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On Limit Distribution of Sums of Random Variables

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Centered Rademacher sequences and centered sequences of lattice random variables with a non-trivial weak limit of the sums $\frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i$ are considered in the article. A general form of limit distribution is found for these sequences. It is shown that the form of limit distribution depends only on the average mixed moments of the first order characterizing random variables of the sequence. In the case of lattice random variables we mean a sequence of Rademacher random variables in which we can distribute the elements of the given sequence.

Keywords: sequences of random variables, sum of random variables, sum of dependent random variables, limit distribution.

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Introduction

Features of limit distribution of sums of random variables are some of the most actively discussed problems [1]. A rather detailed study of the features of limit distribution of sums of random variables is presented in [2]. It provides a general view on the limit distribution for Rademacher and lattice random variables. Characteristic studies in this field are given in [3–6].

We study the sequences of random variables $\xi = (\xi_t)_{t \in I}$, defined on probability space $(\Omega_I, \mathfrak{A}_I, \mathbf{P}_\xi)$, where I is some set of indexes. It is assumed that random variables are defined on similar spaces of elementary events $\Omega_t = \Omega, t \in I$, and $\Omega_I = \Omega_1 \times \Omega_2 \times \dots \times \Omega_t \times \dots$ with similar algebras of events $\mathfrak{A}_t = \mathfrak{A}$ and $\mathfrak{A}_I = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \dots \times \mathfrak{A}_t \times \dots$. The values of random variables lie in the value space $\xi_t(\omega) \in \mathfrak{X}_t, \omega \in \Omega_t$. Two kinds of sets are used as value spaces of random variables. In the case of Rademacher random variables $\mathfrak{X}(\theta) = \{-\theta; \theta\}$, that is a set which consists of two numbers θ and $-\theta$. We assume that $\mathbf{P}(\xi_t = \theta) = p_t$, and $\mathbf{P}(\xi_t = -\theta) = 1 - p_t = q_t$. In the case of lattice random variables, it is a set consisting of $s + 1$ numbers $\mathfrak{X}(\theta, s) = \{\theta(2k - s); k = 0, 1, \dots, s\}$, where θ is a step of lattice distribution and $\mathbf{P}(\xi_t = \theta(2k - s)) = p_t(k); k = 0, 1, \dots, s; \sum_{k=0}^s p_t(k) = 1$. There are no additional limits for the values of combined probability of sequence elements ξ . The main result presented in the paper is Theorem 3.

1. Preliminary results

Let us consider sequences of random variables $\xi = (\xi_t)_{t \in N}, N = \{1, 2, \dots\}, |\mathbf{E}\xi_t| < \infty$, where

$$\frac{1}{n} \sum_{t=1}^n \mathbf{E}\xi_t \xrightarrow{n \rightarrow \infty} 0$$

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takes place. The finite sequences of random variables are denoted by

$$\xi_I = (\xi_t)_{t \in I}, \quad I = \{t_1, \dots, t_m\}.$$

In the case when $I = I_m = \{1, 2, \dots, m\}$ we write either $(\xi_t)_{t \in I_m}$ or $\xi_{(m)}$.

Let us consider the subsequences of the finite sequence $\xi_{(n)}$. Let $I = \{t_1, \dots, t_m\} \subset I_n$ and $|I|$ is the cardinality of the set I . The initial mixed moment m of random variables $\xi_{t_1}, \dots, \xi_{t_m}$ of order $|I| = m$ is denoted by v_I :

$$v_I = m_{1, \dots, 1}(\xi_{t_1}, \dots, \xi_{t_m}) = \mathbf{E}(\xi_{t_1})^1 \dots (\xi_{t_m})^1.$$

Let us introduce the total mixed moment of order m as

$$v_m = \sum_{|I|=m} v_I, \quad \forall m = 1, \dots, n, \quad \text{when } m = 0, v_0 = 1.$$

Two kinds of average moments of order m are used:

$$\dot{v}_m = \frac{v_m}{C_n^m}, \quad \forall m = 1, \dots, n;$$

and

$$\ddot{v}_m = \frac{v_m}{\sqrt{C_n^m}}, \quad \forall m = 1, \dots, n.$$

Here C_n^m is a binominal coefficient. It is the number of combinations from n by m . When total mixed moment v_m or some average mixed moments \dot{v}_m, \ddot{v}_m are defined for the sequence ξ we write $v_m(\xi)$ or $\dot{v}_m(\xi), \ddot{v}_m(\xi)$, respectively.

Let us introduce a random variable

$$S_\alpha(\xi_{(n)}) = \frac{1}{n^\alpha} \sum_{t=1}^n \xi_t.$$

It can be also represented as

$$S_0(\pi_{(n)}) = \sum_{t=1}^n \pi_{t,n},$$

where $\pi_{t,n} = \theta_n \xi_t$, and $\theta_n = \frac{1}{n^\alpha}$. It is shown in theorem 2.2 [7] that in the case when the sequence of Rademacher random variables with the values $\pi_t(\omega) \in \{-\theta, \theta\} = \mathfrak{X}(\theta)$, $\omega \in \Omega_t$, $t \in I_n$ is defined then the following relations are satisfied

$$\mathbf{P}(S_0(\pi_{(n)}) = \theta(2k - n)) = \mathbf{P}_{\pi_{(n)}}(k) = \frac{C_n^k}{2^n} \sum_{m=0}^n \theta^{-m} \dot{v}_m(\pi_{(n)}) \cdot B_n(m, k) \quad (1)$$

and

$$v_m(\pi_{(n)}) = \theta^m \sum_{k=0}^n \mathbf{P}_{\pi_{(n)}}(k) \cdot B_n(m, k) \quad \forall m \geq 1, \quad (2)$$

where

$$B_n(m, k) = (-1)^m \sum_{i=0}^m (-1)^i C_k^i \cdot C_{n-k}^{m-i}.$$

Let us note that there is a correlation between the total mixed moment $v_m(\xi_{(n)})$ and the total mixed moment $v_m(\pi_{(n)})$:

$$v_m(\pi_{(n)}) = \sum_{|I|=m} v_I(\pi_{(n)}) = \theta^m \sum_{|I|=m} v_I(\xi_{(n)}) = \theta^m v_m(\xi_{(n)}) \quad \forall m \geq 1,$$

and when $m = 0$ we assume $v_0(\pi(n)) = v_0(\xi(n)) = 1$. When $m \geq 1$ we have similar relations for

$$\dot{v}_m(\pi(n)) = \frac{v_m(\pi(n))}{C_n^m} = \theta^m \dot{v}_m(\xi(n)) \text{ and } \ddot{v}_m(\pi(n)) = \frac{v_m(\pi(n))}{\sqrt{C_n^m}} = \theta^m \ddot{v}_m(\xi(n)). \quad (3)$$

Let η be some absolutely continuous random variable with probability density function μ_η . We denote by $L_m(\eta)$ the value of integral

$$L_m(\eta) = \frac{1}{\sqrt{m!}} \int_{-\infty}^{\infty} H_m(x) \mu_\eta(x) dx = \int_{-\infty}^{\infty} h_m(x) \mu_\eta(x) dx; \quad m \in N.$$

We assume that it exists and has a finite value. Here H_m is the orthogonal and h_m is the orthonormal Hermite polynomial of degree m .

We also introduce

$$\mathbf{m}_k(\eta) = \int_{-\infty}^{\infty} x^k \mu_\eta(x) dx.$$

It is the initial moment of degree k of the random variable η . At the same time we use the following formalism: $h_l(\mathbf{m}_\eta)$ means that all terms x^k in the polynomial $h_l(x)$ are replaced by $\mathbf{m}_k(\eta)$, $k = 0, 1, \dots, m$.

Let us consider the sequence of Rademacher random variables $(\xi_t)_{t \in N}$, $\xi_t(\omega) \in \mathfrak{X}_t(\theta = 1)$, $\omega \in \Omega_t$, $\mathbf{E}\xi_t = 0$. Sums $\frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i$ converge weakly to an absolutely continuous random variable η with probability density function μ_η . Then

Theorem 1. *For any m so that $L_m(\eta)$ exists and has a finite value the following relations are true:*

$$\lim_{n \rightarrow \infty} \frac{v_m(\xi(n))}{\sqrt{n^m}} = \frac{1}{\sqrt{m!}} \cdot L_m(\eta). \quad (4)$$

$$\lim_{n \rightarrow \infty} \dot{v}_m(\xi(n)) \sqrt{n^m} = \sqrt{m!} \cdot L_m(\eta). \quad (5)$$

$$\lim_{n \rightarrow \infty} \ddot{v}_m(\xi(n)) = L_m(\eta). \quad (6)$$

$$\lim_{n \rightarrow \infty} \dot{v}_m(\xi(n)) = 0. \quad (7)$$

Proof. To prove (4) we use relation (2) and the results given in [8]. It follows that $B_n(m, k)$ with fixed n are the Kravchuk orthogonal polynomials [9] ($p = q = \frac{1}{2}$) with respect to integer variable k . They are asymptotically related to the Hermite polynomials. For any converging sequence on the extended domain of real numbers $x_n = \frac{2k_n - n}{\sqrt{n}} \rightarrow x$ and a corresponding sequence (k_n) , $k_n \in \{0, 1, 2, \dots, n\}$ the following statement is true uniformly in relation to x :

$$\lim_{n \rightarrow \infty} \frac{B_n(m, k_n)}{\sqrt{n^m}} = \frac{1}{m!} H_m(x) = \frac{1}{\sqrt{m!}} h_m(x).$$

For a big value n we have

$$\mathbf{P}_{\pi(n)}(k) = \mathbf{P}_{\pi(n)}(x_n) \approx \mu_\eta(x_n) \Delta x_n, \text{ where } x_n = \frac{2k - n}{\sqrt{n}}.$$

Then we obtain

$$\begin{aligned}
v_m(\pi(n)) &= \sum_{k=0}^n \mathbf{P}_{\pi(n)}(k) \cdot \frac{B_n(m, k)}{\sqrt{n^m}} \Big|_{x_n = \frac{2k-n}{\sqrt{n}}}^{\text{exchange}} \\
&= \sum_{x_n = -\sqrt{n}}^{\sqrt{n}} \left(\frac{h_m(x_n)}{\sqrt{m!}} + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \right) \cdot \mathbf{P}_{\pi(n)}(x_n) \\
&= \frac{1}{\sqrt{m!}} \left(\sum_{x_n = -\sqrt{n}}^{\sqrt{n}} h_m(x_n) \mu_\eta(x_n) \Delta \right).
\end{aligned}$$

Thus

$$x_n + \mathcal{O}_1\left(\frac{1}{\sqrt{n}}\right) \rightarrow \frac{1}{\sqrt{m!}} L_m(\eta).$$

It proves relation (4).

To prove (5) we use the relation

$$v_m(\pi(n)) = \theta^m v_m(\xi(n)) = \frac{v_m(\xi(n))}{\sqrt{n^m}} \rightarrow \frac{1}{\sqrt{m!}} L_m(\eta). \quad (8)$$

It is true for any fixed m . Taking into account that $\lim_{n \rightarrow \infty} \frac{n^m}{C_n^m m!} = 1$, for a big value n we have $v_m(\xi(n)) = C_n^m \dot{v}_m(\xi(n))$ and

$$v_m(\xi(n)) \sim \frac{n^m}{m!} \dot{v}_m(\xi(n)).$$

The proof of (6) is the same and (7) follows from the previous relations. \square

Let us assume that the average mixed moments of the sequence ξ exist. Then their limits are

$$\sqrt{n^m} \dot{v}_m(\xi(n)) \sim \sqrt{n^m} \frac{m!}{n^m} \cdot v_m(\xi(n)) = m! \frac{v_m(\xi(n))}{\sqrt{n^m}} \rightarrow \sqrt{m!} L_m(\eta).$$

There is a relation between limited values of the average mixed moments of this sequence and values of moments of the limited random variable η , assuming that it exists and is absolutely continuous.

Theorem 2. *The first r moments $\ddot{v}_k(\xi) = \lim_{n \rightarrow \infty} \ddot{v}_k(\xi(n))$, $k = 1, 2, \dots$, of a random variable η are limited then and only then when the first r average mixed moments $\mathbf{m}_k(\eta)$, $k = 1, 2, \dots, r$ of the sequence ξ are limited and $\ddot{v}_m(\xi) = h_m(\mathbf{m}_\eta)$.*

Proof. The statement follows from the following relation:

$$\ddot{v}_m(\xi) = \int_{-\infty}^{\infty} h_m(x) \mu_\eta(x) dx = \int_{-\infty}^{\infty} \sum_{l=0}^m a_l x^l \mu_\eta(x) dx = \sum_{l=0}^m a_l \int_{-\infty}^{\infty} x^l \mu_\eta(x) dx = h_m(\mathbf{m}_\eta).$$

\square

2. Rademacher random variables

The set Ξ_1 of sequences of random variables $\xi = (\xi_t)_{t \in \mathbb{N}}$, where $\xi_t(\omega) \in \mathfrak{X}(\theta = 1) = \mathfrak{X}(1)$, $\omega \in \Omega_t$ with $\mathbf{P}(\xi_t = \theta) = p_t$ and $\mathbf{P}(\xi_t = -\theta) = 1 - p_t = q_t$ is so defined that for any sequence ξ from this set the following conditions are fulfilled:

1. $\frac{1}{n} \sum_{t=1}^n \mathbf{E} \xi_t \xrightarrow{n \rightarrow \infty} 0$,
2. there is a weak limit $S_{1/2}(\xi)$ having a non-degenerate distribution

$$S_{1/2}(\xi_{(n)}) = \frac{1}{\sqrt{n}} \sum_{t=1}^n \xi_t \xrightarrow{n \rightarrow \infty} S_{1/2}(\xi).$$

Let us also introduce $\hat{\Xi}_1 \subset \Xi_1$. It is a set of sequences of random variables with average links (shortly *sal*) $\hat{\xi} = (\hat{\xi}_t); \hat{\xi}_t(\omega) \in \mathfrak{X}(1)$, $\omega \in \Omega_t$, $\mathbf{E} \hat{\xi}_t = 0$. They are built according to the Theorem 3.1 [7] for sequences $\xi \in \Xi_1$ with $\alpha = 1/2$.

Sequences $\hat{\xi}$ have the following properties:

- i. Distribution functions of random variables $S_{1/2}(\xi)$ and $S_{1/2}(\hat{\xi})$ coincide.
- ii. $\dot{v}_m(\xi) = \dot{v}_m(\hat{\xi})$ and $\ddot{v}_m(\xi) = \ddot{v}_m(\hat{\xi})$, $\forall m \in N$.

Theorem 3. *Let a sequence $\xi \in \Xi_1$ be defined. Then $\eta = S_{1/2}(\xi)$ is an absolutely continuous random variable with the density distribution function*

$$\mu_\eta(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum_{m=0}^{\infty} \ddot{v}_m(\xi) \cdot h_m(x), \quad \forall x \in \mathbf{R}. \quad (9)$$

Proof. Let us consider sequence $\xi \in \Xi_1$ and sequence $\hat{\xi} \in \hat{\Xi}_1$ associated with it. Let us consider random variables

$$S_{1/2}(\hat{\xi}_{(n)}) = \frac{1}{\sqrt{n}} \sum_{t=1}^n \hat{\xi}_t = \sum_{t=1}^n \hat{\pi}_t = S(\hat{\pi}_{(n)}),$$

where $\hat{\xi}_{(n)}$ is the finite subsequence of sequence $\hat{\xi}$, $\hat{\pi}_t = \theta_n \hat{\xi}_t$, $\theta_n = \frac{1}{\sqrt{n}}$.

Random variable $S_{1/2}(\hat{\xi}_{(n)})$ can take the values $x_n = \frac{2k-n}{\sqrt{n}}$; $k = 0, 1, \dots, n$ and, in addition,

$$\mathbf{E} S_{1/2}(\hat{\xi}_{(n)}) = 0.$$

The probability that this random variable falls in the interval $(-\infty, b)$ is

$$\mathbf{P}(S_{1/2}(\hat{\xi}_{(n)}) < b) = \sum_{x_n < b} \mathbf{P}(S_{1/2}(\hat{\xi}_{(n)}) = x_n) = \sum_{x_n < b} \mathbf{P}(S(\hat{\pi}_{(n)}) = x_n),$$

Here b is some real number. Using relations (1) and (3) we obtain

$$\mathbf{P}(S_{1/2}(\hat{\xi}_{(n)}) = x_n) = \mathbf{P}(S(\hat{\pi}_{(n)}) = x_n) = \frac{C_n^k}{2^n} \sum_{m=0}^n \theta^{-m} \dot{v}_m(\hat{\pi}_{(n)}) \cdot B_n(m, k)$$

and

$$\mathbf{P}(S_{1/2}(\hat{\xi}_{(n)}) = x_n) = \frac{C_n^k}{2^n} \sum_{m=0}^n \dot{v}_m(\hat{\xi}_{(n)}) \cdot B_n(m, k).$$

Then we have

$$\mathbf{P}(S_{1/2}(\hat{\xi}_{(n)}) < b) = \sum_{\frac{2k-n}{\sqrt{n}} < b} \frac{C_n^k}{2^n} \sum_{m=0}^n \dot{v}_m(\hat{\xi}_{(n)}) B_{n,m}(k) = \sum_{\frac{2k-n}{\sqrt{n}} < b} \frac{C_n^k}{2^n} \sum_{m=0}^n \ddot{v}_m(\hat{\xi}_{(n)}) \varphi_{n,m}(k), \quad (10)$$

where $\varphi_{n,m}(k)$ are the Kravchuk orthogonal polynomials (see [8,9]). They satisfy the following relations:

$$\lim_{n \rightarrow \infty} \varphi_{n,m}(k) = \lim_{n \rightarrow \infty} \frac{B_n(m, k)}{\sqrt{C_n^m}} = \lim_{n \rightarrow \infty} \varphi_{n,m}(x_n) = \frac{1}{\sqrt{m!}} H_m(x) = h_m(x),$$

where $x_n \rightarrow x$, $x \in \mathbf{R}$. Because $\hat{\xi}$ is strictly stationary sequence with $\mathbf{E}\hat{\xi}_t = 0$ the strong law of large numbers is applicable to it (see, for example, [10, p. 438]). Then for any $\epsilon > 0$ we have

$$\mathbf{P}\left(\sup_{m \geq n} |S_1(\hat{\xi}_{(m)})| \geq \epsilon\right) = \mathbf{P}\left(\sup_{m \geq n} \left|\frac{1}{m} \sum_{t=1}^m \hat{\xi}_t\right| \geq \epsilon\right) \xrightarrow{n \rightarrow \infty} 0.$$

Moreover

$$\mathbf{P}\left(\sup_{m \geq n} \left|\frac{1}{2} S_1(\hat{\xi}_{(m)})\right| \geq \epsilon\right) = \mathbf{P}\left(\sup_{m \geq n} \left|\frac{k}{m} - \frac{1}{2}\right| \geq \epsilon\right) \xrightarrow{n \rightarrow \infty} 0, \text{ where from } \frac{k}{n} \xrightarrow[n \rightarrow \infty]{\text{almost sure}} \frac{1}{2} = p,$$

where random variable $\frac{k}{n}$ is the frequency of positive elements in the sequence $\xi_{(n)}$.

Let us consider the expression

$$\frac{C_n^k}{2^n} = C_n^k p^k q^{n-k}, \text{ where } p = q = \frac{1}{2}.$$

Asymptotically equivalent conversions of this expression as well as the sums of the form

$$\sum_k C_n^k p^k q^{n-k}, \text{ where } p = q = \frac{1}{2}$$

and $\frac{k}{n} \xrightarrow[n \rightarrow \infty]{\mathbf{P}} p$ are considered in the proof of de Moivre-Laplace limit theorem (see, for example, [10,11]). These conversions are also applicable in our case.

As a result, putting $\Delta x_n = \frac{2}{\sqrt{n}}$ and substituting k for x_n in (10), we have

$$\mathbf{P}(\eta < b) \sim \lim_{n \rightarrow \infty} \sum_{x_n < b} \frac{\Delta x_n}{\sqrt{2\pi}} e^{-\frac{x_n^2}{2}} \sum_{m=0}^n \ddot{v}_m(\hat{\xi}_{(n)}) \cdot \varphi_{n,m}(x_n).$$

Let us note that

$$\lim_{n \rightarrow \infty} \ddot{v}_m(\hat{\xi}_{(n)}) = \ddot{v}_m(\hat{\xi}), \quad m = 1, 2, \dots$$

The expression for $\mathbf{P}(\eta < b)$ can be rewritten in the following way:

$$\mathbf{P}(\eta < b) \sim \lim_{n \rightarrow \infty} \sum_{x_n < b} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x_n^2}{2}} \sum_{m=0}^n \ddot{v}_m(\hat{\xi}_{(n)}) \cdot \varphi_{n,m}(x_n) \right) \Delta x_n.$$

Due to the existence of the function \mathbf{F}_η and and taking into account integrability of sums

$$\lim_{n \rightarrow \infty} \sum_{-\sqrt{n} \leq x_n \leq x} \frac{\Delta x_n}{\sqrt{2\pi}} e^{-\frac{x_n^2}{2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy, \quad \forall x \in \mathbf{R},$$

we obtain that the sum

$$\lim_{n \rightarrow \infty} \sum_{m=0}^n \ddot{v}_m(\hat{\xi}_{(n)}) \varphi_{n,m}(x_n)$$

necessarily exists and it is finite. Then, using the notation $x = \lim_{n \rightarrow \infty} x_n$ and taking into account that $\lim_{n \rightarrow \infty} \varphi_{n,m}(x_n) = h_m(x)$, we get that the following inequalities

$$0 \leq \lim_{n \rightarrow \infty} \sum_{m=0}^n \ddot{v}_m(\hat{\xi}_{(n)}) \varphi_{n,m}(x_n) = \sum_{m=0}^{\infty} \ddot{v}_m(\hat{\xi}) h_m(x) < \infty$$

are true for any real x . It also follows that $\ddot{v}_m(\hat{\xi})$, $m = 1, 2, \dots$ exist and they are finite. As a result, we have

$$\mathbf{F}_\eta(x) = \int_{-\infty}^x \mu_\eta(y) dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} \sum_{m=0}^{\infty} \ddot{v}_m(\xi) \cdot h_m(y) dy$$

for any $x \in \mathbf{R}$. □

Corollary 1. *Let us assume that sequence $\xi \in \Xi_1$ is given. Then the random variable $\eta = S_{1/2}(\xi)$ has all moments and they are finite if its distribution density μ_η is a continuous function on the whole real axis.*

Proof. It follows from the continuity of μ_η that the series

$$\mu_\eta(x) = \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{\infty} \ddot{v}_m(\xi) h_m(x) e^{-\frac{x^2}{2}}$$

converges uniformly on the whole real axis. It allows us to integrate this expression term by term and obtain

$$L_k(\eta) = \int_{-\infty}^{\infty} h_k(x) \mu_\eta(x) dx = \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{\infty} \ddot{v}_m(\xi) \int_{-\infty}^{\infty} h_k(x) h_m(x) e^{-\frac{x^2}{2}} dx = \ddot{v}_k(\xi).$$

Taking into account theorem 2, this expression proves the statement. □

Corollary 2. *Let us assume that sequence $\xi \in \Xi_1$ is given. Then the random variable $\eta = S_{1/2}(\xi)$ has the standard normal distribution then and only then when*

$$\lim_{n \rightarrow \infty} \ddot{v}_m(\xi_{(n)}) = 0.$$

Proof. It follows from theorem 3 that the density of the random variable η for an arbitrary value of the argument $x \in \mathbf{R}$ is expressed by (9):

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum_{m=0}^{\infty} \ddot{v}_m(\xi) \cdot h_m(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Leftrightarrow \ddot{v}_m(\xi) = \ddot{v}_m(\hat{\xi}) = 0 \text{ with } m \geq 1.$$

This proves the statement. □

In conclusion let us consider a set Ξ_0 of random variables sequences $\xi = (\xi_t)_{t \in N}$, where $\xi_t(\omega) \in \mathfrak{X}_B = \{0; 1\}$, $\omega \in \Omega_t$. They are defined in the following way: for any sequence ξ from this set the following conditions are true:

- 1) $\mathbf{E}\xi_t = \frac{1}{2}, \forall t \in N$; i.e $p = \mathbf{P}(\xi_t = 1) = \frac{1}{2}$, $q = 1 - p = \mathbf{P}(\xi_t = 0) = \frac{1}{2}$;
- 2) there is a weak limit $S_{1/2}(\tilde{\xi})$ of the sequence $S_{1/2}(\tilde{\xi}_{(n)})$ that has a non-degenerate distribution

$$S_{1/2}(\tilde{\xi}_{(n)}) = \frac{1}{\sqrt{n}} \sum_{t=1}^n \tilde{\xi}_t \xrightarrow{n \rightarrow \infty} S_{1/2}(\tilde{\xi}), \text{ here } \tilde{\xi}_t = \frac{\xi_t - \mathbf{E}\xi_t}{\sqrt{pq}}.$$

For any sequence $\xi \in \Xi_0$ we construct a sequence $\gamma = (\gamma_t)_{t \in N}$, $\gamma_t = 2\xi_t - 1$; $\forall t \in N$. It follows from the features of the sequence $\xi \in \Xi_0$ that sequence γ satisfies the following conditions:

$$1) \mathbf{E}\gamma_t = 0, \forall t \in N, \text{ moreover } \frac{1}{n} \sum_{t=1}^n \mathbf{E}\gamma_t \xrightarrow{n \rightarrow \infty} 0,$$

2) there is a weak limit $S_{1/2}(\gamma)$ of the sequence $S_{1/2}(\gamma_{(n)})$ that has a non-degenerate distribution

$$S_{1/2}(\gamma_{(n)}) = \frac{1}{\sqrt{n}} \sum_{t=1}^n \gamma_t \xrightarrow{n \rightarrow \infty} S_{1/2}(\gamma)$$

and

$$\mathbf{F}_{S_{1/2}(\tilde{\xi})}(x) = \mathbf{F}_{S_{1/2}(\gamma)}(x), \forall x \in \mathbf{R}. \quad (11)$$

Indeed, because $p = q = \frac{1}{2}$ we have $\forall x \in \mathbf{R}$ and

$$\mathbf{F}_{S_{1/2}(\tilde{\xi})_{(n)}}(x) = \mathbf{P}\left(\frac{1}{\sqrt{n}} \sum_{t=1}^n \frac{\xi_t - \mathbf{E}\xi_t}{\sqrt{pq}} < x\right) = \mathbf{P}\left(\frac{1}{\sqrt{n}} \sum_{t=1}^n (2\xi_t - 1) < x\right) = \mathbf{P}\left(\frac{1}{\sqrt{n}} \sum_{t=1}^n \gamma_t < x\right).$$

This proves relation (11).

Then sequence γ is an element of the set Ξ_1 and it satisfies the conditions of **theorem 3**. This means that

$$\mu_{S_{1/2}(\gamma)} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum_{m=0}^{\infty} \ddot{v}_m(\gamma) \cdot h_m(x), \forall x \in \mathbf{R}.$$

The relationship between $\ddot{v}_m(\gamma)$ and $c\ddot{v}_m(\xi)$ is

$$v_I(\gamma) = \mathbf{E}\left(2 * \left(\xi_{t_1} - \frac{1}{2}\right)\right) \left(2 * \left(\xi_{t_2} - \frac{1}{2}\right)\right) \dots \left(2 * \left(\xi_{t_m} - \frac{1}{2}\right)\right) = 2^m c v_I(\xi)$$

and it means that $\ddot{v}_m(\gamma) = 2^m c\ddot{v}_m(\xi)$. As a result, we obtain the following statement.

Corollary 3. *Let us assume that sequence $\xi \in \Xi_0$ is given. Then $\eta = S_{1/2}(\tilde{\xi})$ is an absolutely continuous random variable with the distribution density function given by*

$$\mu_\eta(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum_{m=0}^{\infty} 2^m c\ddot{v}_m(\xi) \cdot h_m(x), \forall x \in \mathbf{R}. \quad (12)$$

□

3. Lattice random variables

Let us consider a sequence of lattice random variables $\xi = (\xi_t)_{t \in N}$, $\xi_t : \Omega_t \rightarrow \{-\theta s, \dots, \theta(2k - s), \dots, \theta s\} = \mathfrak{X}(\theta, s)$; $k = 0, 1, \dots, s$. We assume that $\mathbf{P}(\xi_t = \theta(2k - s)) = p_t(k)$; $\sum_{k=0}^s p_t(k) = 1$.

As before Ξ_2 is a set of sequences of random variables $\xi = (\xi_t)$, $\xi_t : \Omega_t \rightarrow \mathfrak{X}(\theta, s)$, $t = 1, 2, \dots$. Arbitrary sequence ξ from this set satisfies the following conditions:

$$1) \frac{1}{n} \sum_{t=1}^n \mathbf{E}\xi_t \xrightarrow{n \rightarrow \infty} 0,$$

2) there exists a nondegenerate random variable $\eta = S_{1/2}(\xi)$ such that

$$S_{1/2}(\xi_{(n)}) = \frac{1}{\sqrt{n}} \sum_{t=1}^n \xi_t \xrightarrow[n \rightarrow \infty]{\Rightarrow} \eta.$$

Let us denote the set $\hat{\xi} = (\hat{\xi}_t)$; $\hat{\xi}_t : \Omega_t \rightarrow \mathfrak{X}(\theta, s)$, $\mathbf{E}\hat{\xi}_t = 0$ by $\hat{\Xi}_2 \subset \Xi_2$. The existence and construction of such sequences is described in theorem 3.2[†] [7]. The sequences have the following property:

$$\mathbf{F}_{S_{1/2}(\hat{\xi})}(x) = \mathbf{F}_{S_{1/2}(\xi)}(x), \forall x \in \mathbf{R}. \quad (13)$$

Let us consider limit distribution properties of a random variable $\eta = S_{1/2}(\xi)$, where $\xi \in \Xi_2$. We use the fact that every lattice random variable $\xi_t : \Omega_t \rightarrow \mathfrak{X}(\theta, s)$ can be represented as a sum of s Rademacher random variables $\gamma_\tau : \Omega'_\tau \rightarrow \mathfrak{X}(1)$:

$$\xi_t = \theta \sum_{\tau=(t-1) \cdot s+1}^{t \cdot s} \gamma_\tau. \quad (14)$$

According to Theorem 3.2¹ [7] we can construct sequence $\hat{\xi} = (\hat{\xi}_t)_{t \in N}$, $\hat{\xi}_t : \Omega_t \rightarrow \mathfrak{X}(\theta, s)$ and $\mathbf{E}\hat{\xi}_t = 0$ that satisfies (13). Every subsequence $\hat{\xi}_{(n)}$ of the sequence $\hat{\xi}$ can be related to a subsequence of Rademacher random variables $\hat{\gamma}_{(ns)}$ such that

$$\hat{\xi}_t = \theta \sum_{\tau=(t-1) \cdot s+1}^{t \cdot s} \hat{\gamma}_\tau \quad \forall t \in I_n.$$

Let us consider the manner in which the distribution density of random variable $\eta = S_{1/2}(\xi)$ can be expressed in terms of mixed moments of the sequence $\gamma = (\gamma_\tau)_{\tau \in N}$.

Theorem 4. *Let us assume that sequence $\xi \in \Xi_2$ is given. Then $\eta = S_{1/2}(\xi)$ is an absolutely continuous random variable with the distribution density function*

$$\mu_\eta(x) = \frac{1}{\sqrt{2\pi s \theta^2}} e^{-\frac{x^2}{2s\theta^2}} \sum_{m=0}^{\infty} \ddot{v}_m(\gamma) \cdot h_m\left(\frac{x}{\theta\sqrt{s}}\right), \forall x \in \mathbf{R},$$

where $\gamma = (\gamma_\tau)$, $\gamma_\tau : \Omega'_\tau \rightarrow \mathfrak{X}(1)$ is the sequence of Rademacher random variables that satisfies (14).

Proof. Let us consider the sequence $\gamma = (\gamma_\tau)_{\tau \in N}$, that satisfies (14). For a fixed value s the following relations are true:

$$\frac{1}{n} \sum_{t=1}^n \mathbf{E}\xi_t \xrightarrow[n \rightarrow \infty]{} 0 \Rightarrow \frac{\theta s}{ns} \sum_{\tau=1}^{ns} \mathbf{E}\gamma_\tau \xrightarrow[n \rightarrow \infty]{} 0 \Rightarrow \frac{1}{ns} \sum_{\tau=1}^{ns} \mathbf{E}\gamma_\tau \xrightarrow[n \rightarrow \infty]{} 0 \Rightarrow \frac{1}{k} \sum_{\tau=1}^k \mathbf{E}\gamma_\tau \xrightarrow[k \rightarrow \infty]{} 0.$$

Because $|\mathbf{E}\gamma_k| \leq 1 \quad \forall k \in N$ and, assuming that $n' = \max\{n | ns \leq k\}$, we obtain

$$\frac{1}{k} \left| \sum_{\tau=1}^k \mathbf{E}\gamma_\tau \right| \leq \frac{1}{n's} \left| \sum_{\tau=1}^{n's} \mathbf{E}\gamma_\tau \right| + \frac{1}{n's} \left| \sum_{\tau=n's+1}^k \mathbf{E}\gamma_\tau \right| \leq \frac{1}{n's} \left| \sum_{\tau=1}^{n's} \mathbf{E}\gamma_\tau \right| + \frac{1}{n'} \xrightarrow[k \rightarrow \infty]{} 0.$$

[†]There is a misprint in the article. The corrected theorem: Let the sequence $\xi = (\xi_t)_{t \in N}$, $\xi_t : \Omega_t \rightarrow \mathfrak{X}(\theta, s)$ be defined on $(\Omega_N, \mathfrak{A}_N, \mathbf{P}_\xi)$. For some $\alpha \in (0; 1)$ a random variable $S_\alpha(\xi)$ is defined and it is a weak limit of the sums $S_{n,\alpha}(\xi)$. Then there exists $\hat{\xi} = (\hat{\xi}_t)_{t \in N}$, $\hat{\xi}_t : \Omega_t \rightarrow \mathfrak{X}(\theta, s)$, such that ...

In addition, because

$$\frac{1}{\sqrt{n}} \sum_{t=1}^n \xi_t = \theta \sqrt{s} \frac{1}{\sqrt{ns}} \sum_{\tau=1}^{ns} \gamma_\tau, \quad (15)$$

and, taking into account that the weak limit for $\frac{1}{\sqrt{n}} \sum_{t=1}^n \xi_t$ exists, then there exists the weak limit for $\frac{1}{\sqrt{ns}} \sum_{\tau=1}^{ns} \gamma_\tau = \frac{1}{\theta \sqrt{s}} \frac{1}{\sqrt{n}} \sum_{t=1}^n \xi_t$. One can conclude that sums $\frac{1}{\sqrt{k}} \sum_{\tau=1}^k \gamma_\tau$ weakly converge to some random variable. Let us assume that η is the weak limit of sequence

$$S_{1/2}(\xi_{(n)}) = \frac{1}{\sqrt{n}} \sum_{t=1}^n \xi_t$$

and $n'' = n' + 1$. Then from the definition of weak convergence we obtain that in every continuity point $x \in \mathbf{R}$ of the distribution function of the limit random variable η the following relation is true:

$$\mathbf{F}_\eta(x) = \lim_{n \rightarrow \infty} \mathbf{P} \left(\frac{1}{\sqrt{ns}} \sum_{\tau=1}^{ns} \gamma_\tau < x \right).$$

Taking into account

$$\frac{1}{\sqrt{k}} \sum_{\tau=1}^k \gamma_\tau = \frac{1}{\sqrt{k}} \left(\sum_{\tau=1}^{n''s} \gamma_\tau - \sum_{\tau=k+1}^{n''s} \gamma_\tau \right) = \frac{n''s}{\sqrt{k}} \frac{1}{\sqrt{n''s}} \left(\sum_{\tau=1}^{n''s} \gamma_\tau - \sum_{\tau=k+1}^{n''s} \gamma_\tau \right)$$

and

$$\frac{1}{\sqrt{k}} \sum_{\tau=1}^k \gamma_\tau = \frac{1}{\sqrt{k}} \left(\sum_{\tau=1}^{n's} \gamma_\tau + \sum_{\tau=k+1}^{n's} \gamma_\tau \right) = \frac{n's}{\sqrt{k}} \frac{1}{\sqrt{n's}} \left(\sum_{\tau=1}^{n's} \gamma_\tau + \sum_{\tau=k+1}^{n's} \gamma_\tau \right),$$

we obtain that

$$\begin{aligned} \mathbf{P} \left(\frac{1}{\sqrt{n''s}} \left(\sum_{\tau=1}^{n''s} \gamma_\tau - \sum_{\tau=k+1}^{n''s} \gamma_\tau \right) < x \right) &\leq \mathbf{P} \left(\frac{1}{\sqrt{k}} \sum_{\tau=1}^k \gamma_\tau < x \right) \\ &\leq \mathbf{P} \left(\frac{1}{\sqrt{n's}} \left(\sum_{\tau=1}^{n's} \gamma_\tau + \sum_{\tau=k+1}^{n's} \gamma_\tau \right) < x \right) \end{aligned}$$

for $x \geq 0$.

It follows that

$$\mathbf{P} \left(\frac{1}{\sqrt{n''s}} \sum_{\tau=1}^{n''s} \gamma_\tau < x + \frac{1}{\sqrt{n''}} \right) \leq \mathbf{P} \left(\frac{1}{\sqrt{k}} \sum_{\tau=1}^k \gamma_\tau < x \right) \leq \mathbf{P} \left(\frac{1}{\sqrt{n's}} \sum_{\tau=1}^{n's} \gamma_\tau < x - \frac{1}{\sqrt{n'}} \right)$$

and for $x < 0$ we have

$$\mathbf{P} \left(\frac{1}{\sqrt{n''s}} \sum_{\tau=1}^{n''s} \gamma_\tau < x + \frac{1}{\sqrt{n''}} \right) \geq \mathbf{P} \left(\frac{1}{\sqrt{k}} \sum_{\tau=1}^k \gamma_\tau < x \right) \geq \mathbf{P} \left(\frac{1}{\sqrt{n's}} \sum_{\tau=1}^{n's} \gamma_\tau < x - \frac{1}{\sqrt{n'}} \right).$$

Therefore we obtain weak convergence when $k \rightarrow \infty$. Thus all the conditions of Theorem 3 are fulfilled for the sequence $\gamma = (\gamma_\tau)_{\tau \in \mathbf{N}}$ and we can use it to obtain the density distribution function of the limit probability distribution. Let η_1 be the weak limit of the sequence

$$S_{1/2}(\gamma_{(n)}) = \frac{1}{\sqrt{n}} \sum_{\tau=1}^n \gamma_\tau.$$

Then, taking into account (15), we obtain that the value of random variable η is related to the value of random variable η_1 by the equation

$$\eta = \theta\sqrt{s} \cdot \eta_1.$$

Then from Theorem 3 we obtain that $\forall x \in \mathbf{R}$

$$\mu_{\eta_1}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum_{m=0}^{\infty} \ddot{v}_m(\gamma) \cdot h_m(x).$$

Finally we obtain that

$$\mu_{\eta}(x) = \mu_{\theta\sqrt{s}\eta_1}(x) = \frac{1}{\sqrt{2\pi\theta^2 s}} e^{-\frac{x^2}{2\theta^2 s}} \sum_{m=0}^{\infty} \ddot{v}_m(\gamma) \cdot h_m\left(\frac{x}{\theta\sqrt{s}}\right).$$

□

One should note that when $\theta = \frac{1}{\sqrt{s}}$ the density distribution function of random variable η has the simplest form. In this case

$$\mu_{\eta}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum_{m=0}^{\infty} \ddot{v}_m(\gamma) \cdot h_m(x)$$

for any $x \in \mathbf{R}$.

Let us consider how to change from mixed moments of the sequence γ to mixed moments of the sequence $\hat{\xi}$. First of all, let us note that mixed moments of sequences γ and $\hat{\gamma}$ are the same (it follows from Theorem 3.1 [7]).

For convenience we identify the elements of the sequence $\hat{\gamma}$ with the help of two indexes. The first index t defines the number of an element in sequence $\hat{\xi}$. The second index defines the number of an element in the sequence $\hat{\gamma}$ in representation of the element $\hat{\xi}_t$ so that

$$\hat{\xi}_t = \theta \sum_{j=0}^{s-1} \hat{\gamma}_{t,j}.$$

Then, taking into account results obtained above, we have

$$\ddot{v}_m(\hat{\xi}_{(n)}) = \frac{v_m(\hat{\xi}_{(n)})}{\sqrt{C_n^m}} = \frac{v_{I_m}(\hat{\xi}_{(n)})}{\sqrt{C_n^m}},$$

but

$$v_{I_m}(\hat{\xi}_{(n)}) = \mathbf{E} \prod_{t \in I_m} \hat{\xi}_t = \theta^m \mathbf{E} \prod_{t \in I_m} \left(\sum_{j=1}^s \hat{\gamma}_{t,j} \right) = \theta^m s^m \mathbf{E} \hat{\gamma}_{1,0} \hat{\gamma}_{2,0} \dots \hat{\gamma}_{m,0},$$

and

$$\theta^m s^m \mathbf{E} \hat{\gamma}_{1,0} \hat{\gamma}_{2,0} \dots \hat{\gamma}_{m,0} = \theta^m s^m v_{I_m}(\hat{\gamma}_{(ns)}) = \theta^m s^m v_m(\hat{\gamma}_{(ns)}) = \theta^m s^m v_m(\gamma_{(ns)}).$$

Hence

$$\ddot{v}_m(\hat{\xi}_{(n)}) = \frac{\theta^m s^m v_m(\gamma_{(ns)})}{\sqrt{C_n^m}} = \theta^m s^m \ddot{v}_m(\gamma_{(ns)}) \sqrt{\frac{C_{ns}^m}{C_n^m}}.$$

Taking into account (8), in the limit $n \rightarrow \infty$ we obtain

$$\ddot{v}_m(\hat{\xi}) = \theta^m s^{\frac{3m}{2}} \ddot{v}_m(\gamma) \tag{16}$$

which leads to

$$\ddot{v}_m(\gamma) = \theta^{-m} s^{-\frac{3m}{2}} \ddot{v}_m(\hat{\xi}).$$

Then the density of the random variable $\eta \forall x \in \mathbf{R}$ is

$$\mu_\eta(x) = \frac{1}{\sqrt{2\pi\theta^2 s}} e^{-\frac{x^2}{2\theta^2 s}} \sum_{m=0}^{\infty} \frac{\ddot{v}_m(\hat{\xi})}{\theta^m s^{\frac{3m}{2}}} \cdot h_m\left(\frac{x}{\theta\sqrt{s}}\right).$$

Corollary 4.1. *Let us assume that sequence $\xi \in \Xi_2$ is given. Then random variable η*

$$\frac{1}{\sqrt{n}} \sum_{t=1}^n \xi_t \xrightarrow[n \rightarrow \infty]{\Rightarrow} \eta$$

has the normal distribution with parameters $\mathbf{E}\eta = 0$ and $\mathbf{D}\eta = \theta^2 s$ then and only then when

$$\ddot{v}_m(\hat{\xi}) = \lim_{n \rightarrow \infty} \ddot{v}_m(\hat{\xi}_{(n)}) = 0, \quad m \geq 1.$$

Proof. It is similar to the proof of Corollary 2. □

Let us consider two special cases: the expression for the density of the sum of sequence $\xi \in \Xi_2$ when $\theta = \frac{1}{\sqrt{s}}$ and $\theta = \frac{1}{s\sqrt{s}}$. In the first case the change scale of x is conserved but the values of mixed moments are changed. In the second case the values of the moments are conserved but the change scale of x is changed.

Corollary 4.2. *Let us assume that sequence $\xi \in \Xi_2$ is given and $\theta = \frac{1}{\sqrt{s}}$. Then the random variable η*

$$\frac{1}{\sqrt{n}} \sum_{t=1}^n \xi_t \xrightarrow[n \rightarrow \infty]{\Rightarrow} \eta$$

for any $x \in \mathbf{R}$ has the density distribution function

$$\mu_\eta(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum_{m=0}^{\infty} \frac{\ddot{v}_m(\hat{\xi})}{s^m} \cdot h_m(x).$$

Corollary 4.3. *Let us assume that sequence $\xi \in \Xi_2$ is given and $\theta = \frac{1}{s\sqrt{s}}$. Then the random variable η*

$$\frac{1}{\sqrt{n}} \sum_{t=1}^n \xi_t \xrightarrow[n \rightarrow \infty]{\Rightarrow} \eta$$

for any $x \in \mathbf{R}$ has the density distribution function

$$\mu_\eta(x) = \frac{s}{\sqrt{2\pi}} e^{-\frac{(sx)^2}{2}} \sum_{m=0}^{\infty} \ddot{v}_m(\hat{\xi}) \cdot h_m(sx).$$

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О предельном распределении сумм случайных величин

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Рассмотрены центрированные последовательности радемахеровских и решетчатых случайных величин, имеющие нетривиальный слабый предел сумм $\frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i$. Для них найден общий вид предельного распределения. Показано, что вид предельного распределения зависит лишь от усредненных смешанных моментов первого порядка, характеризующих случайные величины последовательности, причем в случае решетчатых случайных величин имеется в виду последовательность радемахеровских случайных величин, в которую можно разложить элементы рассматриваемой последовательности.

Ключевые слова: последовательности случайных величин, сумма случайных величин, сумма зависимых случайных величин, предельное распределение сумм случайных величин.