УДК 517.55

An Integral Formula for the Number of Lattice Points in a Domain

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Received 06.02.2015, received in revised form 06.03.2015, accepted 14.04.2015

Using the multidimensional logarithmic residue we show a simple formula for the difference between the number of integer points in a bounded domain of $\mathbb{R}^n$ and the volume of this domain. The difference proves to be the integral of an explicit differential form over the boundary of the domain.

Keywords: logarithmic residue, lattice point.

Introduction

Classical function theory is of great importance in number theory, let alone the analytical extension of the Riemann zeta function and prime number theorem, see [6, 8, 9], etc.

This work was intended as an attempt at applying the theory of functions of several complex variables to classical problems of number theory. To wit, we apply the multidimensional logarithmic residue which is an efficient numerical tool of algebraic geometry, see [1].

Let $\mathcal{Z}$ be a bounded domain with piecewise smooth boundary in the space $\mathbb{C}^n$ of n complex variables $z = (z_1, \ldots, z_n)$. Consider a holomorphic mapping $w = f(z)$ of the closed domain $\bar{\mathcal{Z}}$ into $\mathbb{C}^n$ which has no zeros at the boundary of $\mathcal{Z}$. Then f has only isolated zeros in $\mathcal{Z}$ and the number of zeros counted with their multiplicity is given by the logarithmic residue formula

$$N(f, \mathcal{Z}) = \int_{\partial \mathcal{Z}} \frac{(n-1)!}{(2\pi\imath)^n} \sum_{j=1}^{n} (-1)^{j-1} \frac{\bar{f}_j}{|f|^{2n}} d\bar{f}[j] \wedge df \qquad (1)$$

(see [1, § 2]), where $|f|^2 = |f_1|^2 + \ldots + |f_n|^2$, f_j being the j th component of f, by $df = df_1 \wedge \ldots \wedge df_n$ is meant the exterior product of the differentials $df_1, \ldots, df_n$, and $d\bar{f}[j]$ stands for the exterior product of the differentials $d\bar{f}_1, \ldots, d\bar{f}_n$ after each other, the differential $d\bar{f}_j$ being omitted. The domain $\mathcal{Z}$ is oriented in such a way that

$$\int_{\mathcal{Z}} \frac{1}{(2\imath)^n} d\bar{z} \wedge dz > 0.$$

We apply formula (1) to get an equality for the difference between the number of lattice points in the domain $\mathcal{Z}$ and its volume. A number of classical problems of number theory, e.g. the

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problem on the number of lattice points in a ball [10], the problems on Dirichlet divisors [4], etc. reduce to evaluating asymptotics of the difference. It is worth pointing out that this asymptotics can not be found by standard methods, such as the Laplace method, stationary phase method, or saddle point method.

The theory of lattice points in large regions has attracted the interest of many mathematicians for more than eleven decades. The monograph [5] presents a broad survey of the main problems and results in lattice point theory.

1. The integral formula

As usual, we write $\mathbb{R}^n$, $n \geqslant 1$, for the n-dimensional real Euclidean space of variables $x = (x_1, \dots, x_n)$ with $x_j \in \mathbb{R}$. Suppose $\mathcal{X}$ is a bounded domain in $\mathbb{R}^n$ whose boundary is piecewise smooth and does not contain any point with integer coordinates. Denote by $N(\mathcal{X})$ the number of integer points in $\mathcal{X}$ and by $V(\mathcal{X})$ its volume.

Theorem 1. *If the boundary $\partial\mathcal{X}$ does not contain lattice points then the difference $N(\mathcal{X}) - V(\mathcal{X})$ can be written in the form*

$$N(\mathcal{X}) - V(\mathcal{X}) = \int_0^\infty \!\!\dots\! \int_0^\infty dt \int_{\partial\mathcal{X}} \frac{2^{n-2}(n-1)!}{\pi} \frac{\sum\limits_{j=1}^n t[j] \sin(2\pi x_j) \nu_j}{\Big(\sum\limits_{j=1}^n (t_j^2 - 2t_j \cos(2\pi x_j)) + n\Big)^n} \, ds, \qquad (2)$$

where $dt = dt_1 \wedge \dots \wedge dt_n$, $t[j] = t_1 \dots t_{j-1} t_{j+1} \dots t_n$, ds is the surface measure of $\partial\mathcal{X}$ and $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ is the unit outward normal vector of the boundary at $x \in \partial\mathcal{X}$.

Proof. Consider the domain $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ in $\mathbb{C}^n$, where $\mathcal{Y}$ is a bounded domain with piecewise smooth boundary in the space $\mathbb{R}^n$ of variables $y = (y_1, \dots, y_n)$. We assume that $0 \in \mathcal{Y}$. The points $z = (z_1, \dots, z_n)$ of $\mathcal{Z}$ have the form $z_j = x_j + \imath y_j$, for $j = 1, \dots, n$. As holomorphic mapping $f : \mathcal{Z} \to \mathbb{C}^n$ vanishing solely at the entire points of $\mathcal{X} \times \{0\}$, we take

$$\begin{array}{rcl} f_1(z) & = & e^{2\pi\imath z_1} - 1, \\ & \dots & \\ f_n(z) & = & e^{2\pi\imath z_n} - 1, \end{array}$$

each zero being simple.

By formula (1), we get

$$N(f, \mathcal{Z}) = \int_{\partial\mathcal{Z}} \frac{(n-1)!}{(2\pi\imath)^n} \frac{\sum\limits_{j=1}^n (-1)^{j-1} (e^{-2\pi\imath\bar{z}_j} - 1) de^{-2\pi\imath\bar{z}}[j] \wedge de^{2\pi\imath z}}{\Big(|e^{2\pi\imath z_1} - 1|^2 + \dots + |e^{2\pi\imath z_n} - 1|^2\Big)^n},$$

where

$$\begin{array}{rcl} de^{-2\pi\imath\bar{z}}[j] & = & de^{-2\pi\imath\bar{z}_1} \wedge \dots \wedge de^{-2\pi\imath\bar{z}_{j-1}} \wedge de^{-2\pi\imath\bar{z}_{j+1}} \wedge \dots \wedge de^{-2\pi\imath\bar{z}_n}, \\ de^{2\pi\imath z} & = & de^{2\pi\imath z_1} \wedge \dots \wedge de^{2\pi\imath z_n}. \end{array}$$

The right-hand side is easily reduced to

$$\int_{\partial\mathcal{Z}} (-2\pi\imath)^{n-1} (n-1)! \, \frac{\sum\limits_{j=1}^n (-1)^{j-1} \Big(e^{-2\pi\imath \sum\limits_{k=1}^n \bar{z}_k} - e^{-2\pi\imath \sum\limits_{k \neq j} \bar{z}_k}\Big) e^{2\pi\imath \sum\limits_{k=1}^n z_k} d\bar{z}[j] \wedge dz}{\Big(|e^{2\pi\imath z_1} - 1|^2 + \dots + |e^{2\pi\imath z_n} - 1|^2\Big)^n}. \qquad (3)$$

A trivial verification shows that

$$d\bar{z}[j] \wedge dz = (2\imath)^{n-1} \left((-1)^{n-1} dx \wedge dy[j] + \imath dx[j] \wedge dy\right) \tag{4}$$

for all $j = 1, \dots, n$. Using (4) one separates the real and imaginary parts of (3), these are

$$\int_{\partial \mathcal{Z}} (4\pi)^{n-1}(n-1)! \frac{\sum_{j=1}^{n} (-1)^{n+j} \left(e^{-4\pi \sum_{k=1}^{n} y_k} - e^{-4\pi \sum_{k \neq j} y_k - 2\pi y_j} \cos(2\pi x_j)\right) dx \wedge dy[j]}{\left(|e^{2\pi \imath z_1} - 1|^2 + \ldots + |e^{2\pi \imath z_n} - 1|^2\right)^n} + \\ + \int_{\partial \mathcal{Z}} (4\pi)^{n-1}(n-1)! \frac{\sum_{j=1}^{n} (-1)^{j-1} e^{-4\pi \sum_{k \neq j} y_k - 2\pi y_j} \sin(2\pi x_j) dx[j] \wedge dy}{\left(|e^{2\pi \imath z_1} - 1|^2 + \ldots + |e^{2\pi \imath z_n} - 1|^2\right)^n} \tag{5}$$

and

$$\int_{\partial \mathcal{Z}} (4\pi)^{n-1}(n-1)! \frac{\sum_{j=1}^{n} (-1)^{j-1} \left(e^{-4\pi \sum_{k=1}^{n} y_k} - e^{-4\pi \sum_{k \neq j} y_k - 2\pi y_j} \cos(2\pi x_j)\right) dx[j] \wedge dy}{\left(|e^{2\pi \imath z_1} - 1|^2 + \ldots + |e^{2\pi \imath z_n} - 1|^2\right)^n} + \\ + \int_{\partial \mathcal{Z}} (4\pi)^{n-1}(n-1)! \frac{\sum_{j=1}^{n} (-1)^{n+j-1} e^{-4\pi \sum_{k \neq j} y_k - 2\pi y_j} \sin(2\pi x_j) dx \wedge dy[j]}{\left(|e^{2\pi \imath z_1} - 1|^2 + \ldots + |e^{2\pi \imath z_n} - 1|^2\right)^n},$$

respectively.

The number $N(f, \mathcal{Z})$ is real, hence it suffices to consider the mere real part (5) of formula (1). Moreover, we make the change of variables

$$\begin{aligned} t_1 &= e^{-2\pi y_1}, \\ &\dots \\ t_n &= e^{-2\pi y_n}, \end{aligned}$$

obtaining

$$N(f, \mathcal{Z}) = \int_{\partial \mathcal{Z}'} 2^{n-1}(n-1)! \frac{\sum_{j=1}^{n} (-1)^{j-1} t_1 \dots t_n (t_j - \cos(2\pi x_j)) dt[j] \wedge dx}{\left(\sum_{j=1}^{n} (t_j^2 - 2t_j \cos(2\pi x_j)) + n\right)^n} + \\ + \int_{\partial \mathcal{Z}'} \frac{2^{n-2}(n-1)!}{\pi} \frac{\sum_{j=1}^{n} (-1)^{n+j-1} t[j] \sin(2\pi x_j) dt \wedge dx[j]}{\left(\sum_{j=1}^{n} (t_j^2 - 2t_j \cos(2\pi x_j)) + n\right)^n} = I_1 + I_2, \tag{6}$$

where $\mathcal{Z}'$ is the image of the domain $\mathcal{Z}$ under the change of variables $t_j = e^{-2\pi y_j}$, for $j = 1, \dots, n$.

This change involves the mere variables y whence $\mathcal{Z}' = \mathcal{X} \times \mathcal{T}$, where $\mathcal{T}$ is the image of $\mathcal{Y}$ by $t_j = e^{-2\pi y_j}$ with $j = 1, \dots, n$. Since $\mathcal{Y}$ contains the origin, the n-tuple with coordinates 1 belongs to $\mathcal{T}$. We now give the domain $\mathcal{T}$ the following concrete form

$$\mathcal{T} = \{t \in \mathbb{R}^n : r^2 < |t|^2 < R^2\} \cap \{t \in \mathbb{R}^n : t_1, \dots, t_n > \varepsilon\},$$

where $r < \sqrt{n}$, $R > \sqrt{n}$ and $\varepsilon > 0$ is small enough. The boundary $\partial\mathcal{T}$ consists of a piece S_r of the $(n-1)$-dimensional sphere $\{t \in \mathbb{R}^n : |t| = r\}$, a piece S_R of the $(n-1)$-dimensional sphere $\{t \in \mathbb{R}^n : |t| = R\}$, and pieces H_j of hypersurfaces $t_j = \varepsilon$ parallel to the coordinates hyperplanes $t_j = 0$. According to this structure of the boundary of $\mathcal{T}$ we represent the integral I_1 as the sum of integrals I_{1,S_r}, I_{1,S_R} and I_{1,H_j} with $j = 1, \ldots, n$.

Let the piece H_1 tend to the hyperplane $\{t_1 = 0\}$. At this hyperplane we obviously get

$$\sum_{j=1}^{n}(t_j^2 - 2t_j \cos(2\pi x_j)) + n = \sum_{j=2}^{n}(t_j - \cos(2\pi x_j))^2 + \sin^2(2\pi x_j) + 1 \geqslant 1.$$

Therefore, the integral I_{1,H_1} tends to zero as H_1 tends to the hyperplane $\{t_1 = 0\}$. Analogously, I_{1,H_j} tends to zero as H_j tends to the hyperplane $\{t_j = 0\}$, for each $j = 2, \ldots, n$.

It remains to consider the limits of the integrals I_{1,S_r} and I_{1,S_R}, when $r \to 0$ and $R \to \infty$. Let $\mathbb{S}^{n-1}_{\geqslant 0}$ be the part of the unit sphere with centre at the origin which lies in the cube $0 \leqslant t_j \leqslant 1$, $j = 1, \ldots, n$. We endow $\mathbb{S}^{n-1}_{\geqslant 0}$ with the usual orientation, then $S_r = -r\,\mathbb{S}^{n-1}_{\geqslant 0}$ and $S_R = R\,\mathbb{S}^{n-1}_{\geqslant 0}$. (When we tended H_j to the hyperplane $\{t_j = 0\}$ for all $j = 1, \ldots, n$, then S_r and S_R became one 2^n-th spheres.) Hence it follows readily that

$$I_{1,S_r} \;=\; -\int_{\mathcal{X}} dx \int_{\mathbb{S}^{n-1}_{\geqslant 0}} 2^{n-1}(n-1)!\, \frac{\displaystyle\sum_{j=1}^{n}(-1)^{j-1}\, r^{2n-1} t_1 \ldots t_n\, (rt_j - \cos(2\pi x_j))dt[j]}{\displaystyle\Big(r^2 - 2\sum_{j=1}^{n} rt_j \cos(2\pi x_j) + n\Big)^n} \to 0$$

as $r \to 0$.

On the other hand, we get

$$\begin{aligned} I_{1,S_r} \;&=\; \int_{\mathcal{X}} dx \int_{\mathbb{S}^{n-1}_{\geqslant 0}} 2^{n-1}(n-1)!\, \frac{\displaystyle\sum_{j=1}^{n}(-1)^{j-1}\, R^{2n-1} t_1 \ldots t_n\, (Rt_j - \cos(2\pi x_j))dt[j]}{\displaystyle\Big(R^2 - 2\sum_{j=1}^{n} Rt_j \cos(2\pi x_j) + n\Big)^n} \to \\ &\to \int_{\mathcal{X}} dx \int_{\mathbb{S}^{n-1}_{\geqslant 0}} 2^{n-1}(n-1)! \sum_{j=1}^{n}(-1)^{j-1}\, t_1 \ldots t_n\, t_j dt[j], \end{aligned}$$

as $R \to \infty$. The last integral just amounts to $V(\mathcal{X})$, for

$$\int_{\mathbb{S}^{n-1}_{\geqslant 0}} 2^{n-1}(n-1)! \sum_{j=1}^{n}(-1)^{j-1}\, t_1 \ldots t_n\, t_j dt[j] \;=\; \int_{\mathbb{S}^{n-1}_{\geqslant 0}} 2^{n-1}(n-1)!\, t_1 \ldots t_n\, ds = 1.$$

Thus, if the domain $\mathcal{T}$ expands to the nonnegative one 2^n-th space as above, the integral I_1 tends to $V(\mathcal{X})$. And the integral I_2 converges to the integral on the right-hand side of formula (2), for $\partial\mathcal{Z}' = (\partial\mathcal{X} \times \mathcal{T}) \cup (\mathcal{X} \times \partial\mathcal{T}$ and $(-1)^{j-1}dx[j] = \nu_j ds$ for all $j = 1, \ldots, n$, as desired. □

For the most practical cases $n = 2$ and $n = 3$ Theorem 1 was first proved in [2].

2. The one-dimensional case

In this section we clarify the structure of formula (2) by directly computing the integral on the right-hand side of this formula in the case $n = 1$. Let $\mathcal{X} = (a, b)$, where $m < a < m + 1$ and

$M < b < M+1$, m and M being integer numbers satisfying $m < M$. Then

$$\begin{aligned} I &= \int_0^\infty dt \int_{\partial\mathcal{X}} \frac{1}{2\pi} \frac{\sin 2\pi x}{(t-\cos 2\pi x)^2 + (\sin 2\pi x)^2} = \\ &= \int_0^\infty \frac{1}{2\pi} \Big(\frac{\sin 2\pi b}{(t-\cos 2\pi b)^2 + (\sin 2\pi b)^2} - \frac{\sin 2\pi a}{(t-\cos 2\pi a)^2 + (\sin 2\pi a)^2} \Big) dt. \end{aligned}$$

Substituting $s = t - \cos 2\pi b$ and $s = t - \cos 2\pi a$ into the first and second terms on the right-hand side, respectively, we get

$$\begin{aligned} I &= \int_{-\cos 2\pi b}^\infty \frac{1}{2\pi} \frac{\sin 2\pi b}{s^2 + (\sin 2\pi b)^2} ds - \int_{-\cos 2\pi a}^\infty \frac{1}{2\pi} \frac{\sin 2\pi a}{s^2 + (\sin 2\pi a)^2} ds = \\ &= \frac{1}{2\pi} \arctan \frac{s}{\sin 2\pi b} \restriction_{-\cos 2\pi b}^\infty - \frac{1}{2\pi} \arctan \frac{s}{\sin 2\pi a} \restriction_{-\cos 2\pi a}^\infty . \end{aligned}$$

To be specific, we consider the case

$$\begin{array}{ccccc} m+1/2 & < & a & < & m+1, \\ M & < & b & < & M+1/2, \end{array}$$

then $\sin 2\pi a < 0$ and $\sin 2\pi b > 0$. Hence it follows that

$$\begin{aligned} I &= \frac{1}{2\pi} \Big(\frac{\pi}{2} - \arctan \Big(-\frac{\cos 2\pi b}{\sin 2\pi b} \Big) - \Big(-\frac{\pi}{2} \Big) + \arctan \Big(-\frac{\cos 2\pi a}{\sin 2\pi a} \Big) \Big) = \\ &= \frac{1}{2\pi} (\pi + \arctan\cot 2\pi b - \arctan\cot 2\pi a). \end{aligned}$$

Finally, on using the equality $\arctan x = \pi/2 - \operatorname{arccot} x$ we deduce

$$I = \frac{1}{2\pi}(\pi - \arctan\cot(2\pi b - 2\pi M) + \arctan\cot(2\pi a - 2\pi(m+1/2)) = (M-m) - (b-a),$$

which just amounts to $N(\mathcal{X}) - V(\mathcal{X})$, as desired.

3. Some comments

It is easy to see that the integrations over $t \in [0,\infty)^n$ and $x \in \partial\mathcal{X}$ in formula (2) can be exchanged. In this way we get

$$N(\mathcal{X}) - V(\mathcal{X}) = \int_{\partial\mathcal{X}} \sum_{j=1}^n (-1)^{j-1} F_j(x) \sin 2\pi x_j \, dx[j], \tag{7}$$

where

$$F_j(x) = \frac{2^{n-2}(n-1)!}{\pi} \int_0^\infty \!\! \dots \! \int_0^\infty \frac{t[j]}{\Big(\sum_{k=1}^n (t_k - \cos 2\pi x_k)^2 + \sum_{k=1}^n (\sin 2\pi x_k)^2 \Big)^n} dt$$

are functions of $\cos 2\pi x_j$ and $\sin 2\pi x_j$, for $j = 1, \dots, n$. The differential form under the integral over $\partial\mathcal{X}$ on the right-hand side of (7) is smooth away from the lattice of half-integer points in $\mathbb{R}^n$. As is seen from Section 2, the differential form is not closed outside this lattice. The coefficients F_j bear certain symmetry in variables $x_1, \dots, x_n$, perhaps, it suffices to compute only one of these coefficients in order to determine the others. Moreover, F_j can be computed in a closed form, however, the expressions are cumbersome, cf. formula (3) in [3]. It is possible that formula (7) can be applied to construct asymptotics of the difference $N(\mathcal{X}) - V(\mathcal{X})$ as $R \to \infty$, where $\mathcal{X}$ is the ball of radius R with centre at 0 or, more generally, an ellipsoid

$$\left(\frac{x_1}{a_1}\right)^2 + \ldots + \left(\frac{x_n}{a_n}\right)^2 < R^2$$

or another expanding domain, cf. [5, 7]. But we will not develop this point here.

This research was supported by the German Research Society (DFG), grant TA 289/4-2. The first author wishes to express his gratitude to Professor Sinai Robins who payed attention to the paper [2], invited the author to attend Singapore for two weeks and thus recommenced his interest to the topic.

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Интегральная формула для числа целых точек в области

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Используя формулу многомерного логарифмического вычета, мы даем простую формулу для разности между числом целых точек в ограниченной области из $\mathbb{R}^n$ и объемом этой области. Эта разность дается интегралом от дифференциальной формы, задаваемой точным выражением, по границе этой области.

Ключевые слова: логарифмический вычет, целая точка.