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Numerical Investigation of Solutions to a Reaction-diffusion System with Variable Density

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In this paper we demonstrate the possibilities of the self-similar and approximately self-similar approaches for studying solutions of a nonlinear mutual reaction-diffusion system. The asymptotic behavior of compactly supported solutions and free boundary is studied. Based on established qualitative properties of solutions numerical computation is carried out. The solutions are presented in visualization form, which allows observing evolution of the studied process in time.

Keywords: double nonlinear reaction-diffusion system, self-similar solutions, asymptotics, numerical calculations.

Introduction

In the domain $Q = \left\{ (t,x) \colon t > 0, \, x \in \mathbb{R}^N \right\}$ properties are investigated of the process of a nonlinear diffusion-reaction with variable density described by the following system:

$$
\begin{aligned}
\frac{\partial \left(\rho(x)u \right)}{\partial t} &= \operatorname{div} \left(|x|^n \, v^{m_1-1} \, |\nabla u|^{p-2} \, \nabla u \right) + \rho(x)\gamma(t)u^{\beta_1}, \\
\frac{\partial \left(\rho(x)v \right)}{\partial t} &= \operatorname{div} \left(|x|^n \, u^{m_2-1} \, |\nabla v|^{p-2} \, \nabla v \right) + \rho(x)\gamma(t)v^{\beta_2},
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
u\left(0,x \right) &= u_0\left(x \right) \geqslant 0, \\
v\left(0,x \right) &= v_0\left(x \right) \geqslant 0, \, x \in \mathbb{R}^N,
\end{aligned}
\tag{2}
$$

where $m_1, \, m_2, \, n \in \mathbb{R}, \; \beta_1, \, \beta_2 \geqslant 1, \; p \geqslant 2$ are given positive numbers.

$$
\nabla(\cdot) - \operatorname*{grad}_{x}(\cdot), \quad u_0\left(x \right), v_0\left(x \right) \geqslant 0, \quad x \in \mathbb{R}^N,
$$
$$
\rho(x) = |x|^{-l}, \quad l > 0, \quad 0 < \gamma(\mathrm{t}) \in C(0,\infty).
$$

System (1) describes different physical processes in two componental inhomogeneous nonlinear media, for example, the processes of mutual reaction-diffusions, heat conductivity, the theory of combustion, the theory of a polytrophic filtration of a liquid and a gas in the presence of a source whose power is equal to $\rho(x)\gamma(t)u^{\beta_1}, \; \rho(x)\gamma(t)v^{\beta_2}$. In works [1–6] were considered particular cases of the system (1), when $\gamma(\mathrm{t}) = 1, \; n = l = 0, \; p = 2$ and we investigated arising different types of solutions depending on the parameters of system (1).

System (1) is degenerate in the domain, where $u = v = 0$ and may have no classical solutions. Therefore, the weak solutions of system (1) are studied, having physical sense. Namely functions $0 \leqslant u, \, v \in C\left(Q \right)$ such that $|x|^n \, v^{m_1-1} \, |\nabla u|^{p-2} \, \nabla u, \; |x|^n \, u^{m_2-1} \, |\nabla v|^{p-2} \, \nabla v \in C(Q)$ satisfying the

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integral identity in the sense of distributions [1]. For solutions of system (1) may take place the phenomena of finite velocity of propagation and space localization of disturbance, i.e. there may exist functions $l_1(t)$, $l_2(t)$, such that $u(t,x) \equiv 0$ and $v(t,x) \equiv 0$ if $|x| \geqslant l_1(t)$ and $|x| \geqslant l_2(t)$. In the case $l_1(t)$, $l_2(t) < \infty$, for $t > 0$ a solution to problem (1)–(2) is called a space localization of disturbance. The surfaces $|x| = l_1(t)$ and $|x| = l_2(t)$ are called a free boundary or a front.

In the present work we suggest a method of construction of self-similar equations system (1) based of splitting of system (1), and study asymptotics of compactly support solutions and a free boundary and asymptotics of self-similar solutions for the quick diffusion case. It is shown that the coefficient of the main member of the asymptotics of the solution satisfies a certain system of nonlinear algebraic system equation. Based on established qualitative properties of the solution, using approximately self-similar solutions, numerical experiments, visualization of processes described by reaction-diffusion system (1) with variable density were carried out.

1.   Construction of a self-similar system of equations

Studying different properties of solutions to system (1) is a complicated problem, even for particular cases of system (1) [2, 6–9]. In these works for particular cases of system (1)–(2) the effectiveness was shown of the self-similar approach for studying different properties of solutions to problem (1)–(2). Below a new method (nonlinear splitting) of construction of a self-similar and an approximately self-similar system is suggested. This method gives us a more simple way of investigation of qualitative properties of solutions to problem (1)–(2).

For construction of a self-similar and an approximately self-similar system for system (1) solutions $u(t,x)$, $v(t,x)$ to system of equations (1) are searched in the form

$$
\begin{aligned}
u(t,x) &= \bar{u}(t)w\left(\tau(t), \varphi(|x|)\right), \\
v(t,x) &= \bar{v}(t)z\left(\tau(t), \varphi(|x|)\right),
\end{aligned}
\tag{3}
$$

where

$$
\bar{u}(t) = \left[T + \int_0^t \gamma(t)dt\right]^{-\frac{1}{\beta_1 - 1}}, \quad \bar{v}(t) = \left[T + \int_0^t \gamma(t)dt\right]^{-\frac{1}{\beta_2 - 1}}, \quad T > 0,
$$

and functions τ, φ will be choosen below.

Substituting (3) in system (1) reduces it to the following system of equations

$$
\begin{aligned}
\frac{\partial w}{\partial \tau} &= \varphi^{1-s}\frac{\partial}{\partial \varphi}\left(\varphi^{s-1}z^{m_1-1}\left|\frac{\partial w}{\partial \varphi}\right|^{p-2}\frac{\partial w}{\partial \varphi}\right) + \gamma(t)\bar{u}^{\beta_1-(p-1)}\bar{v}^{-(m_1-1)}\left(w + w^{\beta_1}\right), \\
\frac{\partial z}{\partial \tau} &= \varphi^{1-s}\frac{\partial}{\partial \varphi}\left(\varphi^{s-1}w^{m_2-1}\left|\frac{\partial z}{\partial \varphi}\right|^{p-2}\frac{\partial z}{\partial \varphi}\right) + \gamma(t)\bar{u}^{-(m_2-1)}\bar{v}^{\beta_2-(p-1)}\left(z + z^{\beta_2}\right),
\end{aligned}
\tag{4}
$$

where the functions τ, φ are chosen as

$$
\tau(t) = \int_0^t \bar{v}^{m_1-1}(\eta)\bar{u}^{p-2}(\eta)d\eta = \int_0^t \bar{u}^{m_2-1}(\eta)\bar{v}^{p-2}(\eta)d\eta,
\tag{5}
$$

$$
\varphi(r) = \frac{1}{p_1}|r|^{p_1}, \quad |r| = \sqrt{\sum_{i=1}^N x_i^2}, \quad p_1 = \frac{p-(n+l)}{p}, \quad s = p\frac{N-l}{p-(n+l)}, \quad l+n < p.
$$

It is easy to establish that system (4) has approximately self-similar solutions of the form

$$
\begin{aligned}
w(\tau, \varphi) &= f(\xi), \\
z(\tau, \varphi) &= \psi(\xi),
\end{aligned}
\tag{6}
$$

where $\xi = \dfrac{\varphi(x)}{\tau^{1/p}}$ and the functions f, ψ satisfy to the following approximately self-similar system of equations

$$\xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}\psi^{m_1-1}\left|\frac{df}{d\xi}\right|^{p-2}\frac{df}{d\xi}\right) + \frac{\xi}{p}\frac{df}{d\xi} + \gamma(t)\tau(t)\bar{u}^{\beta_1-(p-1)}\bar{v}^{-(m_1-1)}\left(f + f^{\beta_1}\right) = 0,$$

$$\xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}f^{m_2-1}\left|\frac{d\psi}{d\xi}\right|^{p-2}\frac{d\psi}{d\xi}\right) + \frac{\xi}{p}\frac{d\psi}{d\xi} + \gamma(t)\tau(t)\bar{u}^{-(m_2-1)}\bar{v}^{\beta_2-(p-1)}\left(\psi + \psi^{\beta_2}\right) = 0,$$
(7)

It is easy to provej that

$$\gamma(t)\bar{u}^{\beta_1-(p-1)}\bar{v}^{-(m_1-1)}\tau(t) \to \text{const}, \ \ \text{as} \ \ t \to \infty,$$
$$\gamma(t)\bar{u}^{-(m_1-1)}\bar{v}^{\beta_2-(p-1)}\tau(t) \to \text{const}, \ \ \text{as} \ \ t \to \infty$$
(8)

if $0 < \gamma(t) \in H$, where H is Hardy's body [2]. In this case system (1) becomes self-similar. Therefore it is possible to say that (7) is an asymptotically self-similar system for system of equations (1).

Let $\gamma(t) = \text{const}$. Then approximately self-similar system (7) has a self-similar form if

$$(\beta_2 - 1)(p - (m_1 + 1)) = (\beta_1 - 1)(p - (m_2 + 1))$$
(9)

In this case for the functions f, ψ we have the following self-similar system

$$\xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}\psi^{m_1-1}\left|\frac{df}{d\xi}\right|^{p-2}\frac{df}{d\xi}\right) + \frac{\xi}{p}\frac{df}{d\xi} + a_1\left(f + f^{\beta_1}\right) = 0,$$

$$\xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}f^{m_2-1}\left|\frac{d\psi}{d\xi}\right|^{p-2}\frac{d\psi}{d\xi}\right) + \frac{\xi}{p}\frac{d\psi}{d\xi} + a_2\left(\psi + \psi^{\beta_2}\right) = 0,$$
(10)

where

$$a_1 = (\beta_1 - 1)(\beta_2 - 1)/[(\beta_1 - 1)(\beta_2 - 1) - ((m_1 - 1)(\beta_1 - 1) + (p - 2)(\beta_2 - 1))],$$

$$a_2 = (\beta_1 - 1)(\beta_2 - 1)/[(\beta_1 - 1)(\beta_2 - 1) - ((m_2 - 1)(\beta_2 - 1) + (p - 2)(\beta_1 - 1))].$$

We notice that in the singular case $\beta_1 = (m_1 + p - 2)$, $\beta_2 = (m_2 + p - 2)$ positive solutions system (10) in the case of one equation were studied in [9].

2. Asymptotics of self-similar solutions

Now we will study asymptotics of weak compactly supported solutions (c.s.) to system (10) when $\gamma(t) = \text{const}$.

Consider system of equations (10) with the following boundary condition

$$f(0) = c_1 > 0, \ \ f(b) = 0,$$
$$\psi(0) = c_2 > 0, \ \ \psi(b) = 0,$$
(11)

where $0 < b < +\infty$.

The existence of a self-similar weak solution to problem (10)–(11) for one equation, in the case $\gamma(t) = 0$, $n = l = 0$, $p = 2$ was studied in [1] and conditions for the existence of c.s. solutions were obtained. Asymptotic behavior of a self similar c.s. solution ere established for the case

one equation in [10] and for other system, but with other nonlinear coefficients in [3–6, 11] when p=2.

Now we will study the asymptotic of a c.s. solution system (10) in the case $p + m_1 - 3 > 0$, $p + m_2 - 3 > 0$. For this goal we will transform system (10) to the convenient form for investigation by using the following transformations

$$f(\xi) = \bar{f}(\xi) y_1(\eta), \quad \eta = -\ln\left(a - \xi^\lambda\right), \quad 0 < \xi < a^{1/\lambda},$$
$$\psi(\xi) = \bar{\psi}(\xi) y_2(\eta), \tag{12}$$

where

$$\bar{f}(\xi) = \left(a - \xi^\lambda\right)_+^{q_1}, \quad \bar{\psi}(\xi) = \left(a - \xi^\lambda\right)_+^{q_2}, \quad \lambda = \frac{p}{p-1},$$

$$q_1 = \frac{(p-1)(p-(m_1+1))}{q}, \quad q_2 = \frac{(p-1)(p-(m_2+1))}{q},$$

$$p > m_1 + 1, \quad p > m_2 + 1, \quad a > 0, \quad q = (p-2)^2 - (m_1-1)(m_2-1).$$

Case $(p-2)^2 - (m_1-1)(m_2-1) = 0$ is a singular case. In this case the asymptotic of a solutions of the system (10) have another behavior. This case is required an additional investigation.

Theorem 1. *Let $q_1 > 0$, $q_2 > 0$. Then c.s. solutions to system (10) as $\eta \to +\infty$ $\left(\xi \to a^{1-\frac{1}{p}}\right)$ have the asymptotic*

$$f(\xi) = y_1^0 \bar{f}(\xi)(1 + o(1)),$$
$$\psi(\xi) = y_2^0 \bar{\psi}(\xi)(1 + o(1)), \tag{13}$$

where $0 < y_i^0 < +\infty$ $(i = 1, 2)$, if one of following conditions is satisfied:

1) $\beta_i > \dfrac{q_i - 1}{q_i}$, $i = 1, 2$, then (y_1^0, y_2^0) are the real roots of the following system of nonlinear algebraic equations

$$(y_2^0)^{m_1-1}(y_1^0)^{p-2} = c_1,$$
$$(y_1^0)^{m_2-1}(y_2^0)^{p-2} = c_2, \tag{14}$$

$$c_i = \frac{1}{p(\lambda q_i)^{p-1}}, \quad i = 1, 2,$$

i.e.

$$y_1^0 = \left(\frac{\lambda^{p-2} q_2^{p-1} p}{q_1(\lambda p q_1)^{\frac{1}{p-2}}}\right)^{\frac{m_1-2}{q}} \left[\frac{1}{\lambda p q_1}\right]^{\frac{1}{p-2}} \frac{1}{\lambda q_1},$$

$$y_2^0 = \left(\frac{q_1(\lambda p q_1)^{\frac{1}{p-2}}}{\lambda^{p-2} q_2^{p-1} p}\right)^{\frac{p-2}{q}},$$

where q_i, $i = 1, 2$ are the numbers defined above;

2) $\beta_i = \dfrac{q_i - 1}{q_i}$, $i = 1, 2$, then (y_1^0, y_2^0) are the real roots of the the following system of nonlinear algebraic equations

$$q_1^{p-1}(y_2^0)^{m_1-1}(y_1^0)^{p-2} + \frac{a_1(y_1^0)^{\beta_1-1}}{a\lambda^p q_1} = \frac{1}{p\lambda^{p-1}},$$
$$q_2^{p-1}(y_1^0)^{m_2-1}(y_2^0)^{p-2} + \frac{a_2(y_2^0)^{\beta_2-1}}{a\lambda^p q_2} = \frac{1}{p\lambda^{p-1}}. \tag{15}$$

Proof. In order to prove Theorem 1 we use transformation (12). Transformations (12) reduces self-similar system (10) to the following form

$$\frac{d}{d\eta}L_1(y_1,y_2) + \left(\frac{s}{\lambda}\phi(\eta) - q_1\right)L_1(y_1,y_2) + \frac{y_1' - q_1 y_1}{\lambda^{p-1}}\phi(\eta) + \frac{a_1}{\lambda^p}\phi(\eta)\left(y_1 + \phi_1(\eta)y_1^{\beta_1}\right) = 0,$$
$$\frac{d}{d\eta}L_1(y_1,y_2) + \left(\frac{s}{\lambda}\phi(\eta) - q_2\right)L_1(y_1,y_2) + \frac{y_2' - q_2 y_2}{\lambda^{p-1}}\phi(\eta) + \frac{a_2}{\lambda^p}\phi(\eta)\left(y_2 + \phi_2(\eta)y_2^{\beta_2}\right) = 0,$$

$$(16)$$

where

$$\phi(\eta) = \frac{e^{-\eta}}{a - e^{-\eta}}, \quad \phi_i(\eta) = e^{-\eta q_i(\beta_i - 1)}, \ \ i = 1, 2.$$
$$L_1(y_1,y_2) = y_2^{m_1-1}(|y_1' - q_1 y_1|)^{p-2}(y_1' - q_1 y_1),$$
$$L_2(y_1,y_2) = y_1^{m_2-1}(|y_2' - q_2 y_2|)^{p-2}(y_2' - q_2 y_2),$$

and λ, q_i, a_i, $i = 1, 2$ are the numbers defined above.

Such transformation (12) allows us to reduce studying of the asymptotics of solutions to system (10) as $\eta \to \infty$ to studying those solutions of system (16), which in some neighborhood of $+\infty$ satisfy the inequalities

$$\frac{dy_i}{d\eta} - q_i y_i \neq 0, \ \ y_i(\eta) > 0, \ i = 1, 2.$$

First we show that solutions $y_1(\eta)$, $y_2(\eta)$ to system (16) have finite limits as $\eta \to \infty$.
We introduce the notation

$$v_i(\eta) = L_i(y_1, y_2), \ \ i = 1, 2.$$

Then system (16) may be rewritten in the form

$$v_1' = -\left(\frac{s}{\lambda}\phi(\eta) - q_1\right)v_1 - \frac{y_1' - q_1 y_1}{\lambda^{p-1}}\phi(\eta) - \frac{a_1}{\lambda^p}\phi(\eta)\left(y_1 + \phi_1(\eta)y_1^{\beta_1}\right),$$
$$v_2' = -\left(\frac{s}{\lambda}\phi(\eta) - q_2\right)v_2 - \frac{y_2' - q_2 y_2}{\lambda^{p-1}}\phi(\eta) - \frac{a_2}{\lambda^p}\phi(\eta)\left(y_2 + \phi_2(\eta)y_2^{\beta_2}\right).$$

To analyze the solutions to this system, we introduce the auxiliary functions

$$\theta_1(\mu_1, \eta) = -\left(\frac{s}{\lambda}\phi(\eta) - q_1\right)\mu_1 - \frac{y_1' - q_1 y_1}{\lambda^{p-1}}\phi(\eta) - \frac{a_1}{\lambda^p}\phi(\eta)\left(y_1 + \phi_1(\eta)y_1^{\beta_1}\right),$$
$$\theta_2(\mu_2, \eta) = -\left(\frac{s}{\lambda}\phi(\eta) - q_2\right)\mu_2 - \frac{y_2' - q_2 y_2}{\lambda^{p-1}}\phi(\eta) - \frac{a_2}{\lambda^p}\phi(\eta)\left(y_2 + \phi_2(\eta)y_2^{\beta_2}\right),$$

where μ_i, $i = 1, 2$, are real numbers. For each value of μ_i the function $\theta_i(\mu_i, \eta)$ preserves the sign on some interval $[\eta_{\mu_i}, +\infty) \subset [\eta_0, +\infty)$ $(0 < \eta_0 < \eta_{\mu_i})$ and for all $\eta \in [\eta_{\mu_i}, +\infty)$ takes place one of the inequalities

$$\theta_1(\mu_i, \eta) > 0, \ \ \theta_1(\mu_i, \eta) < 0.$$

Given the theorem of Bohl [13] for the functions $v_i(\eta)$ there are a limit in $\eta \in (\eta_{\mu_i}, +\infty)$:

$$\lim_{\eta \to +\infty} v_i(\eta) < +\infty, \quad \lim_{\eta \to +\infty} v_i'(\eta) = 0.$$

Therefore

$$\lim_{\eta \to +\infty} y_i(\eta) = y_i^0 < +\infty, \quad \lim_{\eta \to +\infty} y_i'(\eta) = 0.$$

Then we have

$$\lim_{\eta\to+\infty} v_1' = \lim_{\eta\to+\infty} \left\{ -\left(\frac{s}{\lambda}\phi(\eta) - q_1\right)\mu_1 - \frac{y_1' - q_1 y_1}{\lambda^{p-1}}\phi(\eta) - \frac{a_1}{\lambda^p}\phi(\eta)\left(y_1 + \phi_1(\eta)y_1^{\beta_1}\right)\right\} = 0,$$

$$\lim_{\eta\to+\infty} v_2' = \lim_{\eta\to+\infty} \left\{ -\left(\frac{s}{\lambda}\phi(\eta) - q_2\right)\mu_2 - \frac{y_2' - q_2 y_2}{\lambda^{p-1}}\phi(\eta) - \frac{a_2}{\lambda^p}\phi(\eta)\left(y_2 + \phi_2(\eta)y_2^{\beta_2}\right)\right\} = 0. \tag{17}$$

Note that as $\eta \to +\infty$

$$\lim_{\eta\to+\infty}\phi(\eta) \to 0, \quad \lim_{\eta\to+\infty}\phi(\eta)\phi_i(\eta) = \begin{cases} 0, & \text{if } \beta_i > \dfrac{q_i - 1}{q_i}, \ i = 1,2, \\ 1/a, & \text{if } \beta_i = \dfrac{q_i - 1}{q_i}, \ i = 1,2. \end{cases}$$

With into account the latest limits we will obtain from (17) the following system of algebraic equations:

$$\begin{aligned}(y_2^0)^{m_1-1}(y_1^0)^{p-2} &= c_1, \\ (y_1^0)^{m_2-1}(y_2^0)^{p-2} &= c_2,\end{aligned} \tag{18}$$

if $\beta_i > \dfrac{q_i - 1}{q_i}$ and

$$q_1^{p-1}(y_2^0)^{m_1-1}(y_1^0)^{p-2} + \frac{a_1(y_1^0)^{\beta_1-1}}{a\lambda^p q_1} = \frac{1}{p\lambda^{p-1}},$$

$$q_2^{p-1}(y_1^0)^{m_2-1}(y_2^0)^{p-2} + \frac{a_2(y_2^0)^{\beta_2-1}}{a\lambda^p q_2} = \frac{1}{p\lambda^{p-1}}, \tag{18'}$$

if $\beta_i = \dfrac{q_i - 1}{q_i}$.

Therefore we have the asymptotical representation (13). □

In the case p = 2 or m = 1 in (10), the properties of the different solutions as computing aspects of the system equation (10) were studied by many authors [12].

3. Fast diffusion case $(q_i < 0, \ i = 1, 2.)$

Consider now the case of fast diffusion. In this case we will study asymptotics of the regular solutions of self-similar system (10) as $\xi \to +\infty$ with the boundary condition

$$\begin{aligned} f(0) &= c_1 \geqslant 0, \ f(\infty) = 0, \\ \psi(0) &= c_2 \geqslant 0, \ \psi(\infty) = 0. \end{aligned} \tag{19}$$

We will replace in (10)

$$\begin{aligned} f(\xi) &= \bar{f}(\xi)y_1(\eta), \\ \psi(\xi) &= \bar{\psi}(\xi)y_2(\eta), \end{aligned} \tag{20}$$

where $\eta = \ln(a + \xi^\gamma)$, $\bar{f} = (a + \xi^\gamma)^{q_1}$, $\bar{\psi} = (a + \xi^\gamma)^{q_2}$, $a > 0$, γ, q_1, q_2 are the numbers defined above.

Theorem 2. *Let $q_1 < 0$, $q_2 < 0$, $\beta_i > 1$, $i = 1, 2$. Then solutions of system (10) have the asymptotics*

$$\begin{aligned} f(\xi) &= y_1^0 \bar{f}(\xi)(1 + o(1)), \\ \psi(\xi) &= y_2^0 \bar{\psi}(\xi)(1 + o(1)) \end{aligned} \tag{21}$$

as $\eta \to \infty$ when one of the following conditions is satisfied:

I. $(N - l) \left[(m_i - 1)(m_{3-i} - 1) - (p - 2)^2 \right] - (p - n - l)(p - m_i - 1) > 0$

and

$$a_i > \frac{p - (m_i + 1)}{(m_i - 1)(m_{3-i} - 1) - (p - 2)^2}, \quad i = 1, 2,$$

or

II. $(N - l) \left[(m_i - 1)(m_{3-i} - 1) - (p - 2)^2 \right] - (p - n - l)(p - m_i - 1) < 0$

and

$$a_i < \frac{p - (m_i + 1)}{(m_i - 1)(m_{3-i} - 1) - (p - 2)^2}, \quad i = 1, 2,$$

numbers y_1^0, y_2^0 are the roots of the following nonlinear algebraic equations

$$\begin{aligned}
(s + \lambda q_1) \left(|q_1 y_1^0| \right)^{p-2} (y_2^0)^{m_1 - 1} + \frac{1}{p\lambda^{p-2}} - \frac{a_1}{q_1 \lambda^{p-1}} = 0, \\
(s + \lambda q_2) \left(|q_2 y_2^0| \right)^{p-2} (y_1^0)^{m_2 - 1} + \frac{1}{p\lambda^{p-2}} - \frac{a_2}{q_2 \lambda^{p-1}} = 0.
\end{aligned} \tag{22}$$

Proof. After replacement (20) system (10) take the following form

$$\begin{aligned}
\frac{d}{d\eta} L_i(y_1, y_2) + \left(\frac{s}{\lambda} \phi_1(\eta) + q_i \right) L_i(y_1, y_2) + \frac{1}{p\gamma^{p-1}} \phi_1(\eta) \left(\frac{dy_i}{d\eta} + q_i y_i \right) + \\
+ \frac{a_i}{\gamma^p} \phi_1(\eta) \left(y_i + \phi_2(\eta) y_i^{\beta_i} \right) = 0,
\end{aligned} \tag{23}$$

where

$$L_i(y_1, y_2) = y_{3-i}^{m_i - 1} \left| \frac{dy_i}{d\eta} + q_i y_i \right|^{p-2} \left(\frac{dy_i}{d\eta} + q_i y_i \right),$$

$$\phi_1(\eta) = \frac{e^\eta}{e^\eta - a}, \quad \phi_2(\eta) = e^{\eta q_i(\beta_i - 1)} \phi_1(\eta), \quad i = 1, 2.$$

Such studying of solutions of a system (10) as $\eta \to +\infty$ is reduced to studying of those solutions of system (10), each of which in some neighborhood of ∞ satisfies to inequalities

$$\frac{dy_i}{d\eta} + q_i y_i \neq 0, \quad y_i(\eta) > 0, \ i = 1, 2.$$

Passing in (23) to the limit at $\eta \to +\infty$ we obtain necessary conditions and system of algebraic equations (22). Rest of a proving is similar to above mentioned manner.   □

4.   Case $\beta_1 = \beta_2 = 1$.

Now consider the Cauchy problem for system

$$\begin{aligned}
\frac{\partial (\rho(x)u)}{\partial t} = div \left(|x|^n \, v^{m_1 - 1} \, |\nabla u|^{p-2} \, \nabla u \right) + \rho(x)\gamma_1(t)u, \\
\frac{\partial (\rho(x)v)}{\partial t} = div \left(|x|^n \, u^{m_2 - 1} \, |\nabla v|^{p-2} \, \nabla v \right) + \rho(x)\gamma_2(t)v.
\end{aligned} \tag{24}$$

We will prove a condition of localization of the problem (24),(2).

Introduce the notation

$$\bar{u}(t) = \exp \int_0^t \gamma_1(s)ds, \quad \bar{v}(t) = \exp \int_0^t \gamma_2(s)ds,$$

$$\varphi(|x|) = \frac{1}{p_1}|r|^{p_1}, \quad p_1 = \frac{p - (n+l)}{p},$$

$$|x| = \sqrt{\sum_{i=1}^N x_i^2}, \quad \tau(t) = \int_0^t \bar{v}^{m_1-1}(\eta)\bar{u}^{p-2}(\eta)d\eta = \int_0^t \bar{u}^{m_2-1}(\eta)\bar{v}^{p-2}(\eta)d\eta$$

$$\begin{aligned} u_+(t,x) &= \bar{u}(t)\bar{f}(\xi), \\ v_+(t,x) &= \bar{v}(t)\bar{\psi}(\xi), \end{aligned} \quad \xi = |x|/[\tau(t)]^{1/p}, \tag{25}$$

where

$$\bar{f}(\xi) = A\left(a - \xi^\gamma\right)_+^{q_1}, \quad \bar{\psi}(\xi) = B\left(a - \xi^\gamma\right)_+^{q_2},$$

$$\gamma = \frac{p}{p-1}, \quad q_1 = \frac{(p-1)(p-(m_1+1)),}{(p-2)^2 - (m_1-1)(m_2-1)}, \quad q_2 = \frac{(p-1)(p-(m_2+1))}{(p-2)^2 - (m_1-1)(m_2-1)},$$

$a > 0$, A and B are positive constants.

We notice that functions $u_+(t,x), v_+(t,x)$ are Zeldovich-Barenblatt type solutions to system (24). Here it constructed for the system (24) at the first.

It is hold the following

Theorem 3. *Let $p > n+1$, $N > 1$, $\exp\left(\int_0^t \gamma_i(\mu)d\mu\right) < +\infty$, $i = 1, 2$, $\tau(t) < +\infty$ for $\forall t > 0$ and*

$$u_0(x) \leqslant u_+(0,x), \quad v_0(x) \leqslant v_+(0,x), \quad x \in \mathbb{R}^N.$$

Then the solution of the system (1) is specially localized.

Proof. In order to prove Theorem 3 we use the following change of variables in system (1)

$$u(t,x) = \bar{u}(t)w\left(\tau(t), \varphi(|x|)\right),$$
$$v(t,x) = \bar{v}(t)z\left(\tau(t), \varphi(|x|)\right).$$

Then the system reduces to the form

$$L_3(w,z) = -\frac{\partial w}{\partial \tau} + \varphi^{1-s}\frac{\partial}{\partial \varphi}\left(\varphi^{s-1}z^{m_1-1}\left|\frac{\partial w}{\partial \varphi}\right|^{p-2}\frac{\partial w}{\partial \varphi}\right) = 0,$$

$$L_4(w,z) = -\frac{\partial z}{\partial \tau} + \varphi^{1-s}\frac{\partial}{\partial \varphi}\left(\varphi^{s-1}w^{m_2-1}\left|\frac{\partial z}{\partial \varphi}\right|^{p-2}\frac{\partial z}{\partial \varphi}\right) = 0, \tag{26}$$

where $s = p\dfrac{N-l}{p-(n+l)}$, $p > n+l$.

Set in (26)

$$\begin{aligned} w(\tau,x) &= f(\xi), \\ z(\tau,x) &= \psi(\xi), \end{aligned} \quad \xi = \frac{\varphi|x|}{\tau^{1/p}}, \tag{27}$$

Then we have from (26) the self similar system equation

$$L_3(f,\psi) = \xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}\psi^{m_1-1}\left|\frac{df}{d\xi}\right|^{p-2}\frac{df}{d\xi}\right) + \frac{\xi}{p}\frac{df}{d\xi} = 0,$$

$$L_4(f,\psi) = \xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}f^{m_2-1}\left|\frac{d\psi}{d\xi}\right|^{p-2}\frac{d\psi}{d\xi}\right) + \frac{\xi}{p}\frac{d\psi}{d\xi} = 0. \tag{28}$$

Notice the functions $\bar{f}(\xi)$, $\bar{\psi}(\xi)$ have properties

$$0 \leqslant \bar{f}(\xi), \ \xi^{s-1}\bar{\psi}^{m_1-1}\left|\frac{d\bar{f}}{d\xi}\right|^{p-2}\frac{d\bar{f}}{d\xi} = -A^{p-1}B^{m_1-1}(\lambda\gamma_1)^{p-1}\xi^s\bar{f}(\xi) \in C(0,\infty),$$

$$0 \leqslant \bar{\psi}(\xi), \ \xi^{s-1}\bar{f}^{m_2-1}\left|\frac{d\bar{\psi}}{d\xi}\right|^{p-2}\frac{d\bar{\psi}}{d\xi} = -B^{p-1}A^{m_2-1}(\lambda\gamma_2)^{p-1}\xi^s\bar{\psi}(\xi) \in C(0,\infty).$$

To complete proof of the Theorem 3 it is sufficient to check conditions

$$L_3(\bar{f},\bar{\psi}) \leqslant 0, \ \ L_4(\bar{f},\bar{\psi}) \leqslant 0, \ \ \text{in } |\xi| < a^{(p-1)/p}$$

according to a comparison principle [11].

In fact, since

$$\xi^{N-1}\bar{\psi}^{m_1-1}\left|\frac{d\bar{f}}{d\xi}\right|^{p-2}\frac{d\bar{f}}{d\xi} = -A^{p-1}B^{m_1-1}(\lambda\gamma_1)^{p-1}\xi^N\bar{f}(\xi),$$

$$\xi^{N-1}\bar{f}^{m_2-1}\left|\frac{d\bar{\psi}}{d\xi}\right|^{p-2}\frac{d\bar{\psi}}{d\xi} = -B^{p-1}A^{m_2-1}(\lambda\gamma_2)^{p-1}\xi^N\bar{\psi}(\xi),$$

$(\gamma_1 - 1)(p - 1) + (m_1 - 1)\gamma_2 = \gamma_1, ((m_2 - 1)\gamma_1 + (p - 1)\gamma_2 = \gamma_2$, we have

$$\xi^{1-N}\frac{d}{d\xi}\left(\xi^{N-1}\bar{\psi}^{m_1-1}\left|\frac{d\bar{f}}{d\xi}\right|^{p-2}\frac{d\bar{f}}{d\xi}\right) = -A^{p-1}B^{m_1-1}(\lambda\gamma_1)^{p-1}\left(N + \xi\frac{d\bar{f}}{d\xi}\right),$$

$$\xi^{1-N}\frac{d}{d\xi}\left(\xi^{N-1}\bar{f}^{m_2-1}\left|\frac{d\bar{\psi}}{d\xi}\right|^{p-2}\frac{d\bar{\psi}}{d\xi}\right) = -B^{p-1}A^{m_2-1}(\lambda\gamma_2)^{p-1}\left(N + \xi\frac{d\bar{\psi}}{d\xi}\right).$$

Therefore

$$L_3(\bar{f},\bar{\psi}) = -\frac{N}{p}\bar{f} \leqslant 0, \ \ L_4(\bar{f},\bar{\psi}) = -\frac{N}{p}\bar{\psi} \leqslant 0, \ \ if|\xi| < a^{(p-1)/p}$$

if numbers A, B are the roots of the system of algebraic equations

$$A^{p-1}B^{m_1-1}(\lambda\gamma_1)^{p-1} = 1/p,$$

$$B^{p-1}A^{m_2-1}(\lambda\gamma_2)^{p-1} = 1/p.$$

Hence

$$L_3(f,\psi) \leqslant 0, \ \ L_4(f,\psi) \leqslant 0,$$

in $D = (t,x) : t > 0, |x| < l(t), \ l(t) = a^{(p-1)/p}[\tau(t)]^{1/(p-l-n)}$.

Applying comparison principle [11] we have for solution of the problem (24), (2)

$$u(t,x) \leqslant u_+(t,x), \ \ v(t,x) \leqslant v_+(t,x), \ \ \text{in } Q.$$

We notice that the functions $w(\tau,x)$, $z(\tau,x)$ have the properties

$$w(\tau,x) = z(\tau,x) \equiv 0, \ \ \text{when } |x| \geqslant a^{(p-1)/p}[\tau(t)]^{1/(p-l-n)}.$$

In this case for the free boundary we have an estimate

$$|x(t)| \leqslant a^{(p-1)/p}[\tau(t)]^{1/(p-l-n)} \ \ \text{for } t > 0.$$

Therefore if $\tau(t) < \infty$ for all $t > 0$, then there place a space localization of a solution of the system (1) according condition of the theorem (3). Theorem 3 is proved. $\square$

We notice that in the case

$$w(\tau,x) = z(\tau,x) \equiv 0, \ \ \text{when } |x| \geqslant a^{(p-1)/p}[\tau(t)]^{1/(p-l-n)}.$$

and $\tau(t) \to \infty$ when $t \to \infty$ there is a finite speed of perturbation.
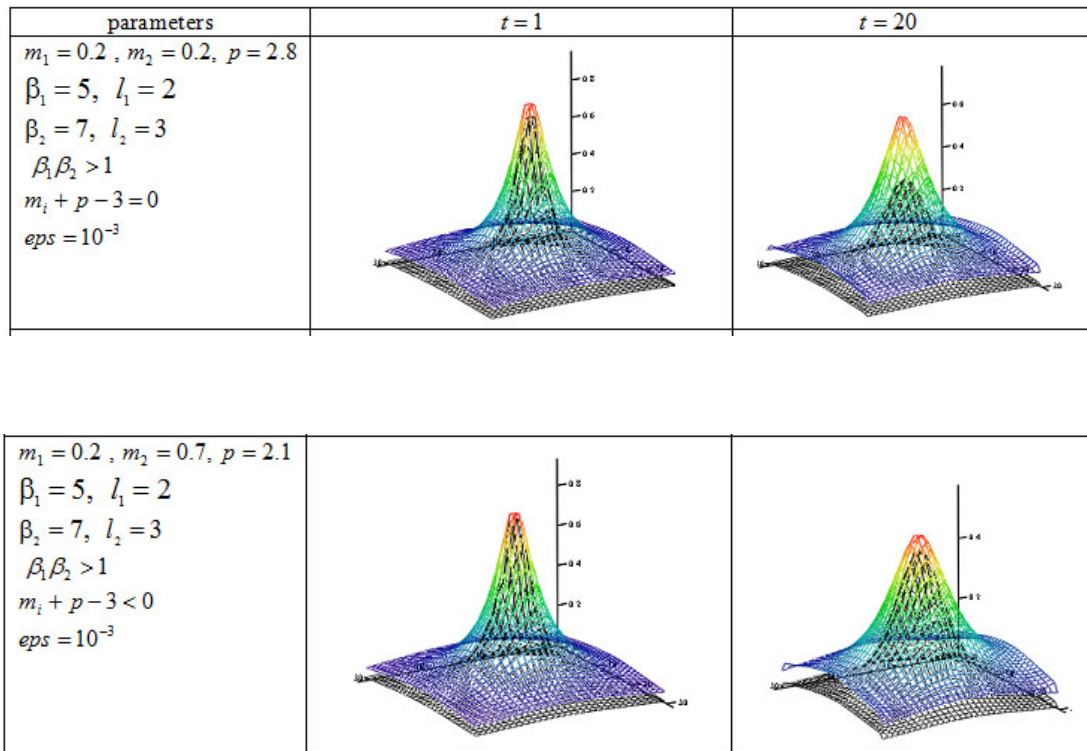
5. Results of numerical experiments and visualization of solutions.

For numerical solution of the problem the equation in two dimensions was used method Alternating Direction Implicit (ADI) in a combination to the method of balance. Iterative processes were constructed on the method, with Picard, Newton and a special method. In special linearization the members u^{β_1}, v^{β_2}, in system (1) are presented as $u_k^{\beta_1} = u_{k-1}^{\beta_1-1} u_k$, $v_k^{\beta_2} = v_{k-1}^{\beta_2-1} v_k$, where u_0, v_0, the solutions of the system of ordinary differential equation

$$\frac{du_0}{dt} = u_0^{\beta_1}, \quad \frac{dv_0}{dt} = v_0^{\beta_2}.$$

Results of computational experiments show, that all listed iterative methods are effective for solving nonlinear problems and lead to nonlinear effects if we use as initial approximation the solutions of self-similar equations constructed by the method of nonlinear splitting and by the method of standard equation [1, 5, 6]. As it was expected, for obtaining some fixed accuracy the method of Newton requires smaller number of iterations, than the methods of Picard and the special method due to a good choice of an initial approximation. We observe that in each of the considered cases Newton's method has the best convergence due to the good initial approximation.

Below numerical results for one of solutions to problem (1)–(2) in the two-dimensional cases (Fig. 1). The results of the numerical experiment gives the effect of a finite speed of perturbation of a solution, and localization of a solution depending on the values of numerical parameters. The computational experiments were carried out for a slow and a quick diffusion cases.
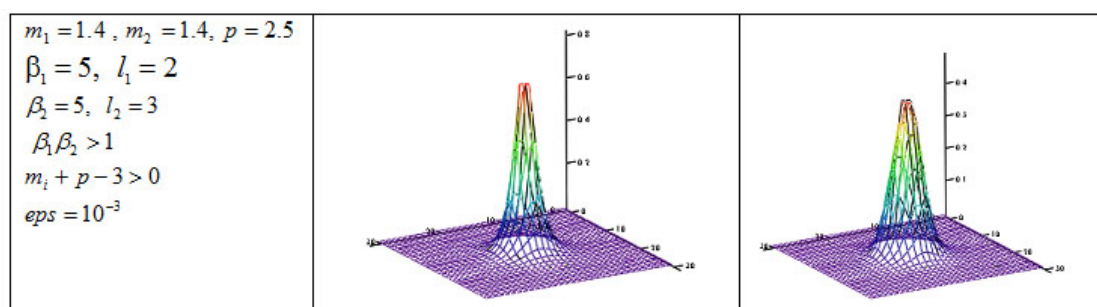
parameters	$t = 1$	$t = 20$
$m_1 = 0.2$, $m_2 = 0.2$, $p = 2.8$ $\beta_1 = 5$, $l_1 = 2$ $\beta_2 = 7$, $l_2 = 3$ $\beta_1\beta_2 > 1$ $m_i + p - 3 = 0$ $eps = 10^{-3}$		
$m_1 = 0.2$, $m_2 = 0.7$, $p = 2.1$ $\beta_1 = 5$, $l_1 = 2$ $\beta_2 = 7$, $l_2 = 3$ $\beta_1\beta_2 > 1$ $m_i + p - 3 < 0$ $eps = 10^{-3}$		

Fig. 1.

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Численное исследование решений системы реакции-диффузии с переменной плотностью

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В статье мы покажем возможности автомодельного и приближенно-автомодельного подхода к исследованию взаимной системы реакции-диффузии нелинейных уравнений. Изучаются асимптотические поведения компактно распространенных решений и свободные границы. На основе полученных качественных свойств проведены численные расчеты. При этом решения представляются в визуальной форме по времени.

Ключевые слова: двойная нелинейность, система реакции-диффузии, автомодельное уравнение, асимптотика решений, численное решение.