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The Two-Square Lemma and the Connecting Morphism in a Preabelian Category

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We obtain a generalization of the Two-Square Lemma proved for abelian categories by Fay, Hardie, and Hilton in 1989 and (in a special case) for preabelian categories by Generalov in 1994. We also prove the equivalence up to sign of two definitions of a connecting morphism of the Snake Lemma.

Keywords: strict morphism, preabelian category, pullback, pushout, semi-stable (co)kernel, Snake Lemma, connecting morphism.

Introduction

One of the most important diagram assertions in homological algebra is the so-called Snake Lemma which makes it possible to obtain homological sequences from short exact sequences of complexes. It always holds in an abelian category. However, in the more general context of preabelian categories, The Snake Lemma fails without additional assumptions on the initial diagram. The main reasons are that the notions of kernel and monomorphism (respectively, of cokernel and epimorphism) do not coincide in a preabelian category and that kernels (respectively, cokernels) do not "survive" under pushouts (respectively, pullbacks).

The question of the validity of the Snake Lemma in the nonabelian case was studied by several authors for classes of additive categories (see, e.g., [1–5]) and in some classes of nonabelian categories (see, e.g., [6,7]). The key properties of the morphisms in the initial diagram required for the exactness of the Ker-Coker-sequence are "strictness" and stability under pushouts (pullbacks) of some monomorphisms (epimorphisms), or their weaker analogs "exactness" and "modularity" [7].

Even the existence of a connecting morphism in the Ker-Coker-sequence, valid in abelian categories (and even in quasi-abelian categories [4] and in their nonadditive counterpart, Grandis homological categories [7]), cannot be guaranteed in general preabelian categories without extra "semi-stability" assumptions (see [2]). The construction of the connecting morphism in [2] involves a preabelian version of a special case of the Two-Square Lemma of Fay–Hardie-Hilton [8, Lemma 3].

Theorem 0.1 (The Two-Square Lemma)**.** *Suppose that the following diagram in an abelian*

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category has exact rows:

$$\begin{array}{ccccc} A & \xrightarrow{\psi} & B & \xrightarrow{\varphi} & C \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow \\ A' & \xrightarrow[\psi']{} & B' & \xrightarrow[\varphi']{} & C'\,. \end{array} \tag{1}$$

Let

$$\begin{array}{ccc} Q' & \xrightarrow{\sigma} & C \\ {\scriptstyle\sigma'}\downarrow & & {\scriptstyle\gamma}\downarrow \\ B' & \xrightarrow[\varphi']{} & C' \end{array} \tag{2}$$

be a pullback and let

$$\begin{array}{ccc} A & \xrightarrow{\psi} & B \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\tau}\downarrow \\ A' & \xrightarrow[\tau']{} & Q \end{array} \tag{3}$$

be a pushout.

Then

(i) there exists a unique morphism $\theta : Q \to B'$ such that $\theta\tau = \beta$, $\theta\tau' = \psi'$;

(ii) there exists a unique morphism $\rho : B \to Q'$ such that $\sigma\rho = \varphi$, $\sigma'\rho = \beta$;

(iii) there exists a unique morphism $\eta : Q \to Q'$ such that $\eta\tau = \rho$, $\sigma'\eta = \theta$, $\sigma\eta\tau' = 0$.

The proof in [8] remains valid in any preabelian category. The Two-Square Lemma of [8] also claims that *if ψ' is a monomorphism then so is η* and *if φ is an epimorphism then so is η.*

In [2], Generalov proved the following assertion:

Theorem 0.2. *Consider a diagram of the form (1) in a preabelian category. If ψ' is a semi-stable kernel and φ is a semi-stable cokernel then η is an isomorphism.*

Below we study the question when η is a monomorphism, an epimorphism, a kernel, a cokernel in a preabelian category.

The article is organized as follows. In Sec. 1., we give basic definitions and facts about preabelian categories. In Sec. 2., we prove the main assertion of the article, Theorem 2.1, explaining what conditions on the initial diagram (1) guarantee each of the above-mentioned properties of η. In Sec. 3., we prove the equivalence up to sign of two definitions of a connecting morphism of the Ker-Coker-sequence in a preabelian category.

1. Preabelian Categories

A *preabelian category* is an additive category with kernels and cokernels.

In a preabelian category, every morphism α admits a canonical decomposition

$$\alpha = (\operatorname{im}\alpha)\bar{\alpha}(\operatorname{coim}\alpha), \quad \text{where } \operatorname{im}\alpha = \ker\operatorname{coker}\alpha,\ \operatorname{coim}\alpha = \operatorname{coker}\ker\alpha.$$

A morphism α is called *strict* if $\bar{\alpha}$ is an isomorphism. A preabelian category is abelian if and only if every morphism in it is strict. Note that

$$\text{strict monomorphisms} = \text{kernels},$$
$$\text{strict epimorphisms} = \text{cokernels}.$$

Lemma 1.1. [4, 9–11] *The following hold in a preabelian category.*

(i) A morphism α is a kernel if and only if $\alpha = \operatorname{im}\alpha$, a morphism α is a cokernel if and only if $\alpha = \operatorname{coim}\alpha$.

(ii) A morphism α is strict if and only if α is representable as $\alpha = \alpha_1\alpha_0$, where α_0 is a cokernel, α_1 is a kernel; in this case, $\alpha_0 = \operatorname{coim}\alpha$ and $\alpha_1 = \operatorname{im}\alpha$.

(iii) Suppose that the commutative square

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[d, "g"'] & D \arrow[d, "f"] \\ A \arrow[r, "\beta"'] & B \end{tikzcd} \tag{4}$$

is a pullback. Then $\ker f = \alpha\ker g$. If $f = \ker h$ for some h then $g = \ker(h\beta)$. If f is a monomorphism then g is a monomorphism; if f is a kernel then g is a kernel.

In the dual manner, assume that (4) is a pushout. Then $\operatorname{coker} g = (\operatorname{coker} f)\beta$. If $g = \operatorname{coker} e$ for some e then $f = \operatorname{coker}(\alpha e)$. If g is an epimorphism then f is an epimorphism; if g is a cokernel then f is a cokernel.

A kernel g in a preabelian category is called *semi-stable* [10] if, for every pushout of the form (4), f is a kernel too. A *semi-stable cokernel* is defined in the dual way. Examples of non-semi-stable cokernels may be found, for example, in [12–15] and non-semi-stable kernels are shown in [10]. If all kernels and cokernels are semi-stable then the preabelian category is called *quasi-abelian* [16].

Lemma 1.2. [10, 17] *The following hold in a preabelian category:*

(i) if gf is a semi-stable kernel then so is f; if gf is a semi-stable cokernel then so is g;

(ii) if f and g are semi-stable kernels (cokernels) and the composition gf is defined then gf is a semi-stable kernel (cokernel);

(iii) if (4) is a pushout and f is a semi-stable kernel then so is g; if (4) is a pullback and g is a semi-stable cokernel then so is f.

If a preabelian category satisfies the following two weaker axioms dual to one another then it is called *P-semi-abelian* or *semi-abelian* in the sense of Palamodov [18]: if (4) is a pushout and g is a kernel then f is a monomorphism; if (4) is a pullback and f is a cokernel then g is an epimorphism. Until recently it was unclear whether every P-semi-abelian category is quasi-abelian (Raikov's Conjecture); this was disproved by Bonet and Dierolf [12] and Rump [14, 15]. It turned out that, for example, the categories of barrelled and bornological locally convex spaces are P-semi-abelian but not quasi-abelian (see [15]). In general preabelian categories, kernels (cokernels) may push out (pull back) even to zero morphisms (see [10, 13]).

In [19] Kuz′minov and Cherevikin proved that a preabelian category is P-semi-abelian in the above sense if and only if, in the canonical decomposition of every morphism α, $\alpha = (\operatorname{im}\alpha)\bar{\alpha}\operatorname{coim}\alpha$, the central morphism $\bar{\alpha}$ is a bimorphism, that is, a monomorphism and an epimorphism simultaneously.

Lemma 1.3. [4, 19] *The following hold in any P-semi-abelian category:*

(i) if gf is a kernel then f is a kernel; if gf is a cokernel then g is a cokernel;

(ii) if f, g are kernels and the composition gf is defined then gf is a kernel; if f, g are cokernels and the composition gf is defined then gf is a cokernel;

(iii) if gf is strict and g is a monomorphism then f is strict; if gf is strict and f is an epimorphism then g is strict.

We observe that, in fact, in a preabelian category, items (i) and (ii) of Lemma 1.3 are equivalent to P-semi-abelianity (see [20] for details).

The following lemma is due to Yakovlev [21].

Lemma 1.4. *For every morphism α in a preabelian category,* $\ker\alpha = \ker\operatorname{coim}\alpha$*,* $\operatorname{coker}\alpha = \operatorname{coker}\operatorname{im}\alpha$*.*

A sequence $\ldots \xrightarrow{a} B \xrightarrow{b} \ldots$ in a preabelian category is said to be *exact at* B if $\operatorname{im} a = \ker b$. As follows from Lemma 1.4, this is equivalent to the fact that $\operatorname{coker} a = \operatorname{coim} b$.

2. The Two-Square Lemma

We begin with a lemma which, being itself of an independent interest, will be used below. It is a generalization of [19, Theorem 3] and [22, Lemma 6].

Lemma 2.1. *Let*

$$\begin{tikzcd} A \arrow[r, "p_1"] \arrow[d, equal] & B_1 \arrow[r, "q_1"] \arrow[d, "r"'] & C \arrow[d, equal] \\ A \arrow[r, "p_2"'] & B_2 \arrow[r, "q_2"'] & C \end{tikzcd}$$

be a commutative diagram in a preabelian category.

(i) If $p_1 = \ker q_1$, $q_2p_2 = 0$, p_2 is a monomorphism then r is a monomorphism.

(ii) Suppose that $p_1 = \ker q_1$, $p_2 = \ker q_2$, p_2 and $\operatorname{im} q_1$ are semi-stable kernels, and q_1 is strict. Then r is a semi-stable kernel.

The dual assertions also hold.

Proof. (i) Suppose that $rx = 0$ and show that then $x = 0$. We have $q_1x = q_2rx = 0$. Since $p_1 = \ker q_1$, we infer that $x = p_1y$ for some y. Then $p_2y = rp_1y = rx = 0$. Since p_2 is a monomorphism, y=0 and, thus, $x = p_1y = 0$.

(ii) Represent q_1 as $q_1 = q_1'q_1''$, $q_2 = q_2'q_2''$, where $q_j'' = \operatorname{coim} q_j : B_j \to K_j$, $j = 1, 2$. By assumption, $q_1' = \operatorname{im} q_1$. Since $\operatorname{coim} q_1 = \operatorname{coker} p_1$ and $\operatorname{coim} q_2 = \operatorname{coker} p_2$, there exists a unique morphism $w : K_1 \to K_2$ with $w \operatorname{coim} q_1 = (\operatorname{coim} q_2)r$. For this w, we have $q_1' = q_2'w$. Since, by hypothesis, q_1' is a semi-stable kernel, w is also a semi-stable kernel (Lemma 1.2).

Consider the pushout

$$\begin{tikzcd} A \arrow[r, "p_1"] \arrow[d, "p_2"'] & B_1 \arrow[d, "u_2"'] \\ B_2 \arrow[r, "u_1"] & F. \end{tikzcd}$$

Since $u_1rp_1 = u_1p_2 = u_2p_1$, we have $(u_1r - u_2)p_1 = 0$. Therefore, there exists a unique morphism $s : K_1 \to F$ such that $u_1r - u_2 = s \operatorname{coim} q_1$.

Consider the pushout

$$\begin{tikzcd} K_1 \arrow[r, "w"] \arrow[d, "s"'] & K_2 \arrow[d, "s'"'] \\ F \arrow[r, "w'"] & S. \end{tikzcd}$$

Put $\mu = w'u_1 - s'\operatorname{coim} q_2$. We infer

$$\mu r = (w'u_1 - s'\operatorname{coim} q_2)r = w'u_2 + w's \operatorname{coim} q_1 - s'(\operatorname{coim} q_2)r =$$
$$= w'u_2 + w's \operatorname{coim} q_1 - w's \operatorname{coim} q_1 = w'u_2.$$

Thus, $\mu r = w'u_2$. Since p_2 and w are semi-stable kernels, so are u_2 and w'. Now, by Lemma 1.2(ii), $\mu r = w'u_2$ is a semi-stable kernel as a composition of semi-stable kernels. Thus, by Lemma 1.2(i), r is a semi-stable kernel. The lemma is proved. □

We will also need the following preabelian version of Lemma 1 of [8]. It also generalizes Lemma 1.1(iii).

Lemma 2.2. *The following hold.*

(i) If in the commutative diagram

$$\begin{array}{ccccc} & & B & \xrightarrow{\varphi} & C \\ & & \beta\Big\downarrow & & \gamma\Big\downarrow \\ A' & \xrightarrow[\psi']{} & B' & \xrightarrow[\varphi']{} & C' \end{array} \tag{5}$$

the square $\varphi'\beta = \gamma\varphi$ is a pullback and the lower row in (5) is exact then there exists a unique morphism $\psi : A' \to B$ such that $\beta\psi = \psi'$, $\varphi\psi = 0$. If, in addition, $\bar{\psi}'$ is an epimorphism then the sequence

$$A' \xrightarrow{\psi} B \xrightarrow{\varphi} C \tag{6}$$

is exact.

(ii) If in the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\psi} & B & \xrightarrow{\varphi} & C \\ \alpha\Big\downarrow & & \beta\Big\downarrow & & \\ A' & \xrightarrow[\psi']{} & B' & & \end{array} \tag{7}$$

the square $\beta\psi = \psi'\alpha$ is a pushout and the upper row in (7) is exact then there exists a unique morphism $\varphi' : B' \to C$ such that $\varphi'\beta = \varphi$, $\varphi'\psi' = 0$. If, in addition, $\bar{\varphi}$ is a monomorphism then the sequence

$$A' \xrightarrow{\psi'} B' \xrightarrow{\varphi'} C$$

is exact.

Proof. Prove (i) (then (ii) is obtained by duality). The existence and uniqueness follow from the equalities $\varphi'\psi' = \gamma 0$. Suppose now that $\bar{\psi}'$ is an epimorphism. Then, by Lemma 1.1(iii), $\beta \ker\varphi = \ker\varphi' = \operatorname{im}\psi'$. Put $\psi = (\ker\varphi)\bar{\psi}'\operatorname{coim}\psi'$. Then $\operatorname{coker}\psi = \operatorname{coker}\ker\varphi = \operatorname{coim}\varphi$, which is the exactness of (6). □

Theorem 2.1. *Consider a commutative diagram with exact rows of the form (1) in a preabelian category. Keep the notations of Theorem 0.1. The following hold.*

(i) If ψ' is a semi-stable kernel and, in the canonical decomposition $\varphi = (\operatorname{im}\varphi)\bar{\varphi}\operatorname{coim}\varphi$ of φ, the morphism $\bar{\varphi}$ is a monomorphism then so is η.

If φ is a semi-stable cokernel and, in the canonical decomposition $\psi' = (\operatorname{im}\psi')\bar{\psi}'\operatorname{coim}\psi'$ of ψ', the morphism $\bar{\psi}'$ is an epimorphism then so is η.

(ii) If ψ' and $\operatorname{im}\varphi$ are semi-stable kernels and φ is strict then η is a semi-stable kernel.

If φ and $\operatorname{coim}\psi'$ are semi-stable cokernels and ψ' is strict then η is a semi-stable cokernel.

Proof (i) Since $\psi' = \theta\tau'$ is a semi-stable kernel, τ' is a semi-stable kernel too (Lemma 1.2(i)). In the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\psi} & B & \xrightarrow{\varphi} & C \\ \alpha\Big\downarrow & & \Big\downarrow\tau & & \Big\| \\ A' & \xrightarrow[\tau']{} & Q & \xrightarrow[\sigma\eta]{} & C \end{array}$$

the left-hand square is a pushout, the upper row is exact, and $\bar{\varphi}$ is a monomorphism. By Lemma 2.2(ii), we conclude that the sequence

$$A \xrightarrow{\tau'} Q \xrightarrow{\sigma\eta} C$$

is exact. Consequently, recalling that τ' is a kernel, we infer that $\tau' = \ker(\sigma\eta)$. Assume now that $\eta z = 0$ for some $z : Z \to Q$. We have $\sigma\eta z = 0$, and, hence, $z = \tau' z'$ for some z'. Therefore,

$$\psi' z' = \theta\tau' z' = \theta z = \sigma'\eta z = 0.$$

Since ψ' is a monomorphism, $z' = 0$, and hence $z = 0$. Thus, η is a monomorphism.

The second assertion in (i) is dual to the first.

(ii) As we have observed, $\tau' = \ker(\sigma\eta)$. Note also that $\eta\tau' = \ker\sigma$. Indeed, we have the commutative diagram

$$\begin{tikzcd}
A' \arrow[r, "\eta\tau'"] \arrow[d, equal] & Q' \arrow[r, "\sigma"] \arrow[d, "\sigma'"'] & C \arrow[d, "\gamma"] \\
A' \arrow[r, "\psi'"'] & B' \arrow[r, "\varphi'"'] & C',
\end{tikzcd}$$

in which $\psi' = \ker\varphi'$, $\bar{\psi}'$ is an epimorphism, and the square on the right is a pullback. By Lemma 2.2(i), we infer that the sequence

$$A' \xrightarrow{\eta\tau'} Q' \xrightarrow{\sigma} C$$

is exact. Since $\psi' = \sigma'\eta\tau'$ is a semi-stable kernel, by Lemma 1.2(i) $\eta\tau'$ is also a semi-stable kernel. Consequently, $\eta\tau' = \ker\sigma$.

Since the square $\tau\psi = \tau'\alpha$ is a pushout, we have $(\operatorname{coker}\tau')\tau = \operatorname{coker}\psi = \operatorname{coim}\varphi$. Therefore,

$$(\operatorname{im}\varphi)(\operatorname{coker}\tau')\tau = (\operatorname{im}\varphi)\operatorname{coim}\varphi = \varphi = \sigma\eta\tau.$$

Moreover, $(\operatorname{im}\varphi)(\operatorname{coker}\tau')\tau' = 0$ and $\sigma\eta\tau' = 0$. Hence,

$$(\sigma\eta - (\operatorname{im}\varphi)\operatorname{coker}\tau')\tau = 0, \quad (\sigma\eta - (\operatorname{im}\varphi)\operatorname{coker}\tau')\tau' = 0.$$

Since the zero morphism $0 : Q \to C$ is the only morphism y with $y\tau = 0$ and $y\tau' = 0$, we see that $(\operatorname{im}\varphi)\operatorname{coker}\tau' - \sigma\eta = 0$. Therefore, the morphism $\sigma\eta = (\operatorname{im}\varphi)\operatorname{coker}\tau'$ is strict.

We come to the commutative diagram

$$\begin{tikzcd}
A' \arrow[r, "\tau'"] \arrow[d, equal] & Q \arrow[r, "\sigma\eta"] \arrow[d, "\eta"'] & C \arrow[d, equal] \\
A' \arrow[r, "\eta\tau'"'] & Q' \arrow[r, "\sigma"'] & C,
\end{tikzcd}$$

where $\tau' = \ker(\sigma\eta)$, $\eta\tau' = \ker\sigma$, $\eta\tau'$ is a semi-stable kernel, the morphism $\sigma\eta$ is strict, and $\operatorname{im}(\sigma\eta) = \operatorname{im}\varphi$ is a semi-stable kernel. Applying Lemma 2.1, we see that η is a semi-stable kernel.

The first assertion in (ii) is proved, and the second follows by duality.

The theorem is proved. □

Note that the only thing we really need from the semi-stability of ψ' (or φ) in the proof of Theorem 2.1(i) is the implication

$$\psi' \text{ is a kernel} \Longrightarrow \tau' \text{ is a kernel} \quad (\text{resp., } \varphi \text{ is a cokernel} \Longrightarrow \sigma \text{ is a cokernel}).$$

By Lemma 1.3(i), this assertion holds also for arbitrary kernels (respectively, cokernels) in a P-semi-abelian category. Thus, we have:

Corollary 2.1. *Consider a commutative diagram with exact rows of the form (1) in a P-semi-abelian category. The following hold.*

(i) If ψ' is a kernel then η is a monomorphism. If φ is a cokernel then η is an epimorphism.

(ii) If ψ' is a semi-stable kernel and φ is a cokernel (or if ψ' is a kernel and φ is a semi-stable cokernel) then η is an isomorphism.

3. Two Definitions of a Connecting Morphism

Consider the commutative diagram

$$\begin{array}{ccccccccc} & & A & \xrightarrow{\psi} & B & \xrightarrow{\varphi} & C & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & \\ 0 & \longrightarrow & A' & \xrightarrow[\psi']{} & B' & \xrightarrow[\varphi']{} & C' & & \end{array} \tag{8}$$

where $\psi' = \ker \varphi'$ and $\varphi = \operatorname{coker} \psi$, in a preabelian category.

As in the abelian case, (8) gives rise to two parts of a Ker-Coker-sequence (the composition of two consecutive arrows is zero):

$$\operatorname{Ker} \alpha \xrightarrow{\varepsilon} \operatorname{Ker} \beta \xrightarrow{\zeta} \operatorname{Ker} \gamma$$

and

$$\operatorname{Coker} \alpha \xrightarrow{\tau} \operatorname{Coker} \beta \xrightarrow{\theta} \operatorname{Coker} \gamma.$$

In contrast to the case of an abelian category (or even a Grandis-homological [7] or a quasi-abelian [4] category), for preabelian categories, it is in general impossible to construct a natural connecting morphism $\delta : \operatorname{Ker} \gamma \to \operatorname{Coker} \alpha$. We will duscuss two constructions of δ, one going back to André–MacLane, and the other based on the Two-Square Lemma, which was proposed by Fay–Hardie–Hilton in [8] for abelian categories and adapted to the preabelian case by Generalov in [2].

3.1 The André–MacLane Construction

According to [23], the following construction, described in [24, p. 203] for abelian categories, is due to André–MacLane. It was used in [4, 5] for quasi-abelian and P-semi-abelian categories.

Let

$$\begin{array}{ccc} X & \xrightarrow{s} & \operatorname{Ker} \gamma \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \ker \gamma} \\ B & \xrightarrow[\varphi]{} & C \end{array} \tag{9}$$

be a pullback and let

$$\begin{array}{ccc} A' & \xrightarrow{\psi'} & B' \\ {\scriptstyle \operatorname{coker} \alpha}\downarrow & & \downarrow{\scriptstyle v} \\ \operatorname{Coker} \alpha & \xrightarrow[t]{} & Y \end{array} \tag{10}$$

be a pushout.

Instead of semi-stability conditions of universal nature, impose on our situation ad hoc "modularity" conditions à la Grandis [7]:

Assumptions A. *In (9), s is an epimorphism and in (10), t is a kernel.*

Assumptions A are fulfilled in a semi-abelian category if ψ' is a semi-stable kernel and φ is a semi-stable cokernel. In a P-semi-abelian category, the semi-stability of ψ' is already enough.

Since square (10) is a pushout, $(\operatorname{coker} t)v = \operatorname{coker}\psi' = \operatorname{coim}\varphi'$. Putting $(\operatorname{im}\varphi')\bar{\varphi}' = \chi$, we have $\varphi' = \chi(\operatorname{coker} t)v$. We infer

$$v\beta\psi = v\psi'\alpha = t(\operatorname{coker}\alpha)\alpha = 0.$$

Therefore, $v\beta = n\varphi$ for some unique n. In the dual manner, $\varphi'\beta u = 0$, and, hence, $\beta u = \psi' m$ for a unique morphism m. We have

$$(\operatorname{coker} t)n(\ker\gamma)s = (\operatorname{coker} t)n\varphi u = (\operatorname{coker} t)v\beta u = (\operatorname{coker}\psi')\psi' m = 0.$$

Since s is an epimorphism, this implies that $(\operatorname{coker} t)n\ker\gamma = 0$. Since $t = \ker\operatorname{coker} t$, we conclude that $n\ker\gamma = t\delta_I$ for some unique δ_I. The morphism δ_I is uniquely characterized by the property

$$t\delta_I s = v\beta u. \tag{11}$$

By duality, consider

Assumptions A*. *In (9), s is a cokernel and, in (10), t is a monomorphism.*

In this case, we also obtain a morphism δ_I defined by (11). Therefore, the two morphisms coincide if s is a cokernel and t is a kernel.

3.2 The Fay–Hardie–Hilton–Generalov Construction

Consider the diagram (8) and assume the fulfillment of one of the following conditions (i) and (ii).

(i) The ambient category is preabelian, ψ' is a semi-stable kernel, and φ is a semi-stable cokernel.

(ii) The ambient category is P-semi-abelian and ψ' is a semi-stable kernel *or* φ is a semi-stable cokernel.

Below we use all notations of the previous subsection and Section 2..

From Generalov's Theorem (Theorem 0.2) or Theorem 2.1 for (i) or Corollary 2.1 for (ii) it follows that, in these cases, the morphism $\eta : Q \to Q'$ is an isomorphism and, therefore, we may assume that $Q = Q'$, $\eta = \mathrm{id}_Q$. Since (3) is a pushout, $\operatorname{coker}\alpha = (\operatorname{coker}\tau)\tau'$; since (2) is a pullback, $\ker\gamma = \sigma(\ker\sigma')$. Put

$$\delta_{II} = (\operatorname{coker}\tau)\ker\sigma'.$$

Theorem 3.1. *The equality $\delta_{II} = -\delta_I$ holds.*

Proof. Prove that $-\delta_{II}$ satisfies (11), i.e., that $t\delta_{II}s = -v\beta u$.

Following [2], put for brevity $\delta_1 = \operatorname{coker}\tau$, $\delta_2 = \ker\sigma'$. Then, by definition, $\delta_{II} = \delta_1\delta_2$.

We have the following "multiplication table":

$$\sigma\tau = \varphi; \quad \sigma\tau' = 0;$$
$$\sigma'\tau = \beta; \quad \sigma'\tau' = \psi'.$$

Hence,

$$(v\sigma' - n\sigma)\tau' = v\sigma'\tau' - n\sigma\tau' = v\psi' = t\operatorname{coker}\alpha = t\delta_1\tau',$$
$$(v\sigma' - n\sigma)\tau = v\beta - n\varphi = v\beta - v\beta = 0.$$

Thus, $(t\delta_1 - (v\sigma' - n\sigma))\tau' = 0, \quad (t\delta_1 - (v\sigma' - n\sigma))\tau = 0$. Therefore, since the square $\tau\psi = \tau'\alpha$ is a pushout, this implies that $t\delta_1 = v\sigma' - n\sigma$. By duality, $\delta_2 s = \tau u - \tau' m$. Consequently,

$$t\delta_{II}s = t\delta_1\delta_2 s = (v\sigma' - n\sigma)(\tau u - \tau' m)$$
$$= v\sigma'\tau u - v\sigma'\tau' m - n\sigma\tau u + n\sigma\tau' m = v\beta u - v\psi' m - v\beta u = -v\beta u.$$

The theorem is proved. □

Even having a connecting morphism $\delta : \operatorname{Ker}\gamma \to \operatorname{Coker}\alpha$, we in general cannot assert that the corresponding Ker-Coker-sequence is exact. For its exactness, one usually has to impose extra conditions like strictness or semi-stability (see [2–5, 7]).

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Лемма о двух квадратах и связывающий морфизм в предабелевой категории

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*В работе получено обобщение леммы о двух квадратах, доказанной для абелевых категорий Фэем, Харди и Хилтоном в 1989 г. и (в специальном случае) для предабелевых категорий Генераловым в 1994 г. Также доказана эквивалентность с точностью до знака двух определений связывающего морфизма в лемме о змее (*Ker-Coker*-последовательности).*

Ключевые слова: строгий морфизм, предабелева категория, (ко)универсальный квадрат, полустабильное (ко)ядро, лемма о змее, связывающий морфизм.