 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$ be the corresponding harmonic function. If*

$$\int_{\mathbb{S}} |(g * P_{y'})(rx')|^p dx' \leqslant C, \qquad y' \in \mathbb{S}, \quad 0 \leqslant r < 1,$$

then $c \in M_H(H^1, H^p)$.

More generally, if $1/q + 1/p = 1 + 1/r$, where $1 \leqslant p, q, r \leqslant \infty$, $\alpha + \gamma = \beta$, $\alpha, \beta, \gamma \geqslant 0$ and $g \in H^p_\gamma$, then $c \in M_H(H^q_\alpha, H^r_\beta)$.

The first part of the following lemma, which gives necessary conditions for c to be a multiplier, is based on [9].

Lemma 8. *Let $0 < p, q \leqslant \infty$, $1 \leqslant s \leqslant \infty$ and $m > \alpha - 1$. Assume a double indexed sequence $c = \{c_k^j : k \geqslant 0, 1 \leqslant j \leqslant d_k\}$ is a multiplier from $B_\alpha^{p,1}$ to $B_\beta^{q,s}$ and $g = g_c$ is defined in* (12). *Then the following condition is satisfied:*

$$N_s(g) = \sup_{0 \leqslant \rho < 1} \sup_{y' \in \mathbb{S}} (1-\rho)^{m+1-\alpha+\beta} \left(\int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(\rho y')|^s dx' \right)^{1/s} < \infty, \tag{14}$$

where the case $s = \infty$ requires usual modification.

Also, let $0 < p \leqslant \infty$, $1 \leqslant s \leqslant \infty$ and $m > \alpha - 1$. If a double indexed sequence $c = \{c_k^j : k \geqslant 0, 1 \leqslant j \leqslant d_k\}$ is a multiplier from $B_\alpha^{p,1}$ to H_β^s, then the above function g satisfies condition (14).

Proof. Let $c \in M_H(B_\alpha^{p,1}, B_\beta^{q,s})$, and assume both p and q are finite, the infinite cases require only small modifications. We have $\|M_c f\|_{B_\beta^{q,s}} \leqslant C\|f\|_{B_\alpha^{p,1}}$ for f in $B_\alpha^{p,1}$. Set $h_y = M_c f_y$, then we have

$$h_y(x) = \sum_{k \geqslant 0} r^k \rho^k \sum_{j=1}^{k} \frac{\Gamma(k + n/2 + m + 1)}{\Gamma(k + n/2)\Gamma(m+1)} c_k^j Y_j^{(k)}(y') Y_j^{(k)}(x'), \quad x = rx' \in \mathbb{B}, \tag{15}$$

moreover

$$\|h_y\|_{B_\beta^{q,s}} \leqslant C\|f_y\|_{B_\alpha^{p,1}}. \tag{16}$$

This estimate and Lemma 8 give

$$\|h_y\|_{B_\beta^{q,s}} \leqslant C(1-|y|)^{\alpha-m-1}, \qquad y \in \mathbb{B}. \tag{17}$$

Note that $h_y(x) = \Lambda_{m+1}(g * P_{y'})(\rho x)$, using monotonicity of $M_s(h_y, r)$ we obtain:

$$\begin{aligned} I_{y'}(\rho^2) &= \left(\int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(\rho^2 y')|^s dx' \right)^{1/s} = \left(\int_\rho^1 (1-r)^{\beta q - 1} r^{n-1} dr \right)^{-1/q} \times \\ &\times \left(\int_\rho^1 (1-r)^{\beta q-1} r^{n-1} \left(\int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{y'})(\rho^2 x')|^s dx' \right)^{q/s} dr \right)^{1/q} \leqslant \\ &\leqslant C(1-\rho)^{-\beta} \left(\int_\rho^1 (1-r)^{\beta q - 1} r^{n-1} M_s^q(h_y, r) dr \right)^{1/q} \leqslant \\ &\leqslant C(1-\rho)^{-\beta} \|h_y\|_{B_\beta^{q,s}}. \end{aligned} \tag{18}$$

Combining (18) and (17) we obtain

$$\left(\int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(\rho^2 y')|^s dx' \right)^{1/s} \leqslant C(1-\rho)^{\alpha-\beta-m-1},$$

which is equivalent to (14). The case $s = \infty$ is treated similarly.

Next we consider $c \in M_H(B_\alpha^{p,1}, H_\beta^s)$, assuming $0 < p \leqslant \infty$. Set $h_y = M_c h_y = g * f_y$. We have, by Lemma 7,

$$\|f_y\|_{B_\alpha^{p,1}} \leqslant C(1-|y|)^{\alpha-m-1}, \qquad y \in \mathbb{B},$$

and, by continuity of M_c, $\|h_y\|_{H_\beta^s} \leqslant C\|f_y\|_{B_\alpha^{p,1}}$. Therefore

$$\|h_y\|_{H_\beta^s} \leqslant C(1-|y|)^{\alpha-m-1}, \qquad y \in \mathbb{B}.$$

Setting $y = \rho y'$ we have

$$I_{y'}(\rho^2) = \left(\int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(\rho^2 y')|^s dx' \right)^{1/s} = \left(\int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{y})(\rho x')|^s dx' \right)^{1/s} =$$
$$= M_s(h_y, \rho) \leqslant (1-|y|)^{-\beta} \|h_y\|_{H^s_\beta}.$$

The last two estimates yield

$$\left(\int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(\rho^2 y')|^s dx' \right)^{1/s} \leqslant C(1-|y|)^{\alpha-\beta-m-1}, \qquad |y| = \rho$$

which is equivalent to (14). $\square$

Theorem 3. *Let $1 < p \leqslant q \leqslant \infty$, and $m > \alpha - 1$. Then for a double indexed sequence $c = \{c^j_k : k \geqslant 0, 1 \leqslant j \leqslant d_k\}$ the following conditions are equivalent:*

1. $c \in M_H(B^{p,1}_\alpha, B^{q,1}_\beta)$.

2. The function $g(x) = \sum_{k\geqslant 0} r^k \sum_{j=1}^{d_k} c^j_k Y^{(k)}_j(x')$ is harmonic in $\mathbb{B}$ and satisfies the following condition

$$N_1(g) < \infty. \tag{19}$$

Proof. Since necessity of (19) is contained in Lemma 8 we prove sufficiency of condition (19). We assume p and q are finite, the remaining cases can be treated in a similar manner. Take $f \in B^{p,1}_\alpha$ and set $h = M_c f$. Applying the operator Λ_{m+1} to both sides of equation (13) we obtain

$$\Lambda_{m+1} h(rx) = \int_{\mathbb{S}} \Lambda_{m+1}(g * P_{y'})(x) f(ry') dy'. \tag{20}$$

Now we estimate the L^1 norm of the above function on $|x| = r$:

$$\begin{aligned} M_1(\Lambda_{m+1}h, r^2) &\leqslant \int_{\mathbb{S}} M_1(\Lambda_{m+1}(g * P_{y'}), r) |f(ry')| dy' \leqslant \\ &\leqslant M_1(f, r) \sup_{y' \in \mathbb{S}} \int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{y'})(rx')| dx' \leqslant \\ &\leqslant M_1(f, r) N_1(g) (1-r)^{\alpha-\beta-m-1}. \end{aligned} \tag{21}$$

Since,

$$\int_0^1 M_1^p(h, r^2)(1-r)^{\beta p-1} r^{n-1} dr \leqslant C \int_0^1 (1-r)^{p(m+1)} M_1^p(\Lambda_{m+1}h, r^2)(1-r)^{\beta p-1} r^{n-1} dr,$$

see [7], we have

$$\begin{aligned} \|h\|^p_{B^{p,1}_\beta} &\leqslant C \int_0^1 (1-r)^{p(m+1)} M_1^p(\Lambda_{m+1}h, r^2)(1-r)^{\beta p-1} r^{n-1} dr \leqslant \\ &\leqslant C N_1^p(g) \int_0^1 M_1^p(f, r)(1-r)^{\alpha p-1} r^{n-1} dr = C N_1^p(g) \|f\|^p_{B^{p,1}_\alpha}, \end{aligned}$$

and therefore $\|h\|_{B^{p,1}_\beta} \leqslant \|f\|_{B^{p,1}_\alpha}$. Since $\|h\|_{B^{q,1}_\beta} \leqslant C\|h\|_{B^{p,1}_\beta}$ the proof is complete. $\square$

Next we consider multipliers from $B^{p,1}_\alpha$ to H^s_β, in the case $0 < p \leqslant 1$ we obtain a characterization of the corresponding space.

Theorem 4. *Let $\beta \geqslant 0$, $0 < p \leqslant 1$, $s \geqslant 1$ and $m > \alpha - 1$. Then, for a double indexed sequence $c = \{c_k^j : k \geqslant 0, 1 \leqslant j \leqslant d_k\}$ the following two conditions are equivalent:*

1. $c \in M_H(B_\alpha^{p,1}, H_\beta^s)$.

2. The function $g(x) = \sum_{k \geqslant 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$ is harmonic in $\mathbb{B}$ and satisfies the following condition:

$$N_s(g) < \infty. \tag{22}$$

Proof. The necessity of condition (22) is contained in Lemma 8. Now we turn to the sufficiency of (22). We chose $f \in B_\alpha^{p,1}$ and set $h = c * f$. Then, by Lemma 6:

$$h(r^2 x') = 2 \int_0^1 \int_{\mathbb{S}} \Lambda_{m+1}(g * P_\xi)(rRx') f(rR\xi)(1 - R^2)^m R^{n-1} d\xi dR \tag{23}$$

and this allows us to obtain the following estimate:

$$\begin{aligned} M_s(h, r^2) &\leqslant 2 \int_0^1 (1 - R^2)^m R^{n-1} \left\| \int_{\mathbb{S}} \Lambda_{m+1}(g * P_\xi)(rRx') f(rR\xi) d\xi \right\|_{L^s(\mathbb{S}, dx')} dR \leqslant \\ &\leqslant 2 \int_0^1 (1 - R^2)^m R^{n-1} M_1(f, rR) \sup_{\xi \in \mathbb{S}} \|\Lambda_{m+1}(g * P_\xi)(rRx')\|_{L^s} dR \leqslant \\ &\leqslant C N_s(g) \int_0^1 (1 - R)^m M_1(f, rR)(1 - rR)^{\alpha - \beta - m - 1} dR \leqslant \\ &\leqslant C N_s(g)(1 - r)^{-\beta} \int_0^1 M_1(f, rR)(1 - R)^m (1 - rR)^{\alpha - m - 1} dR. \end{aligned}$$

Note that $M_1(f, rR)$ is increasing in $0 \leqslant R < 1$, therefore we can combine Lemma 3 and the above estimate to obtain:

$$\begin{aligned} M_s^p(h, r^2) &\leqslant C N_s^p(g)(1 - r)^{-\beta p} \int_0^1 M_1^p(f, rR) \frac{(1 - R)^{mp + p - 1}}{(1 - rR)^{pm - \alpha p + p}} dR \leqslant \\ &\leqslant C N_s^p(g)(1 - r)^{-p\beta} \int_0^1 M_1^p(f, R)(1 - R)^{\alpha p - 1} dR \leqslant \\ &\leqslant C N_s^p(g)(1 - r)^{-p\beta} \|f\|_{B_\alpha^{p,1}}^p \end{aligned}$$

Therefore $M_s(h, r^2) \leqslant C N_s(g)(1 - r)^{-\beta} \|f\|_{B_\alpha^{p,1}}$, which completes the proof of the Theorem. □

3. Estimates for Distances in Harmonic Function Spaces in the Unit Ball and Related Problems in $\mathbb{R}_+^{n+1}$

In this section we investigate distance problems both in the case of the unit ball and in the case of the upper half space.

Lemma 9. *Let $0 < p < \infty$ and $\alpha > -1$. Then there is a $C = C_{p,\alpha,n}$ such that for every $f \in A_\alpha^p(\mathbb{B})$ we have*

$$|f(x)| \leqslant C(1 - |x|)^{-\frac{\alpha + n}{p}} \|f\|_{A_\alpha^p}, \qquad x \in \mathbb{B}.$$

Proof. We use subharmonic behavior of $|f|^p$ to obtain

$$|f(x)|^p \leqslant \frac{C}{(1-|x|)^n} \int_{B(x,\frac{1-|x|}{2})} |f(y)|^p dy \leqslant$$
$$\leqslant C \frac{(1-|x|)^{-\alpha}}{(1-|x|)^n} \int_{B(x,\frac{1-|x|}{2})} |f(y)|^p (1-|y|)^\alpha dy \leqslant C(1-|x|)^{-\alpha-n} \|f\|^p_{A^p_\alpha}.$$

□

This lemma shows that A^p_α is continuously embedded in $A^\infty_{\frac{\alpha+n}{p}}$ and motivates the distance problem that is investigated in Theorem 5.

Lemma 10. *Let $0 < p < \infty$ and $\alpha > -1$. Then there is $C = C_{p,\alpha,n}$ such that for every $f \in \tilde{A}^p_\alpha$ and every $(x,t) \in \mathbb{R}^{n+1}_+$ we have*

$$|f(x,t)| \leqslant C y^{-\frac{\alpha+n+1}{p}} \|f\|_{\tilde{A}^p_\alpha}. \tag{24}$$

The above lemma states that $\tilde{A}^p_\alpha$ is continuously embedded in $\tilde{A}^\infty_{\frac{\alpha+n+1}{p}}$, its proof is analogous to that of Lemma 9.

For $\epsilon > 0$, $t > 0$ and $f \in h(\mathbb{B})$ we set

$$U_{\epsilon,t}(f) = U_{\epsilon,t} = \{x \in \mathbb{B} : |f(x)|(1-|x|)^t \geqslant \epsilon\}.$$

Theorem 5. *Let $p > 1$, $\alpha > -1$, $t = \dfrac{\alpha+n}{p}$ and $\beta > \max\left(\dfrac{\alpha+n}{p} - 1, \dfrac{\alpha}{p}\right)$. Set, for $f \in A^\infty_{\frac{\alpha+n}{p}}(\mathbb{B})$:*

$$t_1(f) = \operatorname{dist}_{A^\infty_{\frac{\alpha+n}{p}}}(f, A^p_\alpha),$$

$$t_2(f) = \inf\left\{\epsilon > 0 : \int_{\mathbb{B}} \left(\int_{U_{\epsilon,t}} |Q_\beta(x,y)|(1-|y|)^{\beta-t} dy\right)^p (1-|x|)^\alpha dx < \infty\right\}.$$

Then $t_1(f) \asymp t_2(f)$.

Proof. We begin with inequality $t_1(f) \geqslant t_2(f)$. Assume $t_1(f) < t_2(f)$. Then there are $0 < \epsilon_1 < \epsilon$ and $f_1 \in A^p_\alpha$ such that $\|f - f_1\|_{A^\infty_t} \leqslant \epsilon_1$ and

$$\int_{\mathbb{B}} \left(\int_{U_{\epsilon,t}(f)} |Q_\beta(x,y)|(1-|y|)^{\beta-t} dy\right)^p (1-|x|)^\alpha dx = +\infty.$$

Since $(1-|x|)^t |f_1(x)| \geqslant (1-|x|)^t |f(x)| - (1-|x|)^t |f(x) - f_1(x)|$ for every $x \in \mathbb{B}$ we conclude that $(1-|x|)^t |f_1(x)| \geqslant (1-|x|)^t |f(x)| \geqslant (1-|x|)^t |f(x)| - \epsilon_1$ and therefore

$$(\epsilon - \epsilon_1)\chi_{U_{\epsilon,t}(f)}(x)(1-|x|)^{-t} \leqslant |f_1(x)|, \qquad x \in \mathbb{B}.$$

Hence

$$+\infty = \int_{\mathbb{B}} \left(\int_{U_{\epsilon,t}(f)} |Q_\beta(x,y)|(1-|y|)^{\beta-t} dy\right)^p (1-|x|)^\alpha dx =$$
$$= \int_{\mathbb{B}} \left(\int_{\mathbb{B}} \frac{\chi_{U_{\epsilon,t}(f)}(y)}{(1-|y|)^t} |Q_\beta(x,y)|(1-|y|)^\beta dy\right)^p (1-|x|)^\alpha dx \leqslant$$
$$\leqslant C_{\epsilon,\epsilon_1} \int_{\mathbb{B}} \left(\int_{\mathbb{B}} |f_1(y)||Q_\beta(x,y)|(1-|y|)^\beta dy\right)^p (1-|x|)^\alpha dx = M,$$

and we are going to prove that M is finite, arriving at a contradiction. Let q be the exponent conjugate to p. We have, using Lemma 4,

$$\begin{aligned}I(x) &= \left(\int_{\mathbb{B}} |f_1(y)|(1-|y|)^{\beta}|Q_\beta(x,y)|dy\right)^p = \\ &= \left(\int_{\mathbb{B}} |f_1(y)|(1-|y|)^{\beta}|Q_\beta(x,y)|^{\frac{1}{n+\beta}(\frac{n}{p}+\beta-\epsilon)}|Q_\beta(x,y)|^{\frac{1}{n+\beta}(\frac{n}{q}+\epsilon)}dy\right)^p \leqslant \\ &\leqslant \int_{\mathbb{B}} |f_1(y)|^p(1-|y|)^{p\beta}|Q_\beta(x,y)|^{\frac{n+p\beta-p\epsilon}{n+\beta}}dy\left(\int_{\mathbb{B}}|Q_\beta(x,y)|^{\frac{n+q\epsilon}{n+\beta}}dy\right)^{p/q} \leqslant \\ &\leqslant C(1-|x|)^{-p\epsilon}\int_{\mathbb{B}} |f_1(y)|^p(1-|y|)^{p\beta}|Q_\beta(x,y)|^{\frac{n+p\beta-p\epsilon}{n+\beta}}dy\end{aligned}$$

for every $\epsilon > 0$. Choosing $\epsilon > 0$ such that $\alpha - p\epsilon > -1$ we have, by Fubini's theorem and Lemma 4:

$$\begin{aligned}M &\leqslant C\int_{\mathbb{B}} |f_1(y)|^p(1-|y|)^{p\beta}\int_{\mathbb{B}}(1-|x|)^{\alpha-p\epsilon}|Q_\beta(x,y)|^{\frac{n+p\beta-p\epsilon}{n+\beta}}dxdy \leqslant \\ &\leqslant C\int_{\mathbb{B}} |f_1(y)|^p(1-|y|)^{\alpha}dy < \infty.\end{aligned}$$

In order to prove the remaining estimate $t_1(f) \leqslant Ct_2(f)$ we fix $\epsilon > 0$ such that the integral appearing in the definition of $t_2(f)$ is finite and use Theorem 1, with $\beta > \max(t-1, 0)$:

$$\begin{aligned}f(x) &= \int_{\mathbb{B}\setminus U_{\epsilon,t}(f)} Q_\beta(x,y)f(y)(1-|y|^2)^{\beta}dy + \int_{U_{\epsilon,t}(f)} Q_\beta(x,y)f(y)(1-|y|^2)^{\beta}dy = \\ &= f_1(x) + f_2(x).\end{aligned}$$

Since, by Lemma 4, $|f_1(x)| \leqslant 2^{\beta}\int_{\mathbb{B}} |Q_\beta(x,y)|(1-|w|)^{\beta-t}dy \leqslant C(1-|x|)^{-t}$ we have $\|f_1\|_{A^\infty_t} \leqslant C\epsilon$. Thus it remains to show that $f_2 \in A^p_\alpha$ and this follows from

$$\|f_2\|^p_{A^p_\alpha} \leqslant \|f\|^p_{A^\infty_t}\int_{\mathbb{B}}\left(\int_{U_{\epsilon,t}(f)}|Q_\beta(x,y)|(1-|y|^2)^{\beta-t}dy\right)^p(1-|x|)^{\alpha}dx < \infty. \qquad \square$$

The above theorem has a counterpart in the $\mathbb{R}^{n+1}_+$ setting. As a preparation for this result we need the following analogue of Lemma 4.

Lemma 11. *For $\delta > -1$, $\gamma > n+1+\delta$ and $m \in \mathbb{N}_0$ we have $\int_{\mathbb{R}^{n+1}_+} |Q_m(z,w)|^{\frac{\gamma}{n+m+1}}s^{\delta}dyds \leqslant Ct^{\delta-\gamma+n+1}$, $t > 0$.*

Proof. Using Fubini's theorem and estimate (5) we obtain

$$\begin{aligned}I(t) &= \int_{\mathbb{R}^{n+1}_+} |Q_m(z,w)|^{\frac{\gamma}{n+m+1}}s^{\delta}dyds \leqslant C\int_0^\infty s^{\delta}\left(\int_{\mathbb{R}^n}\frac{dy}{[|y|^2+(s+t)^2]^{\gamma}}\right)ds = \\ &= C\int_0^\infty s^{\delta}(s+t)^{n-\gamma}ds = Ct^{\delta-\gamma+n+1}.\end{aligned} \qquad \square$$

For $\epsilon > 0$, $\lambda > 0$ and $f \in h(\mathbb{R}^{n+1}_+)$ we set: $V_{\epsilon,\lambda}(f) = \{(x,t) \in \mathbb{R}^{n+1}_+ : |f(x,t)|t^{\lambda} \geqslant \epsilon\}$.

Theorem 6. *Let $p > 1$, $\alpha > -1$, $\lambda = \dfrac{\alpha+n+1}{p}$, $m \in \mathbb{N}_0$ and $m > \max\left(\dfrac{\alpha+n+1}{p} - 1, \dfrac{\alpha}{p}\right)$. Set, for $f \in \tilde{A}^\infty_{\frac{\alpha+n+1}{p}}(\mathbb{R}^{n+1}_+)$: $s_1(f) = \operatorname{dist}_{\tilde{A}^\infty_{\frac{\alpha+n+1}{p}}}(f, \tilde{A}^p_\alpha)$,*

$$s_2(f) = \inf\left\{\epsilon > 0 : \int_{\mathbb{R}^{n+1}_+}\left(\int_{V_{\epsilon,\lambda}} Q_m(z,w)s^{m-\lambda}dyds\right)^p t^{\alpha}dxdt < \infty\right\}.$$

Then $s_1(f) \asymp s_2(f)$.

The proof of this theorem closely parallels the proof of the previous one, in fact, the role of Lemma 4 is taken by Lemma 11 and the role of Theorem 1 is taken by Theorem 2. We leave details to the reader.

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Точные теоремы о мульпликаторах и расстояние в пространствах гармонических функций высшей размерности

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Представляются новые точные результаты, связанные с мульпликаторами и оценками расстояния в различных пространствах гармонических функций в единичном шаре из $\mathbb{R}^n$.

Ключевые слова: мультипликаторы, гармонические функции, пространства Бергмана, пространства со смешанной нормой, оценки дистанции.