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On the Structure of the Classical Discriminant

Evgeny N. Mikhalkin*
Avgust K. Tsikh†
Institute of Mathematics and Computer Science
Siberian Federal University
Svobodny, 79, Krasnoyarsk, 660041
Russia

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Consider a general polynomial of degree n with variable coefficients. It is known that the Newton polytope of its discriminant is combinatorially equivalent to an $(n-1)$-dimensional cube. We show that two facets of this Newton polytope are prisms, and that truncations of the discriminant with respect to facets factor into discriminants of polynomials of smaller degree.

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We consider a general polynomial of degree n

$$f(y) = a_0 + a_1 y + \ldots + a_n y^n. \tag{1}$$

By its discriminant we call an irreducible polynomial $\Delta(a_0, a_1, \ldots, a_n)$ with integer coefficients that vanishes if and only if the polynomial (1) has multiple roots. The purpose of this article is to refine some classical results on the structure of facets of the Newton polytope of the discriminant Δ, as well as to study factorization of truncations of the discriminant with respect to the facets. The knowledge of this structure is important in the study of a general algebraic function $y = y(a)$ of roots of the polynomial (1) ([1,2]).

1. The Newton polytope for the classical discriminant

Recall that the *Newton polytope* $\mathcal{N}(\Delta)$ of the polynomial $\Delta(a_0, \ldots, a_n)$ is the convex hull in $\mathbb{R}^{n+1}$ of exponents $k = (k_0, k_1, \ldots k_n)$ of all monomials participating in Δ.

Note that the Newton polytope of the polynomial (1) is the segment $[0, n] \subset \mathbb{R}$. The following theorem shows that each vertex of the Newton polytope $\mathcal{N}(\Delta)$ of the discriminant of f corresponds to an appropriate triangulation (i.e. a partition into segments) of the segment $[0, n]$. Each partition is given by a set of integer points

$$0 = i_0 < i_1 < \ldots < i_s < i_{s+1} = n.$$

It is clear that such a set is identified by a subset $I \subset \{1, 2, \ldots, n-1\}$ of the type $I = \{i_1 < i_2 < \ldots < i_s\}$, $0 \leqslant s \leqslant n-1$. The number of all such subsets equals 2^{n-1}, since we include in the list the empty set too, which corresponds to $s = 0$.

Theorem 1 ([3], p. 412)**.** *The Newton polytope $\mathcal{N}(\Delta)$ of the discriminant Δ is combinatorially equivalent to an $(n-1)$-dimensional cube; its 2^{n-1} vertices are in a bijective correspondence with all possible subsets*

$$I \subset \{1, 2, \ldots, n-1\}.$$

*mikhalkin@bk.ru
†atsikh@sfu-kras.ru

The vertex v_I corresponding to subset $I = \{i_1 < i_2 < \ldots < i_s\}$ has the coordinates

$$k_0 = i_1 - i_0 - 1,\ k_n = i_{s+1} - i_s - 1 \tag{2}$$

$$k_{i_q} = i_{q+1} - i_{q-1} \quad \text{for} \quad i_q \in I,$$

$$k_i = 0 \quad \text{for} \quad i \notin I \cup \{0, n\}.$$

Let $l_q = i_{q+1} - i_q$ $(0 \leqslant q \leqslant s)$. Then the monomial

$$a^{v_I} = a_0^{l_0 - 1} a_{i_1}^{l_1 + l_0} a_{i_2}^{l_2 + l_1} \ldots a_{i_s}^{l_s + l_{s-1}} a_n^{l_s - 1}$$

is in Δ with the coefficient

$$c_{v_I} = \prod_{q=0}^{s} (-1)^{\frac{l_q(l_q - 1)}{2}} l_q^{l_q}. \tag{3}$$

Let us illustrate the theorem by an example of a cubic polynomial

$$f(y) = a_0 + a_1 y + a_2 y^2 + a_3 y^3.$$

Its discriminant is

$$\Delta = -27a_0^2a_3^2 - 4a_1^3a_3 - 4a_0a_2^3 + a_1^2a_2^2 + 18a_0a_1a_2a_3.$$

In this case there are 4 subsets $I \subset \{1, 2\}$:

$$I_0 = \varnothing, \quad I_1 = \{1\}, \quad I_2 = \{2\}, \quad I_3 = \{1, 2\}.$$

The corresponding monomials are the following

$$-27a_0^2a_3^2, \ -4a_1^3a_3, \ -4a_0a_2^3, \ a_1^2a_2^2.$$

As for the monomial $18a_0a_1a_2a_3$, it corresponds to an interior integer point $(1, 1, 1, 1) \in \mathcal{N}(\Delta)$, and the theorem says nothing about it.

Further on we consider (1) with $a_0 = a_n = 1$, i.e. *a reduced polynomial*

$$f_{red}(y) = 1 + a_1 y + \ldots + a_{n-1} y^{n-1} + y^n. \tag{4}$$

The Newton polytope of the discriminant of this polynomial lies in $\mathbb{R}^{n-1}$ (the coordinates k_0 and k_n in the expression (2) are missing). The discriminant of the polynomial (4) we call a *reduced discriminant.* For example, for a cubic equation the reduced discriminant equals

$$\Delta = 27 + 4a_1^3 + 4a_2^3 - 18a_1a_2 - a_1^2a_2^2.$$

The paper [4] gives the inequalities defining the Newton polytope $\mathcal{N}(\Delta)$ for the discriminant of the polynomial (4). There are $2 \cdot (n-1)$ such inequalities:

$$t_k \geqslant 0, \quad \sum_{j=1}^{n-1} \min(j, k)[n - \max(j, k)] t_j \leqslant nk(n - k), \quad k = 1, 2, \ldots, n-1. \tag{5}$$

Thus, the Newton polytope of the reduced discriminant of a cubic equation is given by the system of inequalities

$$t_1 \geqslant 0, \quad t_2 \geqslant 0, \quad 2t_1 + t_2 \leqslant 6, \quad t_1 + 2t_2 \leqslant 6$$

(see Fig. 1).

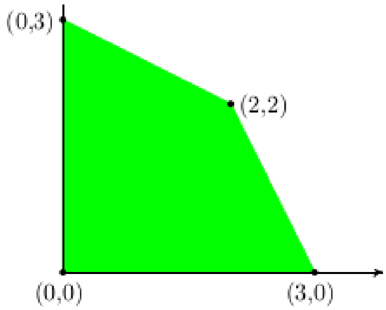

Fig. 1. The Newton polytope of the reduced discriminant of a cubic equation

2. The prism facets of the Newton polytope $\mathcal{N}(\Delta)$

Recall that we consider a reduced discriminant Δ. Further on it will be convenient for us to denote the facets in (5) that are not in the coordinate hyperplanes as

$$g_k := \left\{ t \in \mathcal{N}(\Delta) : \sum_{j=1}^{n-1} \min(j,k)[n-\max(j,k)]t_j = nk(n-k) \right\}, \tag{6}$$

$k = 1, 2, \ldots, n-1$.

We shall call a polytope G of dimension d a *d-prism*, if it is a Minkowski sum of a $(d-1)$-dimensional polytope and a segment. This $(d-1)$-dimensional polytope and its translate we call the prism's *bases*.

Theorem 2. *The facets g_2 and g_{n-2} of the Newton polytope $\mathcal{N}(\Delta)$ of the reduced discriminant Δ are $(n-2)$-prisms.*

To prove Theorem 2 we need two lemmas. Recall that according to Theorem 1 each vertex $v_I = v_{i_1,\ldots,i_s}$ of the Newton polytope $\mathcal{N}(\Delta)$ of the reduced discriminant of the polynomial (4) is written down as follows

$$\Big(0, \ldots, \overset{i_1}{\widehat{i_2}}, \ldots, \overset{i_2}{\overbrace{i_3 - i_1}}, \ldots, \overset{i_3}{\overbrace{i_4 - i_2}}, \ldots, \overset{i_{s-1}}{\overbrace{i_s - i_{s-2}}}, \ldots, \overset{i_s}{\overbrace{n - i_{s-1}}}, \ldots, 0\Big) \tag{7}$$

(there are zeroes on vacant places).

Lemma 1. *Each vertex $v_I = v_{i_1,i_2,\ldots,i_s}$ of the Newton polytope $\mathcal{N}(\Delta)$ belongs to all s facets g_k, $k = i_1, \ldots, i_s$ of the list (6) and doesn't lie in any remaining $n-1-s$ facets of this list.*

Proof of Lemma 1. First we show that the vertex $v_{i_1,\ldots,i_s}$ lies in the facets g_k with $k = i_1, \ldots, i_s$. According to (6), if it belongs to g_{i_p}, $p = 1, 2, \ldots, s$ then

$$(n - i_p) \sum_{\nu=1}^{p} i_\nu t_{i_\nu} + i_p \sum_{\nu=p+1}^{s} (n - i_\nu) t_{i_\nu} = n(n - i_p) i_p, \tag{8}$$

where t_{i_ν} are nonzero coordinates of $v_{i_1,\ldots,i_s}$, which in accordance with (7) can be written down as $t_{i_\nu} = i_{\nu+1} - i_{\nu-1}$, $\nu = 1, 2, \ldots, s$, with $i_0 = 0, i_{s+1} = n$.

Transform both sums in the left hand side of the equality (8). The first sum $\sum\limits_{\nu=1}^{p} i_\nu (i_{\nu+1} - i_{\nu-1})$ equals $i_p i_{p+1}$. As for the second sum $\sum\limits_{\nu=p+1}^{s} (n - i_\nu)(i_{\nu+1} - i_{\nu-1})$, here all the terms except four of them vanish, and as a result we get

$$\sum_{\nu=p+1}^{s}(n-i_\nu)(i_{\nu+1}-i_{\nu-1}) = -ni_p - ni_{p+1} + i_p i_{p+1} + n^2 = i_p(i_{p+1}-n) + n(n-i_{p+1}) =$$
$$= (n-i_p)(n-i_{p+1}).$$

Then for the left hand side of (8) we have

$$(n-i_p)\sum_{\nu=1}^{p} i_\nu t_{i_\nu} + i_p \sum_{\nu=p+1}^{s}(n-i_\nu)t_{i_\nu} = (n-i_p)i_p i_{p+1} + i_p(n-i_p)(n-i_{p+1}) = n(n-i_p)i_p,$$

i.e. the required equality (8).

Further on, for an arbitrary partition of the segment $[0,n]$ by $i_1,\ldots,i_s$ we prove the following fact. If $i' \notin \{i_1,\ldots,i_s\}$ then the vertex $v_{i_1,\ldots,i_s}$ doesn't lie in the facet $g_{i'}$. Let i' lie between i_p and i_{p+1}, then we need to prove that

$$(n-i')\sum_{\nu=1}^{p} i_\nu t_{i_\nu} + i' \sum_{\nu=p+1}^{s}(n-i_\nu)t_{i_\nu} \neq ni'(n-i'),$$

that is that if i' does not coincide with the points $i_1,\ldots,i_s$ of the segment partition, then the function

$$h(i') = (n-i')\sum_{\nu=1}^{p} i_\nu(i_{\nu+1}-i_{\nu-1}) + i' \sum_{\nu=p+1}^{s}(n-i_\nu)(i_{\nu+1}-i_{\nu-1}) - ni'(n-i')$$

does not vanish. It is not hard to see that $h(i')=0$ for $i'=i_p$ and $i'=i_{p+1}$. Indeed, using the equalities proved in this lemma

$$\sum_{\nu=1}^{p} i_\nu t_{i_\nu} = \sum_{\nu=1}^{p} i_\nu(i_{\nu+1}-i_{\nu-1}) = i_p i_{p+1},$$
$$\sum_{\nu=p+1}^{s}(n-i_\nu)t_{i\nu} = \sum_{\nu=p+1}^{s}(n-i_\nu)(i_{\nu+1}-i_{\nu-1}) = (n-i_p)(n-i_{p+1}),$$

we get
$$h(i') = (n-i')i_p i_{p+1} + i'(n-i_p)(n-i_{p+1}) - ni'(n-i').$$

Then for $h(i_p)$ we have

$$h(i_p) = (n-i_p)i_p i_{p+1} + i_p(n-i_p)(n-i_{p+1}) - ni_p(n-i_p) =$$
$$= i_p(n-i_p)(i_{p+1}+n-i_{p+1}-n) = 0.$$

Similarly, we get $h(i_{p+1})=0$.

Since h is a polynomial of degree two in i', we conclude that

$$h(i') = n(i'-i_p)(i'-i_{p+1}).$$

Taking into account the fact that i' is not a point of the partition, i.e. $i' \neq i_p, i' \neq i_{p+1}$, we see that $h(i') \neq 0$. $\square$

Lemma 2. *Let i', i'' be two different integer points of the segment $[0,n]$ that are not in $\{i_1,\ldots,i_s\}$. Then in $\mathbb{R}^{n-1}$ the following vector equalities hold:*

$$\overline{v_{i'}v_{i',i''}} = \overline{v_{i_1,\ldots,i_s,i'}v_{i_1,\ldots,i_s,i',i''}}, \quad i_s < i' < i'';$$

$$\overline{v_{i''}v_{i',i''}} = \overline{v_{i'',i_1,\ldots,i_s}v_{i',i'',i_1,\ldots,i_s}}, \quad i' < i'' < i_s.$$

Proof of Lemma 2. Calculate the coordinates of vertices in the equalities:

$$v_{i'} = \Big(0,\ldots,\overset{i'}{\widehat{n}},\ldots,0\Big), \quad v_{i',i''} = \Big(0,\ldots,\overset{i'}{\widehat{i''}},\ldots,\overset{i''}{\overbrace{n-i'}},\ldots,0\Big),$$

$$v_{i_1,\ldots,i_s,i'} = \Big(0,\ldots,\overset{i_1}{\widehat{i_2}},\ldots,\overset{i_2}{\overbrace{i_3-i_1}},\ldots,\overset{i_{s-1}}{\overbrace{i_s-i_{s-2}}},\ldots,\overset{i'}{\overbrace{n-i_s}},\ldots,0\Big),$$

$$v_{i_1,\ldots,i_s,i',i''} = \Big(0,\ldots,\overset{i_1}{\widehat{i_2}},\ldots,\overset{i_2}{\overbrace{i_3-i_1}},\ldots,\overset{i_{s-1}}{\overbrace{i_s-i_{s-2}}},\ldots,\overset{i'}{\overbrace{i''-i_s}},\ldots,\overset{i''}{\overbrace{n-i'}},\ldots,0\Big)$$

(there are zeroes on vacant places). Hence

$$v_{i',i''} - v_{i'} = v_{i_1,\ldots,i_s,i',i''} - v_{i_1,\ldots,i_s,i'} = \Big(0,\ldots,\overset{i'}{\overbrace{i''-n}},\ldots,\overset{i''}{\overbrace{n-i'}},\ldots,0\Big).$$

Similarly,

$$v_{i',i''} - v_{i''} = v_{i',i'',i_1,\ldots,i_s} - v_{i'',i_1,\ldots,i_s} = \Big(0,\ldots,\overset{i'}{\widehat{i''}},\ldots,\overset{i''}{\widehat{-i'}},\ldots,0\Big).$$

Thus, the lemma is proved. □

Now we can turn to the proof of Theorem 2.

Proof of Theorem 2. First, we prove the theorem for the facet g_2. According to Lemma 1, it contains all vertices of the polytope $\mathcal{N}(\Delta)$ of the type $v_{\ldots,2,\ldots}$ (i.e. corresponding to partitions of $[0,n]$ having the point 2 in the defining set of integer points) and only of this type. Let us show that these points constitute a prism. In order to do this we divide all the mentioned partitions into two groups. The first group includes all the partitions which do not employ the point 1, the vertices of $\mathcal{N}(\Delta)$ corresponding to them are of the type $v_{2,\ldots}$; the second group includes all partitions employing the point 1, the corresponding vertices of $\mathcal{N}(\Delta)$ are of the type $v_{1,2,\ldots}$.

Show that vertices of each group lie in two parallel planes of dimension $n-3$. Namely, the vertices of the first group lie in the plane given by

$$t_1 = 0, \ (n-2)t_2 + (n-3)t_3 + \ldots + 1t_{n-1} = n(n-2). \tag{9}$$

The coordinates of the vertices $v_{2,\ldots}$ can be written down as (7) with $i_1 = 2$. Substituting these coordinates into the left hand side of the second equation of (9) we get

$$\sum_{\nu=1}^{s}(n-i_\nu)(i_{\nu+1}-i_{\nu-1}) = \sum_{\nu=1}^{s} n(i_{\nu+1}-i_{\nu-1}-i_\nu i_{\nu+1}+i_{\nu-1}i_\nu) =$$
$$= ni_s + ni_{s+1} - ni_0 - ni_1 - i_s i_{s+1} + i_0 i_1 = n(n-2).$$

Here we take into account that $i_{s+1} = n$, $i_0 = 0$, $i_1 = 2$. Thus the second equality in (9) holds.

Now, according to Lemma 2, each vertex of $\mathcal{N}(\Delta)$ corresponding to a partition from the second group, i.e. each point of the form $v_{1,2,i_1,\ldots,i_s}$, $s = 0,\ldots,n-3$, is a translate of the point $v_{2,i_1,\ldots,i_s}$ corresponding to a partition from the first group by the vector $(2,-1,0,\ldots,0)$. It follows from here that two faces, one formed by the points of the form $v_{1,2,i_1,\ldots,i_s}$ and another formed by the points $v_{2,i_1,\dots,i_s}$, $s = 0,\dots,n-3$, are conguent. Also, it is clear that the other edges of the facet are parallel. So, for the facet g_2 the theorem is proved.

The bases are parallel, and the base formed by the vertices of $\mathcal{N}(\Delta)$ of the type $v_{1,2,\dots}$ passes through the point $v_{1,2,\dots,n-1} = (2,\dots,2)$ of the polytope, from these facts it follows that the equation of this base is

$$t_1 = 2, \quad (n-2)t_2 + (n-3)t_3 + \ldots + 1 \cdot t_{n-1} = (n-1)(n-2).$$

Similar arguments are applicable to the facet g_{n-2}, whose each point looks like $v_{\dots,n-2,\dots}$. For this facet each vertex of $\mathcal{N}(\Delta)$ of the form $v_{i_1,\dots,i_s,n-2,n-1}$, $s = 0,\dots,n-3$, belonging to one of the prism's bases is a translate by the vector $(0,\dots,-1,2)$ of a point of the form $v_{i_1,\dots,i_s,n-2}$, belonging to another base. Herewith, the equations of the bases of the prism g_{n-2} are

$$t_{n-1} = 0, \quad 1t_1 + 2t_2 + \ldots + (n-2)t_{n-2} = n(n-2),$$

$$t_{n-1} = 2, \quad 1t_1 + 2t_2 + \ldots + (n-2)t_{n-2} = (n-1)(n-2).$$

The theorem is proved. □

3. Factorability of the discriminant's truncations $\Delta\big|_{g_k}$

A discriminant truncation with respect to a facet g_k of the Newton polytope $\mathcal{N}(\Delta)$ is the polynomial $\Delta\big|_{g_k}$ consisting of all monomials Δ with exponents from g_k. The main observation of this section is the following statement about discriminants of polynomials of degree not greater than 6.

Theorem 3. *For $n=2,3,4,5,6$ the truncation $\Delta_n\big|_{g_k}$ of the reduced discriminant is a product(up to a monomial factor) $\Delta_k(1,a_1,\dots,a_k)\Delta_{n-k}(1,a_{n-1},\dots,a_k)$ of the discriminants of polynomials of degrees k and $n-k$.*

Conjecture. *The statement of Theorem 3 seems to be valid for any degree n.*

Before we begin to prove Theorem 3, remark that $\Delta_1 \equiv 1$ since $\Delta_k(1,a_1)$ is the discriminant of the polynomial $f(y) = a_1y + 1$. Therefore the statement of Theorem 3 for $n = 2, 3$ is trivial. So, we consider only the cases when $n = 4, 5, 6$. We begin our proof for facets that are prisms.

3.1. Truncations of the discriminant with respect to prism facets

In this section we study truncations of the reduced discriminant with respect to facets g_2 and g_{n-2}, which according to Theorem 2 are prisms.

Consider a reduced polynomial of **degree 4**:

$$f_{red}(y) = 1 + a_1y + a_2y^2 + a_3y^3 + y^4.$$

In this case prisms g_2 and g_{n-2} coincide, therefore only one of the facets of $\mathcal{N}(\Delta)$ is a 2-prism, i.e. a parallelogram. According to (6) the facet g_2 is given by

$$\begin{cases} 3t_1 + 2t_2 + t_3 \leqslant 12, \\ t_1 + 2t_2 + t_3 = 8, \\ t_1 + 2t_2 + 3t_3 \leqslant 12 \end{cases}$$

in the positive orthant. Integer solutions of this system are the four points

$$v_2 = (0,4,0), \quad v_{12} = (2,3,0), \quad v_{123} = (2,2,2), \quad v_{23} = (0,3,2),$$

which are the vertices of the parallelogram. Using formula (3) we compute the truncation

$$\Delta\Big|_{g_2} = 16a_2^4 - 4a_1^2a_2^3 + a_1^2a_2^2a_3^2 - 4a_2^3a_3^2.$$

It turns out that it factors into

$$\Delta\Big|_{g_2} = a_2^2(4a_2 - a_3^2)(4a_2 - a_1^2), \tag{10}$$

where binomial factors are discriminants of quadratic polynomials

$$a_2y^2 + a_3y + 1 \quad \text{and} \quad a_2y^2 + a_1y + 1.$$

For $\mathbf{n = 5}$ the polynomial (4) takes the form

$$f_{red}(y) = 1 + a_1y + a_2y^2 + a_3y^3 + a_4y^4 + y^5. \tag{11}$$

In this case according to Theorem 2 the facets g_2 and g_3 are 3-prisms. The facet g_3 is defined by the system

$$\begin{cases} 4t_1 + 3t_2 + 2t_3 + t_4 \leqslant 20, \\ 3t_1 + 6t_2 + 4t_3 + 2t_4 \leqslant 30, \\ 2t_1 + 4t_2 + 6t_3 + 3t_4 = 30, \\ t_1 + 2t_2 + 3t_3 + 4t_4 \leqslant 20. \end{cases}$$

Calculations show that a solution of this system which defines the truncation $\Delta\Big|_{g_3}$ of the discriminant of (11) consists of the following 10 points

$$v_3 = (0,0,5,0),\ v_{13} = (3,0,4,0),\ v_{23} = (0,3,3,0),\ v_{123} = (2,2,3,0),$$
$$v_{34} = (0,0,4,2),\ v_{134} = (3,0,3,2),\ v_{234} = (0,3,2,2),\ v_{1234} = (2,2,2,2),$$
$$(1,1,4,0),\ (1,1,3,2).$$

As we see, the first five points lie on the prism's base in the hyperplane $t_4 = 0$, and the rest of these points lie on the base in the hyperplane $t_4 = 2$.

The expression for $\Delta\Big|_{g_3}$ is

$$\Delta\Big|_{g_3} = 108a_3^5 + 16a_1^3a_3^4 + 16a_2^3a_3^3 - 4a_1^2a_2^2a_3^3 - 27a_3^4a_4^2 - 4a_1^3a_3^3a_4^2 - 4a_2^3a_3^2a_4^2+$$
$$+a_1^2a_2^2a_3^2a_4^2 - 72a_1a_2a_3^4 + 18a_1a_2a_3^3a_4^2.$$

It turns out that this truncation also factors:

$$\Delta\Big|_{g_3} = a_3^2(4a_3 - a_4^2)(27a_3^2 + 4a_1^3a_3 + 4a_2^3 - a_1^2a_2^2 - 18a_1a_2a_3). \tag{12}$$

Note that the factor $4a_3 - a_4^2$ in this expression is the discriminant $\Delta_{2;3,4}(a_3, a_4)$ of a quadratic polynomial $f_{2;3,4}(y) = a_3y^2 + a_4y + 1$ and the last factor is the discriminant $\Delta_{3;3,2,1}(a_1, a_2, a_3)$ of a cubic polynomial

$$f_{3;3,2,1}(y) = a_3y^3 + a_2y^2 + a_1y + 1.$$

Consider now the polynomial (4) for $\mathbf{n = 6}$:

$$f_{red}(y) = 1 + a_1y + a_2y^2 + a_3y^3 + a_4y^4 + a_5y^5 + y^6. \tag{13}$$

Compute $\Delta\big|_{g_4}$. According to (6) the integer points of $\mathcal{N}(\Delta)$ that belong to g_4 are defined by the following conditions

$$\begin{cases} 5t_1 + 4t_2 + 3t_3 + 2t_4 + t_5 \leqslant 30, \\ 4t_1 + 8t_2 + 6t_3 + 4t_4 + 2t_5 \leqslant 48, \\ 3t_1 + 6t_2 + 9t_3 + 6t_4 + 3t_5 \leqslant 54, \\ 2t_1 + 4t_2 + 6t_3 + 8t_4 + 4t_5 = 48, \\ t_1 + 2t_2 + 3t_3 + 4t_4 + 5t_5 \leqslant 30. \end{cases}$$

It follows from the equations for the prisms' bases obtained in the previous section that for the level t_5 the values $0, 1, 2$ are possible. For the base in $t_5 = 0$ we get the system:

$$\begin{cases} t_4 \geqslant 3, \\ t_3 + 2t_4 \geqslant 8, \\ t_2 + 2t_3 + 3t_4 \geqslant 15, \\ t_1 + 2t_2 + 3t_3 + 4t_4 = 24, \end{cases} \tag{14}$$

the solution of which are 16 points from $\mathbb{Z}^5_{\geqslant 0}$, and 8 of them are vertices

$$v_{1234} = (2,2,2,3,0),\ v_{234} = (0,3,2,3,0),\ v_{134} = (3,0,3,3,0),$$

$$v_{34} = (0,0,4,3,0),\ v_{124} = (2,3,0,4,0),\ v_{24} = (0,4,0,4,0),$$

$$v_{14} = (4,0,0,5,0),\ v_4 = (0,0,0,6,0).$$

The remaining 8 points

$$(1,1,3,3,0)\ (3,1,1,4,0),\ (1,2,1,4,0),\ (2,0,2,4,0),\ (0,1,2,4,0),$$

$$(2,1,0,5,0),\ (0,2,0,5,0)\ , (1,0,1,5,0)$$

are not vertices but correspond to some monomials of the discriminant of the equation under consideration.

For $t_5 = 1$ for each integer solution of (14) from $\mathbb{Z}^5_{\geqslant 0}$ there is no corresponding monomial participating in the discriminant of the polynomial of degree 6. As for the plane $t_5 = 2$, here we use the fact that each point of this base is a translate from the base in $t_5 = 0$ by the vector $(0,0,0,-1,2)$. As a result we get the following points in the remaining base (in $t_5 = 2$):

$$v_{12345} = (2,2,2,2,2),\ v_{2345} = (0,3,2,2,2),\ v_{1345} = (3,0,3,2,2),$$

$$v_{345} = (0,0,4,2,2),\ v_{1245} = (2,3,0,3,2),\ v_{245} = (0,4,0,3,2),$$

$$v_{145} = (4,0,0,4,2),\ v_{45} = (0,0,0,5,2);$$

$$(1,1,3,2,2)\ (3,1,1,3,2),\ (1,2,1,3,2),\ (2,0,2,3,2),$$

$$(0,1,2,3,2),\ (2,1,0,4,2),\ (0,2,0,4,2)\ , (1,0,1,4,2).$$

Then for $\Delta\big|_{g_4}$ we get the following representation:

$$\Delta\big|_{g_4} = -(4a_1^2a_2^2a_3^2a_4^3 - 16a_2^3a_3^2a_4^3 - 16a_1^3a_3^3a_4^3 - 108a_3^4a_4^3 - 16a_1^2a_2^3a_4^4 + 64a_2^4a_4^4 -$$

$$-108a_1^4a_4^5 + 1024a_4^6) + (72a_1a_2a_3^3a_4^3 + 72a_1^3a_2a_3a_4^4 - 320a_1a_2^2a_3a_4^4 - 24a_1^2a_3^2a_4^4 +$$

$$+576a_2a_3^2a_4^4 + 576a_1^2a_2a_4^5 - 512a_2^2a_4^5 - 768a_1a_3a_4^5) - a_5^2\big((-a_1^2a_2^2a_3^2a_4^2 +$$

$$+4a_2^3a_3^2a_4^2 + 4a_1^3a_3^3a_4^2 + 27a_3^4a_4^2 + 4a_1^2a_2^3a_4^3 - 16a_2^4a_4^3 + 27a_1^4a_4^4 - 256a_4^5) +$$

$$+(-18a_1a_2a_3^3a_4^2 - 18a_1^3a_2a_3a_4^3 + 80a_1a_2^2a_3a_4^3 + 6a_1^2a_3^2a_4^3 - 144a_2a_3^2a_4^3 -$$
$$-144a_1^2a_2a_4^4 + 128a_2^2a_4^4 + 192a_1a_3a_4^4)\big),$$

which admits a factorization

$$\Delta\Big|_{g_4} = -a_4^2(a_5^2 - 4a_4)(6a_1^2a_3^2a_4 + 27a_1^4a_4^2 - 18a_1^3a_2a_3a_4 + 80a_1a_2^2a_3a_4 - 144a_1^2a_2a_4^2 +$$
$$+4a_1^3a_3^3 - 18a_1a_2a_3^3 + 192a_1a_3a_4^2 - 256a_4^3 - a_1^2a_2^2a_3^2 - 16a_2^4a_4 +$$
$$+128a_2^2a_4^2 + 27a_3^4 + 4a_1^2a_2^3a_4 - 144a_2a_3^2a_4 + 4a_2^3a_3^2).$$

In the obtained product the factor $a_5^2 - 4a_4$ is the discriminant $\Delta_{2;4,5}(a_4, a_5)$ of a quadratic polynomial

$$f_{2;4,5}(y) = a_4y^2 + a_5y + 1,$$

and the last factor is the discriminant $\Delta_{4;4,3,2,1}(a_1, a_2, a_3, a_4)$ of a polynomial of degree 4:

$$f_{4;4,3,2,1}(y) = a_4y^4 + a_3y^3 + a_2y^2 + a_1y + 1.$$

Then for $\Delta\Big|_{g_4}$ we get the following representation:

$$\Delta\Big|_{g_4} = a_4^2\,\Delta_{2;4,5}(a_4, a_5)\,\Delta_{4;4,3,2,1}(a_1, a_2, a_3, a_4).$$

3.2. Truncations of the discriminant with respect to the remaining facets

Now we turn again to the polynomial (11) of **degree 5**. Consider the truncation $\Delta\Big|_{g_k}$ of its discriminant with respect to the remaining facets g_k, i.e. g_1 and g_4. The facet g_4 is given by the system

$$\begin{cases} 4t_1 + 3t_2 + 2t_3 + t_4 \leqslant 20, \\ 3t_1 + 6t_2 + 4t_3 + 2t_4 \leqslant 30, \\ 2t_1 + 4t_2 + 6t_3 + 3t_4 \leqslant 30, \\ t_1 + 2t_2 + 3t_3 + 4t_4 = 20. \end{cases}$$

Calculations show that among integer solutions of this system only the following points are exponents of monomials of the discriminant Δ

$$(1,1,3,2),\ (3,1,1,3),\ (2,2,2,2),\ (3,0,3,2),\ (2,0,2,3),\ (1,0,1,4),$$
$$(0,1,2,3),\ (2,1,0,4),\ (0,3,2,2),\ (2,3,0,3),\ (1,2,1,3),\ (4,0,0,4),$$
$$(0,0,0,5),\ (0,2,0,4),\ (0,4,0,3),\ (0,0,4,2).$$

The truncation of Δ with respect to g_4 equals

$$\Delta\Big|_{g_4} = 18a_1a_2a_3^3a_4^2 + 18a_1^3a_2a_3a_4^3 + a_1^2a_2^2a_3^2a_4^2 - 4a_1^3a_3^3a_4^2 - 6a_1^2a_3^2a_4^3 - 192a_1a_3a_4^4 +$$
$$+144a_2a_3^2a_4^3 + 144a_1^2a_2a_4^4 - 4a_2^3a_3^2a_4^2 - 4a_1^2a_2^3a_4^3 - 80a_1a_2^2a_3a_4^3 - 27a_1^4a_4^4 + 256a_4^5 -$$
$$-128a_2^2a_4^4 + 16a_2^4a_4^3 - 27a_3^4a_4^2 = a_4^2(18a_1a_2a_3^3 + 18a_1^3a_2a_3a_4 + a_1^2a_2^2a_3^2 - 4a_1^3a_3^3 -$$
$$-6a_1^2a_3^2a_4 - 192a_1a_3a_4^2 + 144a_2a_3^2a_4 + 144a_1^2a_2a_4^2 - 4a_2^3a_3^2 - 4a_1^2a_2^3a_4 - 80a_1a_2^2a_3a_4 -$$
$$-27a_1^4a_4^2 + 256a_4^3 - 128a_2^2a_4^2 + 16a_2^4a_4 - 27a_3^4) = a_4^2\Delta_{4;4,3,2,1}(a_1, a_2, a_3, a_4)\,,$$

where $\Delta_{4;4,3,2,1}(a_1, a_2, a_3, a_4)$ is the discriminant of a polynomial

$$f_{4;1,2,3}(y) = a_4y^4 + a_3y^3 + a_2y^2 + a_1y + 1.$$

It is known that this discriminant is irreducible [4]. Since $\Delta_1 \equiv 1$, we can write as desired:

$$\Delta\big|_{g_4} = a_4^2 \Delta_{4;4,3,2,1}(a_1, a_2, a_3, a_4)\Delta_{1;4}(a_4).$$

At the end of the section we show factorability of $\nabla\big|_{g_3}$ for the polynomial of **degree 6**. The facet g_3 is given by the system

$$\begin{cases} 5t_1 + 4t_2 + 3t_3 + 2t_4 + t_5 \leqslant 30, \\ 4t_1 + 8t_2 + 6t_3 + 4t_4 + 2t_5 \leqslant 48, \\ 3t_1 + 6t_2 + 9t_3 + 6t_4 + 3t_5 = 54, \\ 2t_1 + 4t_2 + 6t_3 + 8t_4 + 4t_5 \leqslant 48, \\ t_1 + 2t_2 + 3t_3 + 4t_4 + 5t_5 \leqslant 30. \end{cases}$$

Its solution consists of 25 points

$$(1,1,3,3,0),\ (1,1,4,0,3),\ (2,2,2,3,0),\ (2,2,3,0,3),\ (0,3,3,1,1),$$

$$(0,3,2,2,2),\ (3,0,4,1,1),\ (3,0,3,2,2),\ (0,0,4,3,0),\ (0,0,5,0,3),$$

$$(0,3,4,0,0),\ (3,0,5,0,0),\ (2,2,3,1,1),\ (1,1,3,2,2),\ (0,0,5,1,1),$$

$$(0,0,4,2,2),\ (1,1,5,0,0),\ (2,2,4,0,0),\ (0,3,2,3,0),\ (0,3,3,0,3),$$

$$(3,0,3,3,0),\ (3,0,4,0,3),\ (0,0,6,0,0),\ (1,1,4,1,1),\ (2,2,2,2,2).$$

Accordingly, the truncation $\nabla\big|_{g_3}$ is

$$\nabla\big|_{g_3} = 72a_1a_2a_3^3a_4^3 + 72a_1a_2a_3^4a_5^3 + 4a_1^2a_2^2a_3^2a_4^3 + 4a_1^2a_2^2a_3^3a_5^3 + 72a_2^3a_3^3a_4a_5+$$

$$+4a_2^3a_3^2a_4^2a_5^2 - +72a_1^3a_3^4a_4a_5 + 4a_1^3a_3^3a_4^2a_5^2 - 108a_3^4a_4^3 - 108a_3^5a_5^3 - 108a_2^3a_3^4-$$

$$-108a_1^3a_3^5 - 18a_1^2a_2^2a_3^3a_4a_5 - 18a_1a_2a_3^3a_4^2a_5^2 + 486a_3^5a_4a_5 + 27a_3^4a_4^2a_5^2+$$

$$+486a_1a_2a_3^5 + 27a_1^2a_2^2a_3^4 - 16a_2^3a_3^2a_4^3 - 16a_2^3a_3^3a_5^3 - 16a_1^3a_3^3a_4^3-$$

$$-16a_1^3a_3^4a_5^3 - 729a_3^6 - 324a_1a_2a_3^4a_4a_5 - a_1^2a_2^2a_3^2a_4^2a_5^2.$$

Calculations show that this expression admits the following factorization

$$-a_3^2(27a_3^2 - 18a_1a_2a_3^2 - a_1^2a_2^2 + 4a_2^3 + 4a_1^3a_3)\times$$

$$\times(27a_3^2 - 18a_3a_4a_5 - 9a_3a_5^2 + 4a_4^3 - 3a_4a_5^2 - a_5^3).$$

Thus, we get

$$\nabla\big|_{g_3} = -a_3^2\Delta_{3;3,2,1}\Delta_{3;3,4,5},$$

where $\Delta_{3;3,2,1}(a_1, a_2, a_3)$, $\Delta_{3;3,4,5}(a_3, a_4, a_5)$ are the discriminants of polynomials

$$f_{3;3,2,1}(y) = a_3y^3 + a_2y^2 + a_1y + 1, \quad f_{3;3,4,5}(y) = a_3y^3 + a_4y^2 + a_5y + 1.$$

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О структуре классического дискриминанта

Евгений Н. Михалкин
Август К. Цих

Рассматривается полином степени n с переменными коэффициентами. Известно, что многогранник Ньютона дискриминанта такого многочлена комбинаторно эквивалентен $n-1$-мерному кубу. В статье показано, что две гиперграни многогранника Ньютона являются призмами, а срезки дискриминанта на грани факторизуются на дискриминанты полиномов степени меньше n.

Ключевые слова: общее алгебраическое уравнение, дискриминант, многогранник Ньютона.