ON A MIXED PROBLEM FOR THE PARABOLIC LAMÉ TYPE OPERATOR

R. PUZYREV AND A. SHLAPUNOV

Abstract. We consider a boundary value problem for a Lamé type operator, which corresponds to a linearisation of the Navier-Stokes' equations for compressible flow of Newtonian fluids in the case where pressure is known. It consists of recovering a vector function, satisfying the parabolic Lamé type system in a cylindrical domain, via its values and the values of the boundary stress tensor on a given part of the lateral surface of the cylinder. We prove that the problem is ill-posed in the natural spaces of smooth functions and in the corresponding Hölder spaces; besides, additional initial data do not turn the problem to a well-posed one. Using the integral representation's method we obtain a uniqueness theorem and solvability conditions for the problem. We also describe conditions, providing dense solvabilty of the problem.

Introduction

Let, Δ_n be the Laplace operator, ∇_n be the gradient operator and div_n be the divergence operator in $\mathbb{R}^n$, $n \geq 2$. The Navier- Stokes' equations for compressible flow of Newtonian fluids over the four-dimensional domain $\mathcal{D} \subset \mathbb{R}^3_x \times \mathbb{R}_t$ under action of force $F(x,t) = (F_1(x,t), F_2(x,t), F_3(x,t))$ can be written in the following form (see [1, §15, formulas (15.5), (15.6)]):

$$\rho\Big(\frac{\partial v}{\partial t} + v \cdot \nabla_3 v\Big) + \nabla_3 p - \text{div}_3\Big(\mu_1 \nabla_3 v\Big) - \nabla_3\Big(\Big(\frac{\mu_1}{3} + \mu_2\Big)\text{div}_3 v\Big) - av = F, \quad (1)$$

where $v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))$ is the flow velocity, $\rho(x,t)$ is the fluid density, $p(x,t)$ is the pressure, $\mu_j(x,t)$ are (positive) viscosity coefficients,

$$av = [(\nabla_3 \mu_1)^* \otimes \nabla_3 - (\nabla_3 \mu_1)\text{div}_3]v$$

is the linear first order term, M_1^* is the adjoint matrix for a matrix M_1 and $M_1^* \otimes M_2$ is the Kronecker product of matrices M_1^* and M_2. If the boundary ∂D of D is piecewise smooth then the boundary conditions for this system often involve the force $\nu p - \sigma' v$ acting on the unit surface area where the force friction (or the boundary viscosity tensor) σ' has the following entries:

$$\sigma'_{i,j} = \delta_{i,j}\mu_1 \sum_{k=1}^{n} \nu_k \frac{\partial}{\partial_k} + \mu_1 \nu_j \frac{\partial}{\partial x_i} + (\mu_2 - 2\mu_1/3)\nu_i \frac{\partial}{\partial x_j}, \ 1 \leq i,j \leq 3,$$

where $\nu = (\nu_1, \nu_2, \nu_3, \nu_4)$ denotes the unit normal vector to the surface ∂D and $\delta_{i,j}$ means the Kronecker symbol (see [1, §15, formula (15.12)]).

Key words and phrases. Boundary value problems for parabolic equations, ill-posed problems, integral representation's method.

Under given pressure p, since the density ρ is positive, a proper linearisation of the substantial derivative term $v\cdot\nabla_3 v$ turns (1) into a parabolic Lamé type system related to an unknown vector u:

$$L_3 u = \frac{\partial u}{\partial t} - \mathcal{L}_3 u - \sum_{j=1}^{3} a_j(x,t)\frac{\partial u}{\partial x_j} - a_0(x,t)u = f$$

where $a_j(x,t)$, $0\leq j\leq 3$, are (3×3) matrices with functional entries and

$$\mathcal{L}_n = \mathrm{div}_n\Big(\mu\nabla_n\Big) + \nabla_n\Big((\mu+\lambda)\mathrm{div}_n\Big),\ n\geq 2$$

is a strongly elliptic (with respect to the space variables) formally self-adjoint Lamé type operator with the Lamé coefficients satisfying

$$\mu(x,t)>0,\ \ (\mu(x,t)+\lambda(x,t))\geq 0.$$

The smoothness of the Lamé coefficients and the entries of the matrices $a_j(x,t)$ depends upon the regularity of the density ρ and the viscosity coefficients μ_j.

Note that, if μ is constant, $\lambda+\mu=0$ and $a_j=0$, $0\leq j\leq 3$, then L_4 reduces to the heat operator, though, of course, it is known that the heat equation is not ideal to model the process of the heat conduction.

Let Ω be a bounded domain (i.e. bounded open connected set) in n-dimensional real space $\mathbb{R}^n$ with the coordinates $x=(x_1,\dots,x_n)$. As usual we denote by $\overline{\Omega}$ the closure of Ω, and we denote by $\partial\Omega$ its boundary. We assume that $\partial\Omega$ is piece-wise smooth. Then the unit normal vector $\nu=(\nu_1,...,\nu_n)$ is defined almost everywhere on $\partial\Omega$.

Let $\Omega_T=\{x\in\Omega,\ 0<t<T\}$ be an open cylinder in $(n+1)$-dimensional real space $\mathbb{R}^{n+1}=\mathbb{R}^n\times\{-\infty<t<+\infty\}$, having the altitude $0<T\leq+\infty$ and the base Ω. Let also $\Gamma\subset\partial\Omega$ be a non empty connected relatively open subset of $\partial\Omega$ and $\Gamma_T=\Gamma\times(0,\ T)$.

In the present paper we consider the following mixed boundary problem for the parabolic system in the cylindrical domain Ω_T

$$L_n = \frac{\partial}{\partial t} - \mathcal{L}_n - Au,$$

where

$$Au = \sum_{j=1}^{n} a_j(x,t)\frac{\partial}{\partial x_j} + a_0(x,t),$$

the Lamé coefficients and the entries of the $(n\times n)$-matrices $a_j(x,t)$, $0\leq j\leq n$, are C^∞-smooth in a neighbourhood of $\overline{\Omega_T}$ and real analytic with respect to the space variables in a neighbourhood of $\overline{\Omega}$.

Instead of classical boundary value problems for parabolic equations (see, for instance, [2], [3], [4], [5]), we consider an ill-posed problem, consisting in finding a vector function satisfying the corresponding parabolic equation in the cylinder via its values and the values of the boundary stress tensor with the entries

$$\sigma_{i,j} = \mu\,\delta_{i,j}\sum_{k=1}^{n}\nu_k\frac{\partial}{\partial x_k} + \mu\,\nu_j\frac{\partial}{\partial x_i} + \lambda\,\nu_i\frac{\partial}{\partial x_j},\ \ 1\leq i,j\leq n. \tag{2}$$

on the given part Γ_T of the lateral surface of the cylinder Ω_T (cf. [6]).

Using parabolic potentials we prove a uniqueness theorem and obtain solvability conditions for the problem (cf. [7] related to similar results for the heat equation).

Besides, we describe conditions, providing dense solvabilty of the problem. Actually, the approach was invented for the investigation of the famous ill-posed Cauchy problem for elliptic equations (see, for instance, [8] for the Cauchy-Riemann operator, [9] for the Laplace equation, [10] for the elliptic Lamé operator and [11], [12], [15], for general systems with injective principal symbols).

1. Preliminaries

As usual, for $s \in \mathbb{Z}_+$ and an open subset $D \subset \mathbb{R}^m$ we denote by $C^s(D)$ the set of all s times continuously differentiable functions in D. The standard topology of this metrisable space induces uniform convergence on compact subsets in D together with all the partial derivatives up to order s.

For $S \subset \partial D$ we denote by $C^s(D\cup S)$ the set of such functions from the space $C^s(D)$ that all their derivatives up to order s can be extended continuously onto $D\cup S$. The standard topology of this metrisable space induces uniform convergence on compact subsets in $D\cup S$ together with all the partial derivatives up to order s. In particular, for bounded domains, $C^s(D\cup \partial D) = C^s(\overline{D})$ is a Banach space.

Apart from the standard functional spaces, we need also spaces taking into account the specific properties of parabolic equations in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \{-\infty < t < +\infty\}$. Namely, let $C^{1,0}(\Omega_T)$ be the set of continuous functions u in Ω_T, having in Ω_T continuous partial derivatives u_{x_i}, and let $C^{2,1}(\Omega_T)$ denote the set of continuous functions in Ω_T, having in Ω_T continuous partial derivatives u_{x_i}, $u_{x_ix_j}$, u_t. The standard topology of this metrisable space induces uniform convergence on compact subsets in D together with all the partial derivatives used in its definition.

As before, for $S \subset \partial\Omega_T$ we denote by $C^{1,0}(\Omega_T\cup S)$ the set of such functions u from the space $C^{1,0}(\Omega_T)$ that their derivatives u_{x_i} can be extended continuously onto $\Omega_T\cup S$. The standard topology of this metrisable space induces uniform convergence on compact subsets of $\Omega_T\cup S$ of both the functional sequences and the corresponding sequences of first partial derivatives x_i. Clearly, $C^{1,0}(\Omega_T \cup \partial\Omega_T) = C^{1,0}(\overline{\Omega_T})$ is a Banach space.

Let also $L^q(D)$, $1 \leq q \leq +\infty$, stand for the Lebesgue space of functions in D. This is a Banach space with the standard norm.

The space of n-vector functions $u = (u_1, \dots, u_n)$ of a class $\mathfrak{C}$ will be denoted by $[\mathfrak{C}]^n$.

Let now θ be such positive constant that we have

$$\mu(x,t) \geq \theta,\ (\lambda(x,t) + 2\mu(x,t)) \geq \theta \text{ for all } (x,t) \in \overline{\Omega}_T.$$

Then we have

$$\det\Big(\mu(x,t)|\zeta|^2(\sqrt{-1})^2 I_n + (\lambda(x,t)+\mu(x,t))\zeta\zeta^T(\sqrt{-1})^2 - \kappa I_n\Big) =$$
$$(-1)^n(\mu(x,t)|\zeta|^2+\kappa)^{n-1}((2\mu(x,t)+\lambda(x,t))|\zeta|^2+\kappa)$$

for all $\zeta \in \mathbb{R}^n$, where I_n is the unit $(n\times n)$ matrix and ζ^T is the transposed vector for ζ. Hence the roots of this polynomial (with respect to κ) are

$$\kappa_1(x,t,\zeta) = -(2\mu(x,t)+\lambda(x,t))|\zeta|^2,\ \kappa_2(x,t,\zeta) = -\mu(x,t)|\zeta|^2$$

and, for all $(x,t) \in \overline{\Omega}_T$, we have

$$\max\Big(\sup_{|\zeta|=1}\kappa_1(x,t,\zeta), \sup_{|\zeta|=1}\kappa_2(x,t,\zeta)\Big) \leq -\theta,$$

i.e. the operator L_n is uniformly parabolic (according to Petrovskii) on $\overline{\Omega}_T$.

Now we assume that there is a n-dimensional domain $U \supset \overline{\Omega}$ such that the Lamé coefficients $\mu(x,t)$, $\lambda(x,t)$ and the entries of the $(n\times n)$-matrices $a_j(x,t)$, $0 \le j \le n$, are C^∞-smooth in $\overline{U_T}$ and real analytic with respect to the space variables in U.

Under the assumptions, the following properties hold true for parabolic operator L_n, which will be crucial for the approach below (see, for instance, [4, ch. 2]).

Theorem 1. *Each weak solution u to $L_n u = 0$ in the domain $\Omega_T \subset U_T$ belongs to $C^\infty(\Omega_T)$ and it is actually real analytic with respect to variables x in Ω.*

Theorem 2. *The operator L_n has a fundamental solution in U_T, i.e. a $(n\times n)$-matrix $\Phi(x,t,y,\tau)$ satisfying*

$$(L_n)_{x,t}\Phi(x,t,y,\tau) = 0,\ (L_n^*)_{y,\tau}\Phi(x,t,y,\tau) = 0, \quad \text{if } (x,t) \ne (y,\tau), \tag{3}$$

with the formal adjoint operators

$$(L_n^*)_{y,\tau} = -\frac{\partial}{\partial \tau} - (\mathcal{L}_n)_y - A^*,\ A^* = -\sum_{k=1}^n \frac{\partial}{\partial y_k}(a_k^*(y,\tau)\cdot) + a_0^*(y,\tau),$$

and such that, for each fixed $\tau > 0$, the integral

$$u(x,t,\tau) = \int_0^t \Phi^*(x,t,y,\tau)\phi(y)dy$$

satisfies

$$L_n(x,t)u(x,t,\tau) = 0 \text{ for all } x \in \mathbb{R}^n \text{ and } t > \tau.$$
$$u(x,t,\tau) = \phi(x) \text{ for all } x \in \mathbb{R}^n \text{ and } t = \tau$$

if ϕ is a bounded continuous function in $\mathbb{R}^n$.

We need a sort of an integral representation, similar to the famous Green formula for the Laplace operator, constructed with the use of the fundamental solutions. More precisely, consider the cylinder type domain $\Omega_{T_1,T_2} = \Omega_{T_2} \setminus \overline{\Omega_{T_1}}$ and a closed measurable set $S \subset \partial\Omega$.

Let σ be the tensor with the entries given in (2) and

$$\tilde\sigma = \sigma - \sum_{k=1}^n a_k^*(x,t)\nu_k(x) + [(\nabla_n \mu(x,t))\nu^*(x) - \nu(x)(\nabla_n\mu(x,t))^*].$$

For vector functions $f \in [C(\overline{\Omega_{T_1,T_2}})]^n$, $v \in [C(\overline{S_T})]^n$, $w \in [C(\overline{S_T})]^n$, $h \in [C(\overline{\Omega})]^n$ we set

$$I_{\Omega,T_1}h(x,t) = -\int_\Omega \Phi^*(x,t,y,T_1)h(y)dy, \tag{4}$$

$$G_{\Omega,T_1}f(x,t) = \int_{T_1}^t \int_\Omega \Phi^*(x,t,y,\tau)f(y,\tau)dyd\tau, \tag{5}$$

$$V_{S,T_1}v(x,t) = \int_{T_1}^t \int_S \Phi^*(x,t,y,\tau)v(y,\tau)ds(y)d\tau, \tag{6}$$

$$W_{S,T_1}w(x,t) = -\int_{T_1}^t \int_S [\tilde\sigma_y \Phi(x,t,y,\tau)]^* w(y,\tau)ds(y)d\tau, \tag{7}$$

where ds is the volume form on S induced from $\mathbb{R}^n$. All these functions are called *parabolic potentials* with densities f, v, w and h, respectively. In our situation these are convergent improper integrals depending on the vector parameter (x,t) in the neighbourhood U of the cylinder $\overline{\Omega_{T_1,T_2}}$ in $\mathbb{R}^{n+1}$ (see, for instance, [2, ch. 4, §1], [16, ch. 3, §10], [3, ch. 1, §3 and ch. 5, §2]). The potential $I_{\Omega,T_1}(h)$ is sometimes called *Poisson type integral* for the Lam'e type operator, the functions $G_{\Omega,T_1}(f)$, $V_{S,T_1}(v)$, $W_{S,T_1}(w)$ are often referred to as *parabolic volume potential*, *parabolic single layer potential* and *parabolic double layer potential*, respectively.

Lemma 1. *The following formula holds:*

$$\left(I_{\Omega,T_1}u + G_{\Omega,T_1}L_n u + V_{\partial\Omega,T_1}\sigma u + W_{\partial\Omega,T_1}u\right)(x,t) = \begin{cases} u(x,t), & (x,t)\in\Omega_{T_1,T_2} \\ 0, & (x,t)\in U_T\setminus\overline{\Omega_{T_1,T_2}}. \end{cases} \tag{8}$$

for all $0\le T_1 < T_2\le T$ *and all* $u\in[C^{2,1}(\Omega_{T_1,T_2})\cap C^{1,0}(\overline{\Omega_{T_1,T_2}})]^n$ *with* $L_n u\in[C(\overline{\Omega_{T_1,T_2}})]^n$.

Proof. Indeed, it follows from Gauß-Ostrogradskii formula that

$$\int_{\partial\Omega} v^*\sigma u = \int_{\partial\Omega} v^*(\mathcal{L}_n u + au)dy + \mathfrak{D}_\Omega(u,v) \tag{9}$$

for all $u,v\in[C^{1,0}(\overline{\Omega}_{T_1,T_2})]^n$ with $L_n u\in[C(\overline{\Omega}_{T_1,T_2})]^n$, where

$$au = [(\nabla_n\mu)^*\otimes\nabla_n - (\nabla_n\mu)\mathrm{div}_n]u, \tag{10}$$

$$\mathfrak{D}_\Omega(u,v) = \int_\Omega \Big(\mu(\nabla_n v)^*\nabla_n u + \mu(\nabla_n v)^*\otimes\nabla_n u) + \lambda(\mathrm{div}_n v)^*\mathrm{div}_n u\Big)dy. \tag{11}$$

On the other hand, by Gauß-Ostrogradskii formula,

$$\int_{\partial\Omega} v^*(y)[(\nabla_n\mu(y,\tau))\nu^*(x) - \nu(y)(\nabla_n\mu(y,\tau))^*]u(y)ds(y) = \int_\Omega v^*(au-(a^*v)^*u)\,dy.$$

Therefore

$$\int_{\partial\Omega}\Big(v^*\sigma u - (\tilde\sigma v)^*u\Big)ds(y) = \int_\Omega\Big(v^*(\mathcal{L}_n u + Au) - (\mathcal{L}_n v + A^*v)^*u\Big)dy$$

for all $u,v\in[C^1(\overline{\Omega})]^n$ with $\mathcal{L}_n u, \mathcal{L}_n v\in[C(\overline{\Omega})]^n$. Hence, again by Gauß-Ostrogradskii fgormula, we obtain the (first) Green formula for the Lamé type operator:

$$\int_\Omega[v^*(y,T_1)u(y,T_1) - v^*(y,T_2)u(y,T_2)]dy - \int_{T_1}^{T_2}\int_{\partial\Omega}\Big(v^*\sigma u - (\tilde\sigma v)^*u\Big)ds(y)\,d\tau = \int_{\Omega_{T_1,T_2}}\Big(v^*L_n u - (L_n^*v)^*u\Big)d\tau\,dy \tag{12}$$

for all $u,v\in[C^{1,0}(\overline{\Omega}_{T_1,T_2})]^n$ with $L_n u, L_n^* v\in[C(\overline{\Omega}_{T_1,T_2})]^n$.

It follows from the definition of the fundamental solution, that

$$(L_n)_{x,t}\Phi(x,t,y,\tau) = \delta(x-y,t-\tau),\ (L_n^*)_{y,\tau}\Phi(x,t,y,\tau) = \delta(x-y,t-\tau),$$

$$\Phi(x,t,y,\tau) = 0 \text{ for } \tau > t,$$

see, for instance, [4, Theorem 2.2]). Then, using the standard arguments (see, for instance, [17, ch. 6, §12] for the heat equation), we see that Green's formula (8) follows from (12) and Fubini theorem. □

Theorem 3 (Uniqueness Theorem)**.** *If Γ has at least one interior point (on $\partial\Omega$), and function $u \in C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \overline{\Gamma_T})$ satisfies $L_n u \equiv 0$ in Ω, $u \equiv 0$ on Γ_T, $\sigma u \equiv 0$ on Γ_T, then $u \equiv 0$ in Ω_T.*

Proof. Under the hypothesis of the theorem there is an interior point x_0 on Γ. Then there is such a number $r > 0$ that $B(x_0,\ r) \cap \partial\Omega \subset \Gamma$ where $B(x_0,\ r)$ is a ball in $U \subset \mathbb{R}^n$ with centre at x_0 and radius r. Fix an arbitrary point $(x', t') \in \Omega_T$. It is clear that there is a domain $\Omega' \ni x'$ satisfying $\Omega' \subset \Omega$ and $\Omega' \cap \partial\Omega \subset \Gamma \cap B(x_0,\ r)$. Then $(x', t') \in \Omega'_{T_1,T_2}$ with some $0 < T_1 < T_2 < T$.

But $u \in C^{2,1}(\Omega'_{T_1,T_2}) \cap C^{1,0}(\overline{\Omega'_{T_1,T_2}})$ and $L_n u = 0$ in Ω'_{T_1,T_2} under the hypothesis of the theorem. Hence formula (8) implies:

$$I_{\Omega',T_1} u(x,t) + V_{\partial\Omega'\setminus\Gamma,T_1} \sigma u(x,t) + W_{\partial\Omega'\setminus\Gamma,T_1} u(x,t) = \begin{cases} u(x,t),\ (x,t) \in \Omega'_{T_1,T_2}, \\ 0,\ (x,t) \in U_T \setminus \overline{\Omega'_{T_1,T_2}}, \end{cases} \tag{13}$$

because $u \equiv \sigma u \equiv 0$ on Γ_T.

Taking into account the character of the singularity of the kernel $\Phi(x, y, t, \tau)$ (see [4, Theorem 2.2]) , we conclude that the following properties are fulfilled for the integrals, depending on parameter, in the right hand side of identity (13):

$$I_{\Omega',T_1}(u) \in C^{2,1}(U_{T_1,T_2}),$$

$$W_{\partial\Omega'\setminus\Gamma,T_1} u,\ V_{\partial\Omega'\setminus\Gamma,T_1} \sigma u \in C^{2,1}((U \setminus (\partial\Omega' \setminus \Gamma))_{T_1,T_2})$$

(see, for instance, [2, ch. 4, §1], [16, ch. 3, §10] or [3, ch. 1, §3 and ch. 5, §2]). Moreover, as Φ is a fundamental solution to Lamé type operator then using (3) and Leibniz rule for differentiation of integrals depending on parameter we obtain:

$$L_n I_{\Omega',T_1} u = 0 \text{ in } U_{T_1,T_2},$$

$$L_n V_{\partial\Omega'\setminus\Gamma,T_1} \sigma u = L_n W_{\partial\Omega'\setminus\Gamma,T_1} u = 0 \text{ in } (U \setminus (\partial\Omega' \setminus \Gamma))_{T_1,T_2}.$$

Hence the function

$$P(x,t) = I_{\Omega',T_1} u(x,t) + V_{\partial\Omega'\setminus\Gamma,T_1} \sigma u(x,t) + W_{\partial\Omega'\setminus\Gamma,T_1} u(x,t),$$

satisfies the Lamé type equation

$$(L_n P)(x,t) = 0 \text{ in } (U \setminus (\partial\Omega' \setminus \Gamma))_{T_1,T_2}.$$

This implies that the function $P(x,t)$ is real analytic with respect to the space variable $x \in U \setminus (\partial\Omega' \setminus \Gamma)$ for any $T_1 < t < T_2$ (see, for instance, [19, ch. VI, §1, theorem 1]). In particular, by the construction the function $P(x,t)$ is real analytic with respect to x in the ball $B(x_0, r)$ and it equals to zero for $x \in B(x_0, R) \setminus \overline{\Omega}$ for all $T_1 < t < T_2$. Therefore, the Uniqueness Theorem for real analytic functions yields $P(x,t) \equiv 0$ in $(U \setminus (\partial\Omega' \setminus \Gamma))_{T_1,T_2}$, and in the cylinder Ω'_{T_1,T_2}, containing the point (x', t'). Now it follows from (13) that $u(x', t') = P(x', t') = 0$ and then, since the point $(x', t') \in \Omega_T$ is arbitrary we conclude that $u \equiv 0$ in Ω_T. The proof is complete. □

Example 1. *Let $\mu = 1$, $\lambda = -1$ and $a_j = 0$, $0 \le j \le n$. Then L_n reduces to the heat operator:*

$$L_n = \frac{\partial}{\partial t} - \Delta_n$$

and corresponding fundamental solution is given by $\Phi(x,y,t,\tau) = \varphi_0(x-y,t-\tau)I_n$ *where*

$$\varphi_0(x,t) = \begin{cases} \frac{1}{\left(2\sqrt{\pi t}\right)^n}\ e^{-\frac{|x|^2}{4t}} & \text{if } t > 0, \\ 0 & \text{if } t \le 0. \end{cases}$$

In this case $\tilde{\sigma} = \sigma = \frac{\partial}{\partial \nu}$ *is the normal derivative with respect to* ∂D.

Example 2. *Let* μ, λ *be constants and* $a_j = 0$, $0 \le j \le n$. *Then* L_n *reduces to the parabolic Lamé operator*

$$L_n = \frac{\partial}{\partial t} - \mathcal{L}_n$$

and corresponding fundamental solution $\Phi(x,y,t,\tau)$ *is given by* $(n \times n)$*-matrix with entries* $\Phi_{i,j}(x,y,t,\tau) = \varphi_{i,j}(x-y,t-\tau)$ *where*

$$\varphi_{i,j}(x,t) = \varphi_0(x,\mu t)\delta_{i,j} + \int_{\mu t}^{(2\mu+\lambda)t} \frac{\partial^2 \varphi_0(x,s)}{\partial x_j \partial x_i} ds,$$

(see, for instance, [4]*). In this case* $\tilde{\sigma} = \sigma = \mu \frac{\partial}{\partial \nu} + \mu\, \nu^* \otimes \nabla_n + \lambda\, \nu\, div_n$ *is the stress operator on* ∂D.

2. The boundary problems

Green formula (8) and the Uniqueness Theorem 3 suggest us to consider two kind of problems for the parabolic Lamé type operator.

Let vector functions

$$u^{(0)}(x) \in [C(\overline{\Omega})]^n,\ f(x,t) \in [C(\overline{\Omega_T})]^n,$$

$$u^{(1)}(x,t) \in [C^{1,0}(\overline{\Gamma_T})]^n,\ u^{(2)}(x,t) \in [C(\overline{\Gamma_T})]^n$$

be given.

Problem 1. *Find a vector function* $u(x,\ t) \in [C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \overline{\Gamma_T})]^n$ *satisfying the Lamé type equation*

$$L_n u = f \ \text{ in } \Omega_T \tag{14}$$

and boundary conditions

$$u(x,t) = u^{(1)}(x,\ t) \ \text{ on } \overline{\Gamma_T}, \tag{15}$$

$$\sigma u(x,t) = u^{(2)}(x,\ t) \ \text{ on } \overline{\Gamma_T}. \tag{16}$$

Note that, if the surface Γ and the data of the problem are real analytic then the Cauchy-Kovalevsky Theorem implies that Problem 1 can not have more than one solution in the class of (formal) power series. However the theorem does not imply the existence of solutions to Problem 1 because it grants the solution in a small neighbourhood of the surface Γ_T only (but not in a given domain Ω_T!). In any case, we do not assume the real analyticity of Γ and the data $u^{(1)}$, $u^{(2)}$ and f.

Another problem involves initial data.

Problem 2. *Find a vector function* $u(x,\ t) \in [C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \overline{\Gamma_T})]^n$ *satisfying in* Ω_T *Lamé type equation* (14)*, boundary conditions* (15)*,* (16) *and initial condition*

$$u(x,0) = u^{(0)}(x), \quad x \in \overline{\Omega}. \tag{17}$$

Of course one should also take care on the compatibility of the data $u^{(0)}$, $u^{(1)}$, $u^{(2)}$: at least

$$u^{(0)}(x) = u^{(1)}(x,0) \text{ on } \overline{\Gamma}, \tag{18}$$

and, if $u^{(0)} \in C^1(\overline{\Omega})$,

$$\sigma u^{(0)}(x) = u^{(2)}(x,0) \text{ on } \overline{\Gamma}. \tag{19}$$

The motivation of Problems 1 and 2 is transparent. The first problem describes the situation where for some reasons at each time $t \geq 0$ only part $\overline{\Gamma}$ of the solid surface $\partial\Omega$ bounding the fluid is available for measurements. The second problem describes the situation where the continuity up to $\partial\Omega_T$ is postulated, the "velocity" u is known at every point $x \in \overline{\Omega}$ at the initial time $t = 0$ but the data on $(\partial\Omega \setminus \overline{\Gamma})_T$ were lost for $t > 0$.

Corollary 1. *If Γ has at least one interior point (on $\partial\Omega$) then Problems 1 and 2 have no more than one solution.*

Proof. Let $v(x,\ t)$ and $w(x,\ t)$ be two solutions to Problem 1. Then function $u = (v - w) \in C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \overline{\Gamma_T}) \cap C(\overline{\Omega_T} \setminus (\partial\Omega \setminus \Gamma)_T)$ is a solution to the corresponding problem with $f = 0,\ u_1 = 0,\ u_2 = 0$. Using 3 we conclude that u is identically zero in Ω_T.

Clearly, Problem 2 has no more than one solution, too, if Γ has at least one interior point (on $\partial\Omega$). □

Thus, the Uniqueness Theorem 3 implies that the data of Problems 1 and 2 are suitable in order to define their solutions uniquely.

Example 3. *Let μ and λ be constants. It is not difficult to prove dense solvability of Problem 1 in the case where Γ is an open connected set of the hyperplane $\{x_n = 0\}$. For this purpose, we may use a version of heat polynomilals (cf.* [20] *First let us prove that if in this case the data of Problem 1 are polynomials then the problem is solvable and its solution is a polynomial.*

Indeed, Problem 1 is easily can be reduced to the following one (see Example 4):

$$L_n v = g \text{ in } \Omega_T \tag{20}$$

$$v(x_1, \ldots, x_{n-1}, 0, t) = 0 \text{ on } \overline{\Gamma_T}, \tag{21}$$

$$\mu \frac{\partial v_j}{\partial x_n}(x_1, \ldots, x_{n-1}, 0, t) = 0 \text{ on } \overline{\Gamma_T}, 1 \leq j \leq n-1, \tag{22}$$

$$(2\mu + \lambda)\frac{\partial v_n}{\partial x_n}(x_1, \ldots, x_{n-1}, 0, t) = 0 \text{ on } \overline{\Gamma_T}, 1 \leq j \leq n-1, \tag{23}$$

with

$$g(x,t) = f(x,t) - (L_n u_1)(x_1, \ldots, x_{n-1}, t) - x_n J(\mu, \lambda)(L_n u_2)(x_1, \ldots, x_{n-1}, t).$$

where $J(\mu, \lambda)$ is the diagonal matrix with the components

$$J_{j,j}(\mu, \lambda) = \mu^{-1},\ 1 \leq j \leq n-1,\ J_{n,n}(\mu, \lambda) = (2\mu + \lambda)^{-1}.$$

Besides, $u(x,t) = v(x,t) + u_1(x_1, \ldots, x_{n-1}, t) + J(\mu, \lambda) x_n u_2(x_1, \ldots, x_{n-1}, t)$.

Now consider data $g^{(j,\alpha)}(x,t) = t^j x^\alpha$ with a multi-index $\alpha \in \mathbb{Z}_+^n$.

If $0 \leq \alpha_1 + \ldots \alpha_{n-1} \leq 1$, we easily obtain (unique) polynomial solutions

$$v^{(j,\alpha)}(x,t) = x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} w^{(j,\alpha_n)}(x_n, t),\ \alpha_n, j \in \mathbb{Z}_+, \tag{24}$$

to problem (20)–(22) where

$$w^{(0,k)}(y,t) = -\frac{y^{k+2}k!}{(k+2)!}, \quad w^{(1,k)}(y,t) = -\frac{ty^{k+2}k!}{(k+2)!} - \frac{y^{k+4}k!}{(k+4)!}, \; k \in \mathbb{Z}_+, \; y \in \mathbb{R}$$

and, by the induction with respect to $j \in \mathbb{Z}_+$,

$$w^{(j,k)}(y,t) = -\sum_{\mu=0}^{j} \frac{t^{j-\mu}y^{k+2\mu+2}k!j!}{(k+2\mu+2)!(j-\mu)!}, \; k \in \mathbb{Z}_+, \; y \in \mathbb{R}. \tag{25}$$

To finish the arguments we use the induction with respect to $|\alpha'| \in \mathbb{Z}_+$ *where* $\alpha' = (\alpha_1, \dots, \alpha_{n-1}) \in \mathbb{Z}_+^{n-1}$. *Namely, let for* $s \geq 2$ *and all* α' *with* $|\alpha'| = s$ *the solutions to the problem are polynomial. If* $|\alpha'| = s+1$ *then*

$$L_n \left(x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} w^{(j,\alpha_n)}(x_n,t) \right) = t^j x^\alpha - w^{(j,\alpha_n)}(x_n,t) \Delta_{n-1} \left(x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} \right).$$

Clearly, the degree of the polynomial $p_{j,\alpha}(x,t) = w^{(j,\alpha_n)}(x_n,t)\Delta_{n-1}\left(x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}}\right)$ *with respect to* $x' \in \mathbb{R}^{n-1}$ *equals to* $s-1$. *Then, by the induction, problem (14)–(16) with data* $p_{j,\alpha}(x,t)$ *admits a polynomial solution, say,* $r_{j,\alpha}(x,t)$. *Therefore the solution* $v^{(j,\alpha)}(x,t)$ *to problem (14)–(16) with data* $g^{(j,\alpha)}(x,t) = t^j x^\alpha$, $|\alpha'| = s+1$, *is given as follows:*

$$v^{(j,\alpha)}(x,t) = x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} w^{(j,\alpha_n)}(x_n,t) + r_{j,\alpha}(x,t),$$

i.e. it is a polynomial, too.

Now Problem 1 with zero boundary data in the case $\Gamma \subset \{x_n = 0\}$ *is densely solvable because any continuous function* g *on the compact set* $\overline{\Omega_T}$ *can be approximated by polynomials. But the reducing to zero boundary data was organized in such a way that one easily sees, in this case Problem 1 is densely solvable for non-zero boundary data, too.*

We note that polynomial solutions indicated in Example 3 can be used in order to construct formal solutions to Problem 1.

The dense solvability of Problems 1 and 2 in general setting is natural to expect if the set $\partial\Omega \setminus \overline{\Gamma}$ has at least one interior point in $\partial\Omega$ (cf. [11] in the Cauchy Problem for elliptic equations).

Theorem 4. *If* $\partial\Omega \setminus \overline{\Gamma}$ *has at least one interior point in* $\partial\Omega$ *then Problems 1 and 2 are densely solvable.*

Proof. We begin with Problem 1. According to Khan-Banach Theorem, in order to prove the dense solvability, it sufficient to show that any linear bounded functional $F = (F_1, F_2, F_3)$ on the space

$$Y = [C(\overline{\Omega_T})]^n \oplus [C^{1,0}(\overline{\Gamma_T})]^n \oplus [C(\overline{\Gamma_T})]^n$$

equals to zero if it vanishes on the triple $(L_n u, u, \sigma u)$ for each $u \in [C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \overline{\Gamma_T})]^n$ with $(L_n u, u, \sigma u) \in Y$:

$$\langle F_1, L_n u \rangle_{\Omega_T} + \langle F_2, u \rangle_{\Gamma_T} + \langle F_2, \sigma u \rangle_{\Gamma_T} = 0. \tag{26}$$

Now, applying identity (26) for elements $u \in [C^{2,1}(\Omega_T)]^n$ with compact supports in Ω_T, we see that distribution F_1 satisfies $L_n^* F_1 = 0$ in Ω_T. As the operator L^*n is backward parabolic, its weak solutions keep some uniqueness and regularity properties, similar to the solutions of parabolic equations, see, for instance, [3, Ch.

6, §7]. In particular, F_1 belongs to $C^\infty(\Omega_T)$ and it is real analytic with respect to the space variables.

On the other hand, by Riesz Theorem, the space $C^*(\Omega_T)$, dual to $C(\Omega_T)$, can be interpreted as the space of measures with compact supports in Ω_T. Therefore the elements of the space $C^*(\overline{\Omega_T})$, dual to $C(\overline{\Omega_T})$, can be interpreted as measures in a neighbourhood of $\overline{\Omega_T}$ with supports on $\overline{\Omega_T}$. Similarly, the components F_2 and F_3 can be interpreted as measures on Γ_T.

Then

$$\int_{\Omega_T} F_1(y,\tau)(L_n u)(y,\tau)d\tau dy + \int_{\Gamma_T} (u^*(y,\tau)F_2(y,\tau) + (\sigma u)^*(y,\tau)F_3(y,\tau)) = 0$$

for all $u \in [C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \overline{\Gamma_T})]^n$ with $(L_n u, u, \sigma u) \in Y$.

Let $\{\Omega^{(\varepsilon)}\}_{\varepsilon>0}$ be a family of relatively compact domains in Ω such that:

1) each $\Omega^{(\varepsilon)}$ has a piece-wise smooth boundary;

2) the measure of $\Omega \setminus \Omega^{(\varepsilon)}$ converges to zero as $\varepsilon \to +0$.

Hence, integrating by part with the use of (12), we conclude that

$$\lim_{\varepsilon\to+0} \left(\int_{(\partial\Omega^{(\varepsilon)})_{\varepsilon,T-\varepsilon}} (u^*(\tilde\sigma F_1) - (\sigma u)^* F_1) d\tau ds(y) + \right. \tag{27}$$

$$\left. \int_{\Omega^{(\varepsilon)}} [(u^*F_1)(x,\varepsilon) - (u^*F_1)(y,T-\varepsilon)]dy \right) +$$

$$\int_{\Gamma_T} (u^*(y,\tau)F_2(y,\tau) + (\sigma u)^*(y,\tau)F_3(y,\tau)) = 0$$

for all $u \in [C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \overline{\Gamma_T})]^n$ with $(L_n u, u, \sigma u) \in Y$.

Of course, the properties of backward parabolic equations differ from the properties of the parabolic ones. For instance, the Cauchy problem for this type of equations might be ill-posed. However the kernel Φ^* is a fundamental solution of the operator L_n^* in the sense that identity (3) holds true. This means that a Green formula is still valid for the backward parabolic operator L_n^*. Namely, let

$$\tilde I_{\Omega,T_2} h(x,t) = -\int_\Omega \Phi^*(y,T_2,x,t)h(y)dy,$$

$$\tilde G_{\Omega,T_2} f(x,t) = \int_t^{T_2}\int_\Omega \Phi^*(y,\tau,x,t)f(y,\tau)dyd\tau,$$

$$\tilde V_{S,T_2} v(x,t) = -\int_t^{T_2}\int_S \Phi^*(y,\tau,x,t)v(y,\tau)ds(y)d\tau,$$

$$\tilde W_{S,T_2} w(x,t) = \int_t^{T_2}\int_S [\sigma_y \Phi(y,\tau,x,t)]^* w(y,\tau)ds(y)d\tau,$$

be the corresponding backward parabolic potentials for functions $f \in [C(\overline{\Omega_{T_1,T_2}})]^n$, $v \in [C(\overline{S_T})]^n$, $w \in [C(\overline{S_T} yc)]^n$, $h \in [C(\overline{\Omega})]^n$.

Lemma 2. *The following formula holds:*

$$\left(\tilde I_{\Omega,T_2} v + \tilde G_{\Omega,T_2} L_n^* v + \tilde V_{\partial\Omega,T_2} \tilde\sigma v + \tilde W_{\partial\Omega,T_2} v\right)(x,t) = \begin{cases} v(x,t), \ (x,t)\in \Omega_{T_1,T_2} \\ 0, \ (x,t)\in U_T \setminus \overline{\Omega_{T_1,T_2}}. \end{cases}$$

for all $0 \le T_1 < T_2 \le T$ *and all* $v \in [C^{2,1}(\Omega_{T_1,T_2}) \cap C^{1,0}(\overline{\Omega_{T_1,T_2}})]^n$ *with* $L_n^* v \in [C(\overline{\Omega_{T_1,T_2}})]^n$

Proof. It is similar to the proof of Lemma 1. □

Now, it follows from Lemma 2 that

$$\left(\tilde I_{\Omega^{(\varepsilon)},T-\varepsilon} F_1 + \tilde V_{\partial\Omega^{(\varepsilon)},T-\varepsilon} \tilde\sigma F_1 + \tilde W_{\partial\Omega^{(\varepsilon)},T-\varepsilon} F_1\right)(x,t) = \begin{cases} F_1(x,t), \ (x,t)\in \Omega^{(\varepsilon)}_{\varepsilon,T-\varepsilon} \\ 0, \ (x,t)\in U_T \setminus \overline{\Omega^{(\varepsilon)}_{\varepsilon,T-\varepsilon}}. \end{cases} \tag{28}$$

for $0<\varepsilon<T-\varepsilon<T$ and all sufficiently small $\varepsilon>0$. On the other hand, (27) yields

$$\lim_{\varepsilon\to+0} \left(\tilde I_{\Omega^{(\varepsilon)},T-\varepsilon} F_1 + \tilde V_{\partial\Omega^{(\varepsilon)},T-\varepsilon} \tilde\sigma F_1 + \tilde W_{\partial\Omega^{(\varepsilon)},T-\varepsilon} F_1\right)(x,t) + \tag{29}$$
$$\int_{\Gamma_T} (\Phi^*(y,\tau,x,t) F_2(y,\tau) + (\sigma_y\Phi)^*(y,\tau,x,t) F_3(y,\tau)) = 0$$

for all $(x,t)\in\Omega_T$.

Now, (28) and (29) imply

$$F_1(x,t) = -\int_{\Gamma_T} (\Phi^*(y,\tau,x,t) F_2(y,\tau) + (\sigma_y\Phi)^*(y,\tau,x,t) F_3(y,\tau)) \text{ for all } (x,t)\in\Omega_T.$$

In particular, $F_1 \in C^2((\overline{\Omega_T} \setminus \overline{\Gamma_T})$

Let $\{\tilde\Omega^{(\varepsilon)}\}_{\varepsilon>0}$ be a family of relatively compact domains in $\overline{\Omega}\setminus\Gamma$ such that:

1) each $\tilde\Omega^{(\varepsilon)}$ has a piece-wise smooth boundary;

2) the measure of $\Omega\setminus\tilde\Omega^{(\varepsilon)}$ converges to zero as $\varepsilon\to+0$,

3) the intersection $\partial\Omega\cap\partial\tilde\Omega^{(\varepsilon)}$ contains a relatively open subset $\tilde\Gamma\subset\partial\Omega\setminus\Gamma$ for all $\varepsilon>0$.

Again, it follows from Lemma 2 that

$$\left(\tilde I_{\tilde\Omega^{(\varepsilon)},T-\varepsilon} F_1 + \tilde V_{\partial\tilde\Omega^{(\varepsilon)},T-\varepsilon} \tilde\sigma F_1 + \tilde W_{\partial\tilde\Omega^{(\varepsilon)},T-\varepsilon} F_1\right)(x,t) = \begin{cases} F_1(x,t), \ (x,t)\in \tilde\Omega^{(\varepsilon)}_{\varepsilon,T-\varepsilon} \\ 0, \ (x,t)\in U_T \setminus \overline{\tilde\Omega^{(\varepsilon)}_{\varepsilon,T-\varepsilon}}. \end{cases} \tag{30}$$

for $0<\varepsilon<T-\varepsilon<T$ and all sufficiently small $\varepsilon>0$. Moreover, passing to the limit with respect to $\varepsilon\to+0$ in (30) and using (27), we obtain:

$$-\int_{\Gamma_T} (\Phi^*(y,\tau,x,t) F_2(y,\tau) + (\sigma_y\Phi)^*(y,\tau,x,t) F_3(y,\tau)) = \begin{cases} F_1(x,t), \ (x,t)\in\Omega_T \\ 0, \ (x,t)\in U_T\setminus\overline{\Omega_T}. \end{cases} \tag{31}$$

Clearly, the expression in the left hand side of (31) satisfies the backward parabolic equation

$$L_n^*\left(\int_{\Gamma_T} (\Phi^*(y,\tau,x,t) F_2(y,\tau) + (\sigma_y\Phi)^*(y,\tau,x,t) F_3(y,\tau))\right) = 0 \text{ in } U_T\setminus\overline{\Gamma_T}$$

as a sum of parameter dependent integrals. In particular, it is real analytic with respect to the space variables. Therefore,

$$\int_{\Gamma_T} (\Phi^*(y,\tau,x,t)F_2(y,\tau) + (\sigma_y\Phi)^*(y,\tau,x,t)F_3(y,\tau)) = 0 \text{ in } \Omega_T.$$

Thus, it follows from (31) that $F_1 = 0$ and then

$$\langle F_2, u\rangle_{\Gamma_T} + \langle F_2, \sigma u\rangle_{\Gamma_T} = 0$$

for all $u \in [C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \overline{\Gamma_T})]^n$ with $(L_n u, u, \sigma u) \in Y$.

Finally, as the boundary operators I and σ form a Dirichlet system on $\partial\Omega$ (see, for instance, [15]) we conclude that for each pair $(u_1, u_2) \in [C^2(\overline{\Gamma})]^n \oplus [C^1(\overline{\Gamma})]^n$ there is a vector $u \in [C^2(\overline{D})]^n$ satisfying $u = u_1$ on Γ and $\sigma u = u_1$. Hence, for each pair $(u_1, u_2) \in [C^{2,0}(\overline{\Gamma}_T)]^n \oplus [C^{1,0}(\overline{\Gamma_T})]^n$ there is a vector $u \in [C^{2,0}(\overline{D_T})]^n$ satisfying (15) and (16). Since $[C^{2,0}(\overline{\Gamma}_T)]^n \oplus [C^{1,0}(\overline{\Gamma_T})]^n$ is dense in $[C^{1,0}(\overline{\Gamma}_T)]^n \oplus [C(\overline{\Gamma}_T)]^n$ we conclude that $F_2 = 0$ and $F_3 = 0$.

For Problem 2 the arguments are similar. □

Easily, Problem 1 is ill-posed because this is the property of the Cauchy problem for elliptic systems in $\mathbb{R}^n$ (see, for instance [18] or [19, ch. 1, §2]). Of course, in this case the boundary data should be taken independent on t. The uniqueness theorem clarify why the problem is ill-posed. The reason is the redundant data. Indeed, if Γ has at least one interior point (on $\partial\Omega$), then taking a smaller relatively open set $\Gamma' \subset \Gamma$ we again obtain a problem with no more than one solution.

We note that in classical theory of (initial and) boundary problems for the parabolic equation (14), initial condition (17) and boundary condition $\alpha u + \beta\sigma = u^{(3)}$ on the whole lateral surface $(\partial\Omega)_T$ of the cylinder Ω_T are usually considered. As a rule, such a problem is well-posed in proper spaces (Hölder spaces, Sobolev spaces etc.), see, for instance, [2].

Let us show that Problem 2 is ill-posed.

Example 4. *Let the Lamé coefficients μ, λ be constant and $a_j = 0$, $0 \le j \le n$.*

Take a cube $Q_n = \{0 < x_j < 1, 1 \le j \le n\}$ as base Ω of the cylinder Ω_T. Let Γ be the face $\{x_n = 0\}$ of the cube Q_n. Then $\Gamma_T = Q_{n-1} \times (0,T)$ and the stress tensor σ is given by the diagonal matrix with the non-zero entries

$$\sigma_{j,j} = \mu\frac{\partial}{\partial x_n}, 1 \le j \le n-1, \ \sigma_{n,n} = (2\mu+\lambda)\frac{\partial}{\partial x_n}.$$

Fix $N \in \mathbb{N}$ and consider the following sequence of vector functions $u(x,t,k,r) \in [C^\infty(\mathbb{R}^{n+1})]^n$ with the entries:

$$u_1(x,t,k,r) = 0, \ \ldots, \ u_{n-1}(x,t,k,r) = 0, \ u_n(x,t,k,r) = \frac{e^{k^2(2\mu+\lambda)(t-r)+kx_n}}{k^N},$$

depending on a parameter $0 < r < +\infty$. Consider data $f(x,t,k,r)$, $u^{(0)}(x,t,k,r)$, $u^{(1)}(x,t,k,r)$, $u^{(2)}(x,t,k,r)$ having the following components:

$$f_j(x,t,k,r) = 0, \ 1 \le j \le n,$$

$$u_j^{(0)}(x,k,r) = 0, \ 1 \le j \le n-1, \ u_n^{(0)}(x,k,r) = \frac{e^{-k^2(2\mu+\lambda)r+kx_n}}{k^N},$$

$$u_j^{(1)}(x_1,\ldots,x_{n-1},t,k,r) = 0, \ 1 \le j \le n-1,$$

$$u_n^{(1)}(x_1,\dots,x_{n-1},t,k,r) = \frac{e^{k^2(2\mu+\lambda)(t-r)}}{k^N},$$
$$u_j^{(2)}(x_1,\dots,x_{n-1},t,k,r) = 0,\ 1 \le j \le n-1,$$
$$u_n^{(2)}(x_1,\dots,x_{n-1},t,k) = (2\mu+\lambda)\frac{e^{k^2(2\mu+\lambda)(t-T)}}{k^{N-1}}.$$

Then, for $0 < T_1 < T$, each function $u(x,t,k,T_1)$ is a solution to problem (14), (15), (16), (17) with the data $f(x,t,k,T_1)$, $u^{(0)}(x,t,k,T_1)$, $u^{(1)}(x,t,k,T_1)$, $u^{(2)}(x,t,k,T_1)$ in Ω_{T_1}.

It is clear, that compatibility conditions (18), (19) hold and

$$f(x,t,k,T_1) \underset{k\to\infty}{\longrightarrow} 0 \text{ in } [C^\infty(\overline{\Omega_{T_1}})]^n, \quad u^{(0)}(x,k,T_1) \underset{k\to\infty}{\longrightarrow} 0 \text{ in } [C^\infty(\overline{\Omega})]^n,$$
$$u^{(1)}(x,t,k,T_1) \underset{k\to\infty}{\longrightarrow} 0 \text{ in } [C^s(\overline{\Gamma_{T_1}})]^n, \qquad u^{(2)}(x,t,k,T_1) \underset{k\to\infty}{\longrightarrow} 0 \text{ in } [C^s(\overline{\Gamma_{T_1}})]^n,$$

if $N > 2s+1$. On the other hand we have:

$$u_n(x,T_1,k,T_1) = \frac{e^{k^2(2\mu+\lambda)(T_1-T_1)+kx_n}}{k^N} = \frac{e^{kx_n}}{k^N} \underset{k\to\infty}{\longrightarrow} +\infty.$$

for all $x_n > 0$ and all $N \in \mathbb{N}$. Now, we may consider the following data with a fixed $0 < T_1 < T$:

$$f(x,t,k) = 0 \in [C^\infty(\overline{\Omega_T}]^n,\ u^{(0)}(x,k) = u^{(0)}(x,k,T_1) \in [C(\overline{\Omega})]^n,$$
$$u_j^{(i)}(x,t,k) = 0,\ 1 \le j \le n-1,\ 1 \le i \le 2,$$
$$u_n^{(1)}(x,t,k) = \begin{cases} u_n^{(1)}(x,t,k,T_1), & t \le T_1, \\ \frac{T_1^s}{t^s k^N} & t > T_1, \end{cases}$$
$$u_n^{(2)}(x,t,k) = \begin{cases} u_n^{(1)}(x,t,k,T_1), & t \le T_1, \\ \frac{(2\mu+\lambda)T_1^s}{t^s k^{N-1}} & t > T_1. \end{cases}$$

Obviously,

$$f(x,t,k) \underset{k\to\infty}{\longrightarrow} 0 \text{ in } [C^\infty(\overline{\Omega_T})]^n, \quad u^{(0)},(x,k) \underset{k\to\infty}{\longrightarrow} 0 \text{ in } [C^\infty(\overline{\Omega})]^n,$$
$$u^{(1)}(x,t,k) \underset{k\to\infty}{\longrightarrow} 0 \text{ in } [C^{1,0}(\overline{\Gamma}_T)]^n,$$
$$u^{(2)}(x,t,k) \underset{k\to\infty}{\longrightarrow} 0 \text{ in } [C(\overline{\Gamma_T})]^n,$$

for $N \ge 2$ and $s > 1/q$. The Uniqueness Theorem 3 for Problem 2 implies that

$$u_n(x,t,k) = u_n(x,t,k,T_1) \text{ for } 0 < t \le T_1.$$

Then, for all $x_n > 0$ and all $2 \le N \in \mathbb{N}$, we have $\lim_{k\to+\infty} u_n(x,T_1,k) = +\infty$. Thus, if the data $f(x,t,k)$, $u^{(0)}(x,k)$, $u^{(2)}(x,t,k)$, $u^{(2)}(x,t,k)$ admits the solution to (14) in Ω_T with boundary conditions (15), (16) and the initial condition (17) then there is no continuity with respect to the data in the chosen space. Otherwise there is no solutions to the problem for some data in the data's spaces. In the last case, the problem is ill-posed because it is densely solvable.

As both Problems 1 and 2 are ill-posed, we will not study Problem 2 because in order to investigate it one needs to know both the data related to initial condition (17) and the data (14)-(16). Besides, in the sequel we will consider the case $0 < T < +\infty$ only.

3. Solvability Conditions

From now on we will study Problem 1 under the assumption that its data belong to Hölder spaces (cf., [3, ch. 1, §1] for other boundary problems for parabolic equations). We recall that a function $u(x)$, defined on a set $M \in \mathbb{R}^m$, is called *Hölder continuous with an exponent* $0 < \lambda \leq 1$ on M, if there is such a constant $C > 0$ that

$$|u(x) - u(y)| \leq C|x-y|^\lambda \text{ for all } x, y \in M \tag{32}$$

where $|x-y| = \sqrt{\sum_{j=1}^m (x_j - y_j)^2}$ is Euclidean distance between points x and y in $\mathbb{R}^m$. Let $C^\lambda(\overline{\Omega_T})$ stand for the set of Hölder continuous functions with an exponent λ over $\overline{\Omega_T}$. Besides, let $C^{1+\lambda,\lambda}(\overline{\Omega_T})$ be the set of Hölder continuous functions with an exponent λ over $\overline{\Omega_T}$, having Hölder continuous derivatives u_{x_i}, $1 \leq i \leq n$, with the same exponent $0 < \lambda \leq 1$ in $\overline{\Omega_T}$.

We choose a set Ω^+ in such a way that the set $D = \Omega \cup \Gamma \cup \Omega^+$ would be a bounded domain with piece-wise smooth boundary. It is possible since Γ is an open connected set. It is convenient to set $\Omega^- = \Omega$. For a function v on D_T we denote by v^+ its restriction to Ω_T^+ and, similarly, we denote by v^- its restriction to Ω_T. It is natural to denote by $v^\pm_{|\Gamma_T}$ the limit values of $v^\pm$ on Γ_T, when they are defined.

Set

$$\mathcal{F}(x,t) = G_{\Omega,0}(f) + V_{\overline{\Gamma},0}(u_2) + W_{\overline{\Gamma},0}(u_1) \text{ in } \Omega_T^- \cup \Omega_T^+.$$

Theorem 5 (A solvability criterion)**.** *Let* $\Gamma \in C^{1+\lambda}$,

$$f \in [C^\lambda(\overline{\Omega_T})]^n,\ u_1 \in [C^{1+\lambda,\lambda}(\overline{\Gamma_T})]^n,\ u_2 \in [C^\lambda(\overline{\Gamma_T})]^n.$$

Problem 1 is solvable if and only if there is a vector function $F \in [C^{2,1}(D_T)]^n$ *satisfying the following conditions:*

1) $L_n F = 0$ *in* D_T,
2) $F^+ = \mathcal{F}^+$. *in* Ω_T^+.

Proof. Necessity. Let a function $u(x,t) \in [C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \overline{\Gamma_T})]^n$ satisfy (14), (15), (15). Consider the function

$$F = G_{\Omega,0}(f) + V_{\overline{\Gamma},0}(u_2) + W_{\overline{\Gamma},0}(u_1) - \chi_{\Omega_T} u. \tag{33}$$

in the domain D_T, where χ_M is a characteristic function of the set $M \subset \mathbb{R}^{n+1}$.

By the very construction condition 2) is fulfilled for it. Clearly, the function $u(x,t)$ belongs to the space $[C^{2,1}(\overline{\Omega'_T})]^n$ for each cylindrical domain Ω'_T with such a base Ω' that $\Omega' \subset \Omega$ and $\overline{\Omega'} \cap \partial\Omega \subset \Gamma$. Besides, $L_n u = f \in [C^\lambda(\overline{\Omega'_T})]^n$. Without loss of the generality we may assume that the interior part Γ' of the set $\overline{\Omega'} \cap \partial\Omega$ is non-empty.

We note that $\chi_{\Omega_T} u = \chi_{\Omega'_T} u$ in D'_T, where $D' = \Omega' \cup \Gamma' \cup \Omega^+$. Then using Lemma 3 we obtain:

$$F = G_{\Omega\setminus\overline{\Omega'},0}(f) + V_{\overline{\Gamma}\setminus\Gamma',0}(u_2) + W_{\overline{\Gamma}\setminus\Gamma',0}(u_1) - I_{\Omega',0}(u) \text{ in } D'_T. \tag{34}$$

Arguing as in the proof of Theorem 3 we conclude that each of the integrals in the right hand side of (34) satisfies homogeneous Lamé type equation outside the corresponding integration set. In particular, we see that $L_n F = 0$ in D'_T. Obviously, for any point $(x,t) \in D_T$ there is a domain D'_T containing $(x,\ t)$. That is why $L_n F = 0$ in D_T, and hence F belongs to the space $[C^{2,1}(D_T) \cap L^q(D_T)]^n$. Thus this function satisfies condition 1), too.

Sufficiency. Let there be a function $F \in [C^{2,1}(D_T)]^n$, satisfying conditions 1) and 2) of the theorem. Consider on the set D_T the function

$$U = G_{\Omega,0}(f) + V_{\overline{\Gamma},0}(u_2) + W_{\overline{\Gamma},0}(u_1) - F. \tag{35}$$

As according to [3], the parabolic potentials act continuously in Hölder spaces. As $f \in [C^{\lambda}(\overline{\Omega_T})]^n$ then, using (36) and the definition of the fundamental solution, we see that the results of [3, ch. 1, §3] imply

$$G_{\Omega,0}(f) \in [C^{2+\lambda,1+\lambda}(\Omega^{\pm}_T) \cap C^{2,1}(\Omega^{+}_T) \cap C^{1,0}(D_T) \cap C(\overline{D}_T)]^n \tag{36}$$

and, moreover,

$$L_n G^{-}_{\Omega,0}(f) = f \text{ in } \Omega_T, \quad L_n G^{+}_{\Omega,0}(f) = 0 \text{ in } \Omega^{+}_T. \tag{37}$$

Since $u_2 \in [C^{\lambda}(\overline{\Gamma_T})]^n$ then the results of [3, ch. 5, §2] yield

$$V_{\overline{\Gamma},0}(u_2) \in [C^{2+\lambda,1+\lambda}(\Omega^{\pm}_T) \cap C^{1+\lambda,\lambda}((\Omega^{\pm} \cup \Gamma)_T) \cap C(\overline{D_T} \setminus (\partial\Gamma)_T)]^n, \tag{38}$$

$$L_n V_{\overline{\Gamma},0}(u_2) = 0 \text{ in } \Omega_T \cup \Omega^{+}_T. \tag{39}$$

On the other hand, the behaviour of the double layer potential $W_{\overline{\Gamma},0}(u_1)$ is similar to the behaviour of the normal derivative of the single layer potential $V_{\overline{\Gamma},0}(u_1)$. Hence

$$W_{\overline{\Gamma},0}(u_1) \in [C^{2+\lambda,1+\lambda}(\Omega^{\pm}_T) \cap C(\overline{\Omega^{\pm}_T} \setminus (\partial\Omega^{\pm} \setminus \Gamma)_T)]^n, \tag{40}$$

$$L_n W_{\overline{\Gamma},0}(u_1) = 0 \text{ in } \Omega_T \cup \Omega^{+}_T. \tag{41}$$

Lemma 3. *Let $S \subset \overline{\Gamma} \in C^{1+\lambda}$. If $u_1 \in [C^{1+\lambda,\lambda}(\overline{\Gamma_T})]^n$, then the potential $W^{-}_{\overline{\Gamma},0}(u_1)$ belongs to the space $[C^{1,0}(\Omega_T \cup S_T)]^n$ if and only if $W^{+}_{\overline{\Gamma},0}(u_2) \in [C^{1,0}(\Omega^{+}_T \cup S_T)]^n$.*

Proof. It is similar to the proof of the analogous lemma for Newton double layer potential (see, for instance, [9, lemma 1.1]). Actually, one needs to use Lemma 3 instead of the standard Green formula for the Laplace operator. □

Since $F \in [C^{1,0}(D_T)]^n$ then it follows from the discussion above that $W^{+}_{\overline{\Gamma},0}(u_2) \in [C^{1,0}((\Omega^{+} \cup \Gamma)_T)]^n$. Thus, formulas (35)–(41) and Lemmas 5, 3 imply that

$$U \in [C^{2,1}(\Omega^{\pm}_T) \cap C^{1,0}((\Omega^{\pm} \cup \Gamma)_T) \cap C(\overline{\Omega^{\pm}_T} \setminus (\partial\Omega \setminus \Gamma)_T) \cap L^q(D_T)]^n,$$

$$L_n U = \chi_{D_T} f \text{ in } \Omega_T \cup \Omega^{+}_T.$$

Then $U^{-} \in [C^{2,1}(\Omega_T) \cap C^{1,0}((\Omega \cup \Gamma)_T) \cap L^q(\Omega_T)]^n$ and (14) is fulfilled for U^{-}.

Let us show that the function U^{-} satisfies (15) and (16).

Since $F \in [C^{1,0}(D_T)]^n$ we see that $\partial^{\alpha} F^{-} = \partial^{\alpha} F^{+}$ on Γ_T for $\alpha \in \mathbb{Z}_+$ with $|\alpha| \le 1$ and

$$\partial^{\alpha} F^{+}_{|\Gamma_T} = \left(\partial^{\alpha} G^{+}_{\Omega,0}(f) + \partial^{\alpha} V^{+}_{\overline{\Gamma},0}(u_2) + \partial^{\alpha} W^{+}_{\overline{\Gamma},0}(u_1)\right)_{|\Gamma_T}.$$

It follows from formulas (36) and (38) that the parabolic volume potential and the single layer parabolic potential are continuous if the point (x,t) passes over the surface Γ_T. Then

$$U^{-}_{|\Gamma_T} = W^{-}_{\overline{\Gamma},0}(u_1)_{|\Gamma_T} - W^{+}_{\overline{\Gamma},0}(u_1)_{|\Gamma_T} = u_1.$$

because of the theorem on jump behaviour of the parabolic double layer potential (see, for instance, [3, ch. 5, §2, theorem 1]), i.e. equality (15) is valid for U^{-} .

Formula (36) means that that the surface stress of the parabolic volume potential is continuous if the point (x,t) passes over the surface Γ_T. Therefore

$$(\sigma U)^-_{|\Gamma_T} = \Big(\sigma V^-_{\overline{\Gamma},0} u_2\Big)_{|\Gamma_T} - \Big(\sigma V^+_{\overline{\Gamma},0} u_2\Big)_{|\Gamma_T} + \Big(\sigma W^-_{\overline{\Gamma},0} u_1\Big)_{|\Gamma_T} - \Big(\sigma W^+_{\overline{\Gamma},0} u_1\Big)_{|\Gamma_T}. \quad (42)$$

By theorem on jump behaviour of the stress of the parabolic single layer potential (see, for instance, [16, ch. 3, §10, theorem 10.1])

$$\Big(\sigma V^-_{\overline{\Gamma},0} u_2\Big)_{|\Gamma_T} - \Big(\sigma V^+_{\overline{\Gamma},0} u_2\Big)_{|\Gamma_T} = u_2. \quad (43)$$

Finally, we need the following lemma which is an analogue of the famous theorem on jump behaviour of the normal derivative of the Newton's double layer potential.

Lemma 4. *Let* $\Gamma \in C^{1+\lambda}$ *and* $u_2 \in [C^\lambda(\overline{\Gamma_T})]^n$. *If* $W^-_{\overline{\Gamma},0}(u_1) \in [C^{1,0}((\Omega\cup\Gamma)_T)]^n$ *or* $W^+_{\overline{\Gamma},0}(u_1) \in [C^{1,0}((\Omega^+\cup\Gamma)_T)]^n$ *then*

$$\Big(\sigma W^-_{\overline{\Gamma},0} u_1 - \sigma W^+_{\overline{\Gamma},0} u_1\Big)_{|\Gamma_T} = 0. \quad (44)$$

Proof. Really, let, for instance, $W^-_{\overline{\Gamma},0}(u_1) \in [C^{1,0}((\Omega\cup\Gamma)_T)]^n$. Then using Lemma 3 we obtain $W^+_{\overline{\Gamma},0} u_1 \in [C^{1,0}((\Omega^+\cup\Gamma)_T)]^n$ and $\Big(\sigma W^{\pm}_{\overline{\Gamma},0}(u_1)\Big)_{|\Gamma_T} \in [C(\Gamma_T)]^n$.

Let $\phi \in [C^\infty_0(D_T)]^n$ be a function with compact support in D_T. Then formulas (9)–(11) yield:

$$\int_{\Gamma_T} \phi^* \left(\sigma W^-_{\overline{\Gamma},0} u_1 - \sigma W^+_{\overline{\Gamma},0} u_1\right) ds(x)dt = \quad (45)$$

$$\int_{\Omega_T\cup\Omega^+_T} \phi^* (\mathcal{L}_n + a) W_{\overline{\Gamma},0} u_1 dxdt + \int_{T_1}^{T_2} \mathfrak{D}_{\Omega\cup\Omega^+}(W_{\overline{\Gamma},0} u_1, \phi) dt =$$

$$\int_{\Omega_T\cup\Omega^+_T} \phi^* \Big(\frac{\partial}{\partial t} - A + a\Big) W_{\overline{\Gamma},0} u_1 dxdt + \int_{T_1}^{T_2} \mathfrak{D}_{\Omega\cup\Omega^+}(W_{\overline{\Gamma},0} u_1, \phi) dt$$

because $L_n W^{\pm}_{\overline{\Gamma},0} u_1 = 0$ in $\Omega^{\pm}$ according to (41).

Again, integrating by parts and using formulas (9)–(11) and theorem on jump behaviour of the parabolic double layer potential, we see that

$$\int_{\Omega_T\cup\Omega^+_T} \phi^* \Big(\frac{\partial}{\partial t} - A + a\Big) W_{\overline{\Gamma},0} u_1 dxdt + \int_{T_1}^{T_2} \mathfrak{D}_{\Omega\cup\Omega^+}(W_{\overline{\Gamma},0} u_1, \phi) dt = \quad (46)$$

$$- \int\limits_{\Omega_T\cup\Omega^+_T} \Big(\frac{\partial \phi}{\partial t}\Big)^* W_{\overline{\Gamma},0} u_1 dxdt - \int\limits_{\Omega_T\cup\Omega^+_T} ((\mathcal{L}_n + A^*)\phi)^* W_{\overline{\Gamma},0} u_1 dxdt +$$

$$\int_{\Gamma_T} (\tilde{\sigma}\phi)^* (W^-_{\overline{\Gamma},0} u_1 - W^+_{\overline{\Gamma},0} u_1) ds(x)dt =$$

$$\int_{\Gamma_T} (\tilde{\sigma}\phi)^* u_1 ds(x)dt - \int_{\Omega_T\cup\Omega^+_T} (L^*_n \phi)^* W_{\overline{\Gamma},0} u_1 dxdt.$$

But the kernel $\Phi(x,y,t,\tau)$ is a fundamental solution of the backward parabolic operator L^*_n with respect to variables (y,τ). Hence

$$\int_{D_T} (L^*_n \phi(x,t))^* \Phi(x,y,t,\tau) dxdt = \phi^*(y,\tau), \quad (y,\tau) \in D_T.$$

Then the type of the singularity of the fundamental solution allows us to apply Fubini Theorem and to conclude that

$$\int_{\Omega_T \cup \Omega_T^+} (L_n^* \phi)^* W_{\overline{\Gamma},0} u_1 dx dt = \tag{47}$$

$$\int_{\Gamma_T} \tilde{\sigma} \int_{D_T} (L_n^* \phi(x,t))^* \Phi(x,y,t,\tau) dx dt \, u_1 \, ds(y) d\tau = \int_{\Gamma_T} (\tilde{\sigma}\phi)^* u_1 ds(y) d\tau.$$

Finally, formulas (45)- (47) imply that

$$\int_{\Gamma_T} \phi^* \left(\sigma W^-_{\overline{\Gamma},0} u_1 - \sigma W^+_{\overline{\Gamma},0} u_1 \right) ds = 0$$

for all $\phi \in [C_0^\infty(D_T)]^n$. As such functions are dense in the Lebesgue space $[L^1(K)]^n$ for any compact $K \subset \Gamma_T$ then formula (44) holds true. □

Now using lemma 4 and formulas (42), (43), we conclude that $(\sigma U)^-_{|\Gamma_T} = u_2$, i.e. (16) is fulfilled for U^-.

Thus, function $u(x,t) = U^-(x,t)$ satisfies conditions (14)–(16). The proof is complete. □

Corollary 2. *Let* $\Gamma \in C^{1+\lambda}$,

$$f \in [C^\lambda(\overline{\Omega_T})]^n, \; u_1 \in [C^{1+\lambda,\lambda}(\overline{\Gamma_T})]^n, \; u_2 \in [C^\lambda(\overline{\Gamma_T})]^n.$$

Problem 1 is solvable in the class $[L^q(\Omega_T)]^n$ *if and only if there is a vector function* $F \in [C^{2,1}(D_T) \cap L^q(D_T)]^n$ *satisfying the following conditions:*

1) $L_n F = 0$ *in* D_T,
2) $F = \mathcal{F}^+$ *in* Ω_T^+.

Proof. The proof is based on the following lemma.

Lemma 5. *Let* $\Gamma \in C^{1+\lambda}$, $f \in [C^\lambda(\overline{\Omega_T})]^n$, $u_1 \in [C^{1+\lambda,\lambda}(\overline{\Gamma_T})]^n$, $u_2 \in [C^\lambda(\overline{\Gamma_T})]^n$. *Then* $G_{\Omega,0}(f), V_{\overline{\Gamma},0}(u_2), W_{\overline{\Gamma},0}(u_1) \in [L^q(D_T)]^n$.

Proof. As $f \in C^\lambda(\Omega_T)$, we see that $\chi_{\Omega_T} f \subset [L^q(\mathbb{R}^{n+1})]^n$. Then [5, theorem 3.2] yields $G_{\Omega,0}(f) \in [L^q(D_T)]^n$. Moreover, estimates [4, (2.16), (2.17)] of the fundamental solution Φ and its derivatives, imply that $V_{\overline{\Gamma},0}(u_2)$, and $W_{\overline{\Gamma},0}(u_1)$ belong to $[L^q(D_T)]^n$. □

Necessity. Let Problem 2 be solvable in $L^q(\Omega_T)$. Then, according to Theorem 5, the function $F = G_{\Omega,0}(f) + V_{\overline{\Gamma},0}(u_2) + W_{\overline{\Gamma},0}(u_1)$ extends from Ω_T to D_T as solution to the parabolic system L_n. Moreover, its extension is given by (33). Clearly, $\chi_{\Omega_T} u \in L^q(D_T)$ because $u \in L^q(\Omega_T)$. Thus, it follows from Lemma 5 that F belongs to $[L^q(D_T)]^n$, too.

Sufficiency. Let conditions 1) and 2) be fulfiled. Then Problem 2 is solvable and its unique solution u is given by (35). Since $F \in L^q(D_T)$, Lemma 5 implies that u belongs to $[L^q(D_T)]^n$, too. □

We note that Theorem 5 is an analogue of Theorem by Aizenberg and Kytmanov [8]) describing solvability conditions of the Cauchy problem for the Cauchy–Riemann system (cf. also [9] in the Cauchy Problem for Laplace Equation or [15] in the Cauchy problem for general elliptic systems). Formula (35), obtained in the proof of Theorem 5, gives the unique solution to Problem 1. Clearly, if we will be able to write the extension F of the sum of potentials $\mathcal{F} = G_{\Omega,0}(f) + V_{\overline{\Gamma},0}(u_2) + W_{\overline{\Gamma},0}(u_1)$

from Ω_T^+ onto D_T as a series with respect to special functions or a limit of parameter depending integrals then we will get a Carleman type formula for solutions to Problem 1 (cf. [8]). Moreover, Corollary 2 gives us a possibility to use Hilbert space methods for this purpose in the case where $q = 2$ (cf. [9], [11]). However this is a topic for another paper. Here we will give formulas, involving the Taylor series only.

Example 5. *Let $n = 1$, let D be the interval $(-1, 1)$ on the axis Ox, let $a \in (0, 1)$ be a real number and let Ω be the interval $(a, 1)$. Then $\Omega^+ = (-1, a)$ and $\Gamma = \{a\}$. Since the sum of the potentials $\mathcal{F}(x, t)$ is real analytic for each $t \in (0, T)$ with respect to the variable x in $\Omega^+ \times t$, we have the Taylor decomposition*

$$\mathcal{F}(x, t) = \sum_{j=0}^{\infty} c_j(t) x^j$$

with the Taylor coefficients

$$c_j(t) = \frac{1}{j!} \frac{\partial^j \mathcal{F}}{\partial x^j}(0, t).$$

According to Theorem by Abel, this power series converges absolutely and uniformly on compact subsets of the interval $(-a, a) \times t$ for each $t \in (0, T)$ and

$$\sup_{t \in (0,T)} \limsup_{j \to +\infty} \sqrt[j]{|c_j(t)|} \leq 1/a$$

because of Cauchy-Hadamard Theorem. Hence the function $\mathcal{F}^+$ extends from $\Omega^+ \times t$ to $D \times t$ for each each $t \in (0, T)$ if and only if

$$\sup_{t \in (0,T)} \limsup_{j \to +\infty} \sqrt[j]{|c_j(t)|} \leq 1.$$

Now it follows from (35) and the proof of Theorem 5, that, if Problem 1 is solvable then its unique solution is given by

$$u(x, t) = \mathcal{F}^-(x, t) - \sum_{j=0}^{\infty} c_j(t) x^j,\ t \in (0, T), x \in (a, 1). \tag{48}$$

Example 6. *Let D be a ball $B(0, R_1)$ in $\mathbb{R}^n$ and let Γ be a smooth hyper-surface such that $0 \notin \Gamma$ and $0 \in \Omega^+$. Since the sum of the potentials $\mathcal{F}(x, t)$ is real analytic for each $t \in (0, T)$ with respect to the variable x in $\Omega^+ \times t$, we have the Taylor decomposition*

$$\mathcal{F}(x, t) = \sum_{|\alpha|=0}^{\infty} c_\alpha(t) x^\alpha$$

with the Taylor coefficients

$$c_\alpha(t) = \frac{1}{\alpha!} \frac{\partial^\alpha \mathcal{F}}{\partial x^\alpha}(0, t)$$

in a beighbourhood of $(0, t)$ where, as usual, $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \sum_{j=1}^n \alpha_j$, $\alpha! = \alpha_1! \dots \alpha_n!$, $x^\alpha = x^{\alpha_1} \dots x^{\alpha_n}$. After complexification, we may consider it as the series of complex varaibles $z_j = x_j + \sqrt{-1} y_j$ in $\mathbb{C}^n$:

$$\mathcal{F}_{\mathbb{C}}(x, t) = \sum_{|\alpha|=0}^{\infty} c_\alpha(t) z^\alpha.$$

Set $d_\alpha(D) = \sup_{x\in D} |x^\alpha|$. *Then according to* [22, p. 143] *and* [23], *the function* $\mathcal{F}^+$ *extends from* $\Omega^+ \times t$ *to* $D \times t$ *for each each* $t \in (0,T)$ *if and only if*

$$\sup_{t\in(0,T)} \limsup_{|\alpha|\to+\infty} \sqrt[|\alpha|]{d_\alpha(D)|c_\alpha(t)|} \leq 1.$$

Now it follows from (35) and the proof of Theorem 5, that, if Problem 1 is solvable then its unique solution is given by

$$u(x,t) = \mathcal{F}^-(x,t) - \sum_{|\alpha|=0}^{\infty} c_\alpha(t) x^\alpha,\ t \in (0,T), x \in \Omega. \tag{49}$$

The advantage of formulas (48) and (49) is the simplicity. However they are not so convenient because the partial sums of the corresponding series are not solutions to the homogeneous Lamé type system.

The work was supported by the grant of the Russian Federation Government for scientific research under the supervision of leading scientist at the Siberian federal university, contract N. 14.Y26.31.0006, and by RFBR grant 14-01-00544.

References

[1] Landau, L.D., Lifshitz, E.M., *Fluid Mechanics (Volume 6 of A Course of Theoretical Physics)*, Pergamon Press, 1959.

[2] Ladyzhenskaya, O.A., Solonnikov V.A., Ural'tseva, N.N., *Linear and Quasilinear Equations of Parabolic Type*, Moscow, Nauka, 1967.

[3] Friedman, A., *Partial Differential Equations of Parabolic Type*, Englewood Cliffs, NJ, Prentice-Hall, Inc., 1964.

[4] Eidel'man, S.D., *Parabolic equations*, Partial differential equations 6, Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Fund. Napr., **63**, VINITI, Moscow, 1990, 201-313.

[5] Solonnikov, V.A., *On boundary value problems for linear parabolic systems of differential equations of general form. Boundary value problems of mathematical physics. Part 3. On boundary value problems for linear parabolic systems of differential equations of general form*, Trudy Mat. Inst. Steklov, **83** (1965), 3–163.

[6] Lorenzi, A., Lorenzi, L., *A strongly ill-posed problem for a degenerate parabolic equation with unbounded coefficients in an unbounded domain* $\Omega \times \mathcal{O}$ *of* $\mathbb{R}^{N+M}$, Inverse Problems, **29:2** (2013), doi:10.1088/0266-5611/29/2/025007.

[7] Puzyrev, R., Shlapunov, A., *On an ill-posed problem for the heat equation.* J. Siberian Fed. Univ., **4:2** (2011), 218-229.

[8] Aizenberg, L.A., Kytmanov, A.M., *On the possibility of holomorphic continuation to a domain of functions given on a part of its boundary*, Matem. sbornik. **182:5** (1991), 490–597.

[9] Shlapunov, A.A. *On the Cauchy Problem for the Laplace Equation*, Siberian Math. J., **33:3** (1992), 205–215.

[10] Shlapunov, A., *On the Cauchy Problem for the Lamé system.* Zeitschrift für Angewandte Mathematik und Mechanik, **76:4** (1996), 215–221.

[11] Shlapunov, A.A., Tarkhanov, N., *Bases with double orthogonality in the Cauchy problem for systems with injective symbols*, Proc. London. Math. Soc., **71:1** (1995), 1-54.

[12] Shestakov, I.V., Shlapunov, A.A., *On Cauchy problem for operators with injective symbols in the spaces of distributions*, Journal Inverse and Ill-posed Problems, **19** (2011), 127–150.

[13] Lavrent'ev, M.M., Romanov, V.G., Shishatskii, S.P., *Ill-posed Problems of Mathematical Physics and Analysis*, Nauka, Moscow, 1980.

[14] Tihonov, A.N., Arsenin, V.Ya., *Methods of Solving Ill-posed Problems*, Nauka, Moscow,1986.

[15] Tarkhanov, N.N. *The Cauchy Problem for Solutions of Elliptic Equations.* Berlin: Akademie-Verlag, 1995.

[16] Landis, E.M. *Second Order Equations of Elliptic and Parabolic Types*, Moscow, Nauka, 1971.

[17] Sveshnikov, A.G., Bogolyubov, A.N., Kravtsov, V.V., *Lectures on Mathematical Physics*, Moscow, Nauka, 2004.

[18] Hadamard, J., *Lectures on Cauchy's Problem in Linear Partial Differential Equations. /* J. Hadamard. – Yale Univ. Press, New Haven-London, 1923.
[19] Mihailov, V.P., *Partial Differential Equations*, Moscow, Nauka, 1976.
[20] Rosenbloom, P. C., Widder D. V., *Expansions in terms of heat polynomials and associated functions*, Trans. Amer. Math. Soc. **92** (1959), 220-266.
[21] Tikhonov, A. N., Samarskii, A. A. *Equations of Mathematical Physics*, Moscow: Nauka, 1972.
[22] Vladimirov, V.S., *Methods of function theory of several complex variables*, Moscow: Nauka, 1964.
[23] Kytmanov, A. M., Hodos, O. V., *On conditions of holomorphic extension of smooth CR-functions to a fixed domain*, Izv. VUZ. Mathematics, **6** (1999), 37–40.

(Roman Puzyrev) Siberian Federal University, Institute of Mathematics and Computer Science, pr. Svobodnyi 79, 660041 Krasnoyarsk, Russia
E-mail address: effervesce@mail.ru

(Alexander Shlapunov) Siberian Federal University, Institute of Mathematics and Computer Science, pr. Svobodnyi 79, 660041 Krasnoyarsk, Russia
E-mail address: ashlapunov@sfu-kras.ru