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Unsteady 2D Motions a Viscous Fluid Described by Partially Invariant Solutions to the Navier–Stokes Equations

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3D continuous subalgebra is used to searching partially invariant solution of viscous incompressible fluid equations. It can be interpreted as a 2D motion of one or two immiscible fluids in plane channel. The arising initial boundary value problem for factor-system is an inverse one. Unsteady problem for creeping motions is solved by separating of variables method for one fluid or Laplace transformation method for two fluids.

Keywords: partially invariant solution, viscous fluid, free boundary problem, interface.

Introduction

The Navier–Stokes equations for 2D motions of a viscous fluid are recorded by

$$\begin{aligned} &u_t + uu_x + vu_y + \frac{1}{\rho}\,p_x = \nu(u_{xx} + u_{yy}),\\ &v_t + uv_x + vv_y + \frac{1}{\rho}\,p_y = \nu(v_{xx} + v_{yy}) - g,\\ &u_x + v_y = 0, \end{aligned} \tag{0.1}$$

where ρ is the constant fluid density, u and v are the velocity components in the x and y directions, respectively, p is the pressure and g is the gravity acceleration, ν is the fluid viscosity. The group of point transformations admitted by the system (0.1) is computed in [1, 2]. Corresponding this group basic continuous Lie algebra includes the three parametrical subalgebra $\langle \partial_x, \partial_u + t\partial_x, \partial_p \rangle$. It has the invariants t, y, v and partly invariant solution of (0.1) rang two and defect two necessary to seek in the form $u = u(x, y, t)$, $v = v(y, t)$, $p = p(x, y, t)$. From continuity equation $u_x + v_y = 0$ we obtain the relations

$$u(x, y, t) = w(y, t)x + u_1(y, t), \quad w(y, t) + v_y(y, t) = 0.$$

Navier–Stokes equations (0.1) are equivalent to the system

$$\begin{aligned} &w_t + vw_y + w^2 = f(t) + \nu w_{yy}, \quad \frac{1}{\rho}\,p = l(y, t) - f(t)\frac{x^2}{2} - gy,\\ &v = -\int_0^y w(z, t)\,dz, \quad l_y = \nu v_{yy} - v_t - vv_y,\\ &u_{1t} + vu_{1y} + wu_1 = \nu u_{1yy}. \end{aligned} \tag{0.2}$$

In what follows we assume that $u_1(y, t) = 0$.

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1. Flow in layer with two rigid walls

In this section the solution (0.2) under consideration shall be interpreted as 2D motion viscous liquid fills the layer $0 < y < h$ with a rigid walls $y = 0$, $y = h = \text{const}$. Let us attach the initial and boundary conditions

$$w(y,0) = w_0(y), \quad w_0(0) = w_0(h) = 0, \quad \int_0^h w_0(z)\,dz = 0; \tag{1.1}$$

$$w(0,t) = w(h,t) = 0, \quad \int_0^h w(z,t)\,dz = 0. \tag{1.2}$$

Thus, the function $w(y,t)$ is the solution of integro-differential equation

$$w_t - w_y \int_0^y w(z,t)\,dz + w^2 = \nu w_{yy} + f(t) \tag{1.3}$$

with initial and boundary conditions (1.1), (1.2).

Here and further suppose the Reynolds number $\text{Re} = \max\limits_{y\in[0,1]} |w_0(y)|h^2/\nu \ll 1$. In such case we can neglect the nonlinear terms in equation (1.3) and the following initial boundary value problem is arised

$$w_t = \nu w_{yy} + f(t), \quad y \in (0,h), \quad t > 0; \tag{1.4}$$

$$w(y,0) = w_0(y); \tag{1.5}$$

$$w(0,t) = 0, \quad w(h,t) = 0, \quad \int_0^h w(y,t)\,dy = 0. \tag{1.6}$$

Integrating equation (1.4) we obtain function $f(t)$

$$f(t) = \frac{\nu}{h}\,(w_y(0,t) - w_y(h,t)), \quad f(0) = \frac{\nu}{h}\,(w_{0y}(0) - w_{0y}(h)). \tag{1.7}$$

Hence, we deduce the so-called loaded equation

$$w_t = \nu w_{yy} + w_y(1,t) - w_y(0,t).$$

But we determine new function $W(y,t) = w_y(y,t)$. It satisfies the problem

$$W_t = \nu W_{yy}, \quad y \in (0,1), \quad t > 0; \tag{1.8}$$

$$W(y,0) = w_{0y}; \tag{1.9}$$

$$\int_0^h yW(y,t)\,dy = 0; \tag{1.10}$$

$$\int_0^h W(y,t)\,dy = 0. \tag{1.11}$$

This problem is not classical one for the heat equation (1.4).

The problem (1.8)–(1.11) has the exact solution

$$W(y,t) = \sum_{k=1}^{\infty} a_k \exp\Big(-\frac{4\lambda_k^2 \nu t}{h^2}\Big) \sin\Big[\frac{\lambda_k(2y-h)}{h}\Big].$$

where λ_k is kth positive root of the equation

$$\operatorname{tg}\lambda_k = \lambda_k, \quad \lambda_k \to (2k+1)\pi/2, \quad k \to \infty.$$

A constants a_k are the Fourier series coefficients of known function w_{0y}, i.e.

$$w_{0y} = \sum_{k=1}^{\infty} a_k \sin\left[\frac{\lambda_k(2y-h)}{h}\right].$$

Hence, from (1.7)

$$f(t) = \frac{\nu}{h}\,[W(0,t) - W(h,t)] = -\frac{2\nu}{h}\sum_{k=1}^{\infty} a_k \exp\left(-\frac{4\lambda_k^2\nu t}{h^2}\right)\sin\lambda_k =$$

$$= -\frac{2\nu}{h}\sum_{k=1}^{\infty} a_k \frac{\lambda_k}{\sqrt{1+\lambda_k^2}}\exp\left(-\frac{4\lambda_k^2\nu t}{h^2}\right).$$

Functions $w(y,t)$ and velocity component $v(y,t)$ can be found by the formulae

$$w(y,t) = \frac{h}{2}\sum_{k=1}^{\infty}\frac{a_k}{\lambda_k}\exp\left(-\frac{4\lambda_k^2\nu t}{h^2}\right)\left\{\cos\lambda_k - \cos\left[\frac{\lambda_k(2y-h)}{h}\right]\right\},$$

$$v(y,t) = \frac{h}{2}\sum_{k=1}^{\infty}\frac{a_k}{\lambda_k}\exp\left(-\frac{4\lambda_k^2\nu t}{h^2}\right)\left\{\frac{h}{2\lambda_k}\left[\sin\left(\frac{\lambda_k(2y-h)}{h}\right) + \sin\lambda_k\right] - y\cos\lambda_k\right\}.$$

2. Flow in layer with one rigid wall and free boundary

In the same assumptions like section 1 the function $w(y,t)$ is governed by the equation (1.4). The initial data and boundary conditions are (1.5), (1.6), but it is necessary to change second condition in (1.6) on $w_y(h,t) = 0$. For the function $f(t)$ one obtains

$$f(t) = \frac{\nu}{h}\,w_y(0,t). \tag{2.1}$$

As concerning function $W(y,t) = w_y(y,t)$ it satisfies the equation (1.8) with initial data (1.9) and boundary conditions

$$W(h,t) = 0, \qquad \int_0^h (h-y)W(y,t)\,dy = 0.$$

Using separation of variables technique the problem can be solved to obtain W:

$$W(y,t) = \frac{2}{h}\sum_{k=1}^{\infty} b_k\frac{(1+\lambda_k^2)}{\lambda_k^2}\exp\left(-\frac{\lambda_k^2\nu t}{h^2}\right)\sin\left[\frac{\lambda_k(h-y)}{h}\right].$$

Therefore

$$w(y,t) = 2\sum_{k=1}^{\infty} b_k\frac{(1+\lambda_k^2)}{\lambda_k^3}\exp\left(-\frac{\lambda_k^2\nu t}{h^2}\right)\left\{\cos\left[\frac{\lambda_k(h-y)}{h}\right] - \cos\lambda_k\right\},$$

and from (2.1), (0.2) we get

$$f(t) = \frac{2\nu}{h^2}\sum_{k=1}^{\infty} b_k\frac{\sqrt{1+\lambda_k^2}}{\lambda_k}\exp\left(-\frac{\lambda_k^2\nu t}{h^2}\right),$$

$$v(y,t) = 2\sum_{k=1}^{\infty} b_k \frac{(1+\lambda_k^2)}{\lambda_k^3}\,\exp\Big(-\frac{\lambda_k^2\nu t}{h^2}\Big)\Big\{y\cos\lambda_k + \frac{h}{\lambda_k}\Big[\sin\Big(\frac{\lambda_k(y-h)}{h}\Big) + \sin\lambda_k\Big]\Big\}.$$

A constants b_k are the Fourier series coefficients of function w_{0y}, i.e.

$$w_{0y} = \frac{2}{h}\sum_{k=1}^{\infty} b_k \frac{(1+\lambda_k^2)}{\lambda_k^2}\,\sin\Big[\frac{\lambda_k(h-y)}{\lambda_k}\Big].$$

3. Layered motion of two immiscible fluids

Let us consider a system of two immiscible fluids separated by the interface $y = h_1$. The parameters of the fluid moving in the band $0 < y < h_1$, $x \in R$ are indicated by the subscript "1", and the parameters of the fluid moving in the band $h_1 < y < h$, $x \in R$ are indicated by the subscript "2". In the plane motion considered here, the functions $w_j(y,t)$ and $f_j(t)$, $j = 1, 2$, are the solutions of the equations

$$w_{jt} = \nu_j w_{jyy} + f_j(t), \tag{3.1}$$

related by the conditions on the interface [3]

$$\int_0^{h_1} w_1(z,t)\,dz = 0; \tag{3.2}$$

$$w_1(h_1,t) = w_2(h_1,t), \quad \mu_1 w_{1y}(h_1,t) - \mu_2 w_{2y}(h_1,t) = 0; \tag{3.3}$$

the no-slip conditions on the solid boundaries of the flow domain

$$w_1(0,t) = 0, \quad w_2(h,t) = 0; \tag{3.4}$$

$$\int_{h_1}^{h} w_2(z,t)\,dz = 0, \tag{3.5}$$

and initial data

$$\begin{aligned} w_1(y,0) &= w_{10}(y), \quad 0 \leqslant y \leqslant h_1,\\ w_2(y,0) &= w_{20}(y), \quad h_1 \leqslant y \leqslant h. \end{aligned} \tag{3.6}$$

Remark 1. The initial value problem (3.1)–(3.5) has not a solution expanded into a Fourier series.

A priori estimates. Using equalities (3.2), (3.4) and integrating equation (3.1) we obtain the relations

$$\begin{aligned} f_1(t) &= \frac{\nu_1}{h_1}\,[w_{1y}(0,t) - w_{1y}(h_1,t)], \quad f_1(0) = \frac{\nu_1}{h_1}\,[w_{10y}(0) - w_{10y}(h_1)],\\ f_2(t) &= \frac{\nu_2}{h-h_1}\,[w_{2y}(h_1,t) - w_{2y}(h,t)], \quad f_2(0) = \frac{\nu_2}{h-h_1}\,[w_{20y}(h_1) - w_{20y}(h)]. \end{aligned} \tag{3.7}$$

There exists the energetic identity for the problem (3.1)–(3.5)

$$\frac{\partial}{\partial t}\,E(t) + \mu_1\int_0^{h_1} w_{1y}^2\,dy + \mu_2\int_{h_1}^{h} w_{2y}^2\,dy = 0, \tag{3.8}$$

where

$$E(t) = \frac{1}{2}\,\rho_1\int_0^{h_1} w_1^2\,dy + \frac{1}{2}\,\rho_2\int_{h_1}^{h} w_2^2\,dy. \tag{3.9}$$

Lemma 1. *The following inequality holds*

$$\int_0^{h_1} w_1^2\,dy + \int_{h_1}^{h} w_2^2\,dy \leqslant M\Big(\mu_1 \int_0^{h_1} w_{1y}^2\,dy + \mu_2 \int_{h_1}^{h} w_{2y}^2\,dy\Big),$$

where M is the solution of the variational problem

$$M = \sup_{v_1,v_2\in V}\left[\frac{\displaystyle\int_0^{h_1} v_1^2\,dy + \int_{h_1}^{h} v_2^2\,dy}{\displaystyle\mu_1\int_0^{h_1} v_{1y}^2\,dy + \mu_2\int_{h_1}^{h} v_{2y}^2\,dy}\right].$$

Here $V \subset W_2^1(0,h_1)\times W_2^1(h_1,h)$ and conditions (3.3), (3.4) for v_1, v_2 are satisfied.

The proof is given in [4]. Due to this lemma we get $M = (h-h_1)^2/\mu_1 z_0^2$, where z_0 is the minimal positive root of the equation

$$\sin(a_1 z)\cos(a_2 z) + a_2\sin(a_2 z)\cos(a_1 z) = 0.$$

Here $a_1 = h_1/(h-h_1)$, $a_2 = (\mu_1/\mu_2)^{1/2}$. From (3.8), (3.9) we get inequality

$$\frac{\partial E}{\partial t} + 2\delta E \leqslant 0, \quad \delta = \frac{1}{M}\,\min\left(\frac{1}{\rho_1}, \frac{1}{\rho_2}\right),$$

hence

$$E(t) \leqslant E(0)e^{-2\delta t}, \tag{3.10}$$

with $E(0) = \dfrac{1}{2}\rho_1\displaystyle\int_0^{h_1} w_{10}^2(y)\,dy + \dfrac{1}{2}\rho_2\int_{h_1}^{h} w_{20}^2(y)\,dy.$

Moreovere, there is another identity for the problem (3.1)–(3.6)

$$\rho_1\int_0^{h_1} w_{1t}^2\,dy + \rho_2\int_{h_1}^{h} w_{2t}^2\,dy + \frac{1}{2}\frac{\partial}{\partial t}\left(\mu_1\int_0^{h_1} w_{1y}^2\,dy + \mu_2\int_{h_1}^{h} w_{2y}^2\,dy\right) = 0$$

and then following estimates hold

$$\int_0^{h_1} w_{1y}^2\,dy \leqslant \frac{W_0}{\mu_1}, \quad \int_{h_1}^{h} w_{2y}^2\,dy \leqslant \frac{W_0}{\mu_2}, \tag{3.11}$$

where

$$W_0 = \mu_1\int_0^{h_1} w_{10}^2\,dy + \mu_2\int_{h_1}^{h} w_{20}^2\,dy.$$

From (3.4), (3.10), (3.11) we have the estimates

$$|w_j(y,t)| \leqslant \left(\frac{8E(0)W_0}{\nu_j}\right)^{1/4} e^{-\delta t/2}. \tag{3.12}$$

Therefore, the motion of fluids are slowed down by the viscous friction according to inequalities (3.12).

Now, let us go over to estimate the function $f_j(t)$ defined by (3.7). Firstly, the new unknowns $V_j(y,t) = w_{jt}(y,t)$ are satisfies the problem (3.1)–(3.6) with f_{jt} instead of $f_j(t)$ and initial data at $t = 0$ equal to

$$V_{10}(y) = \nu_1 w_{10yy}(y) + \frac{\nu_1}{h_1}\,[w_{10y}(0) - w_{10y}(h_1)],$$
$$V_{20}(y) = \nu_2 w_{20yy}(y) + \frac{\nu_2}{h-h_1}\,[w_{20y}(h_1) - w_{20y}(h)].$$

Hence, we get estimates like (3.12)

$$|w_{jt}(y,t)| \leqslant \left(\frac{8E^1(0)W_0^1}{\nu_j}\right)^{1/4} e^{-\delta t/2}, \tag{3.13}$$

here

$$E^1(0) = \frac{1}{2}\,\rho_1 \int_0^{h_1} V_{10}^2(y)\,dy + \frac{1}{2}\,\rho_2 \int_{h_1}^{h} V_{20}^2(y)\,dy,$$

$$W_0^1 = \mu_1 \int_0^{h_1} V_{10y}^2(y)\,dy + \mu_2 \int_{h_1}^{h} V_{20y}^2(y)\,dy.$$

If we multiply equation (3.1) by $y(h_1 - y)$ $(j = 1)$ or $(y - h_1)(h - y)$ $(j = 2)$ and integrate, then we obtain equalities

$$\frac{h_1^3}{6}\,f_1(t) = \int_0^{h_1} y(h_1 - y)w_{1t}(y,t)\,dy - \nu_1 h_1 w_1(h_1,t),$$

$$\frac{(h-h_1)^3}{6}\,f_2(t) = \int_{h_1}^{h} (y - h_1)(h - y)w_{2t}(y,t)\,dy - \nu_2(h - h_1)w_2(h,t).$$

Using inequalities (3.12), (3.13) we get estimates

$$|f_j(t)| \leqslant C_j e^{-\delta t/2}, \tag{3.14}$$

with constants are

$$C_1 = 6\left(\frac{8E^1(0)W_0^1}{\nu_1}\right)^{1/4} + \frac{6\nu_1}{h_1^2}\,\left(\frac{8E(0)W_0}{\nu_1}\right)^{1/4},$$

$$C_2 = 6\left(\frac{8E^1(0)W_0^1}{\nu_2}\right)^{1/4} + \frac{6\nu_2}{(h-h_1)^2}\,\left(\frac{8E(0)W_0}{\nu_2}\right)^{1/4}.$$

4. Solution in the Laplace representation

Let us apply the Laplace transform to problem (3.1)–(3.5)

$$\tilde{w}_j(y,p) = \int_0^{\infty} w_j(y,t)e^{-pt}dt. \tag{4.1}$$

As a result, we obtain a boundary-value problem for the ODE:

$$\tilde{w}_{jyy} - \frac{p}{\nu_j}\,\tilde{w}_j = -\frac{\tilde{f}_j(p)}{\nu_j} - \frac{w_{j0}(y)}{\nu_j}\,, \quad j = 1,2; \tag{4.2}$$

$$\tilde{w}_1(0,p) = 0, \quad \tilde{w}_2(h,p) = 0; \tag{4.3}$$

$$\tilde{w}_1(h_1,p) = \tilde{w}_2(h_1,p), \quad \mu_1\tilde{w}_{1y}(h_1,p) - \mu_2\tilde{w}_{2y}(h_1,p) = 0; \tag{4.4}$$

$$\int_0^{h_1} \tilde{w}_1(y,p)\,d = 0, \quad \int_{h_1}^{h} \tilde{w}_2(y,p)\,dy = 0 \tag{4.5}$$

with the exact solution

$$\tilde{w}_j(y,p) = C_j^1 \operatorname{sh}\left(\sqrt{\frac{p}{\nu_j}}\,y\right) + C_j^2 \operatorname{ch}\left(\sqrt{\frac{p}{\nu_j}}\,y\right) -$$

$$- \frac{1}{\sqrt{p\nu_j}} \int_{y_j}^{y} w_{j0}(z) \operatorname{sh}\left(\sqrt{\frac{p}{\nu_j}}\,(y - z)\right) dz + \frac{\tilde{f}_j(p)}{p}\,, \tag{4.6}$$

where $y_1 = 0$, $y_2 = h$,

$$C_1^1 = \left[\mathrm{ch}\left(\sqrt{\frac{p}{\nu_1}}\,h_1\right) - 1\right]^{-1} \left\{\frac{\tilde{f}_1(p)}{p}\left[\mathrm{sh}\left(\sqrt{\frac{p}{\nu_1}}\,h_1\right) - \sqrt{\frac{p}{\nu_1}}\,h_1\right] + \right.$$
$$\left. + \frac{1}{\nu_1}\int_0^{h_1}\left[\int_0^y w_{10}(z)\,\mathrm{sh}\left(\sqrt{\frac{p}{\nu_1}}\,(y-z)\right)dz\right]dy\right\},$$

$$C_2^1 = \frac{1}{\Delta}\left\{G\,\mathrm{ch}\left(\sqrt{\frac{p}{\nu_2}}\,h\right) + \frac{\tilde{f}_2(p)}{p}\left[\mathrm{sh}\left(\sqrt{\frac{p}{\nu_2}}\,h\right) - \mathrm{sh}\left(\sqrt{\frac{p}{\nu_2}}\,h_1\right)\right]\right\},$$
$$C_2^2 = \frac{1}{\Delta}\left\{\frac{\tilde{f}_2(p)}{p}\left[\mathrm{ch}\left(\sqrt{\frac{p}{\nu_2}}\,h_1\right) - \mathrm{ch}\left(\sqrt{\frac{p}{\nu_2}}\,h\right)\right] - G\,\mathrm{sh}\left(\sqrt{\frac{p}{\nu_2}}\,h\right)\right\}, \tag{4.7}$$
$$C_1^2 = -\frac{\tilde{f}_1(p)}{p}, \quad \Delta = 1 - \mathrm{ch}\left(\sqrt{\frac{p}{\nu_2}}\,(h-h_1)\right),$$
$$G = \frac{(h_1-h)}{\sqrt{p\nu_2}}\,\tilde{f}_2(p) + \frac{1}{\nu_2}\int_{h_1}^{h}\left[\int_h^y w_{20}(z)\,\mathrm{sh}\left(\sqrt{\frac{p}{\nu_2}}\,(y-z)\right)dz\right]dy.$$

Taking into account formulae (4.6), (4.7) from (4.5), we get

$$\tilde{f}_1 = \frac{p}{a_1a_4 - a_2a_3}\,(G_1a_4 - G_2a_2), \quad \tilde{f}_2 = \frac{p}{a_1a_4 - a_2a_3}\,(G_2a_1 - G_1a_3), \tag{4.8}$$

here

$$a_1 = \left[\mathrm{ch}\left(\sqrt{\frac{p}{\nu_1}}\,h_1\right) - 1\right]^{-1}\left[-2 + 2\mathrm{ch}\left(\sqrt{\frac{p}{\nu_1}}\,h_1\right) - \sqrt{\frac{p}{\nu_1}}\,h_1\,\mathrm{sh}\left(\sqrt{\frac{p}{\nu_1}}\,h_1\right)\right],$$
$$a_2 = -\frac{1}{\Delta}\left[2 + \sqrt{\frac{p}{\nu_2}}\,(h-h_1)\,\mathrm{sh}\left(\sqrt{\frac{p}{\nu_2}}\,(h-h_1)\right)\right], \tag{4.9}$$
$$a_3 = \mu_1\sqrt{\nu_2}\left\{\sqrt{\nu_1}\left[\mathrm{ch}\left(\sqrt{\frac{p}{\nu_1}}\,h_1\right) - 1\right]\right\}^{-1}\left[\mathrm{sh}\left(\sqrt{\frac{p}{\nu_1}}\,h_1\right) - \sqrt{\frac{p}{\nu_1}}\,h_1\,\mathrm{ch}\left(\sqrt{\frac{p}{\nu_1}}\,h_1\right)\right],$$
$$a_4 = \frac{\mu_2}{\Delta}\left[\sqrt{\frac{p}{\nu_2}}\,(h-h_1)\mathrm{ch}\left(\sqrt{\frac{p}{\nu_2}}\,(h-h_1)\right) - \mathrm{sh}\left(\sqrt{\frac{p}{\nu_2}}\,(h-h_1)\right)\right];$$

$$G_1 = \frac{1}{\sqrt{p\nu_1}}\int_0^{h_1} w_{10}(z)\,\mathrm{sh}\left(\sqrt{\frac{p}{\nu_1}}\,(h_1-z)\right)dz + \frac{1}{\sqrt{p\nu_2}}\int_{h_1}^{h} w_{20}(z)\,\mathrm{sh}\left(\sqrt{\frac{p}{\nu_2}}\,(h_1-z)\right)dz -$$
$$- \frac{1}{\nu_1}\,\mathrm{sh}\left(\sqrt{\frac{p}{\nu_1}}\,h_1\right)\left[\mathrm{ch}\left(\sqrt{\frac{p}{\nu_1}}\,h_1\right) - 1\right]^{-1}\int_0^{h_1}\left[\int_0^y w_{10}(z)\,\mathrm{sh}\left(\sqrt{\frac{p}{\nu_1}}\,(y-z)\right)dz\right]dy -$$
$$- \frac{1}{\nu_2\Delta}\,\mathrm{sh}\left(\sqrt{\frac{p}{\nu_2}}\,(h-h_1)\right)\int_{h_1}^{h}\left[\int_h^y w_{20}(z)\,\mathrm{sh}\left(\sqrt{\frac{p}{\nu_2}}\,(y-z)\right)dz\right]dy,$$
$$G_2 = \frac{\mu_1\sqrt{\nu_2}}{\nu_1\sqrt{p}}\int_0^{h_1} w_{10}(z)\,\mathrm{ch}\left(\sqrt{\frac{p}{\nu_1}}\,(h_1-z)\right)dz + \frac{\mu_2}{\sqrt{p\nu_2}}\int_{h_1}^{h} w_{20}(z)\,\mathrm{ch}\left(\sqrt{\frac{p}{\nu_2}}\,(h_1-z)\right)dz -$$
$$- \frac{\mu_1\sqrt{\nu_2}}{\nu_1^{3/2}}\,\mathrm{ch}\left(\sqrt{\frac{p}{\nu_2}}\,h_1\right)\left[\mathrm{ch}\left(\sqrt{\frac{p}{\nu_1}}\,h_1\right) - 1\right]^{-1}\int_0^{h_1}\left[\int_0^y w_{10}(z)\,\mathrm{sh}\left(\sqrt{\frac{p}{\nu_1}}\,(y-z)\right)dz\right]dy +$$
$$+ \frac{\mu_2}{\nu_2\Delta}\,\mathrm{ch}\left(\sqrt{\frac{p}{\nu_2}}\,(h-h_1)\right)\int_{h_1}^{h}\left[\int_h^y w_{20}(z)\,\mathrm{sh}\left(\sqrt{\frac{p}{\nu_2}}\,(y-z)\right)dz\right]dy.$$

Simple, but cumbersome calculations with the use of asymptotic representations for functions $\operatorname{sh} x$ and $\operatorname{ch} x$ show that

$$\lim_{t\to\infty} w_j(y,t) = \lim_{p\to 0} p\,\tilde{w}_j(y,p) = 0, \quad \lim_{t\to\infty} f_j(t) = \lim_{p\to 0} p\tilde{f}(p) = 0.$$

Lust results obtained are good agrement with the a priori estimates (3.13) and (3.14).

Conclusions

The partly invariant solution of Navier–Stokes equations is investigated. This solution may describes the plane unsteady motions of a viscous fluid in a strip with two rigid walls, the fluid motion with one rigid wall and free boundary or the motion of a two immiscible fluids with interface in a strip bounded rigid walls. The motion arised due to initial velocity field. It was shown that this problem can be reduced for creeping motions to the linear initial boundary inverse problem for parabolic equations. Two problem were solved by Fourier method. At that time, the interface problem is solved by using some properties of the Laplace transformation. For any cases the motions are retarded by viscous friction.

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Частично инвариантные решения уравнений Навье–Стокса, описывающие нестационарные двумерные движения вязкой жидкости

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Трехмерная непрерывная подалгебра используется для нахождения частично инвариантного решения уравнений вязкой несжимаемой жидкости. Оно интерпретируется как двумерное движение одной или двух несмешивающихся жидкостей в плоском канале. Возникающая начально-краевая задача для фактор-системы является обратной. Нестационарная задача для ползущих движений решена методом разделения переменной для одной жидкости и методом преобразования Лапласа для двух жидкостей.

Ключевые слова: частично инвариантное решение, вязкая жидкость, задача со свободной границей, поверхность раздела.