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Generation of the Chevalley Group of Type G_2 over the Ring of Integers by Three Involutions Two of which Commute

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It is proved that $G_2(\mathbb{Z})$ is generated by three involutions. Two of these involutions commute.

Keywords: ring of integers, generating involutions, Chevalley group

Introduction

The main result of this article is

Theorem 1. *The Chevalley group $G_2(\mathbb{Z})$ over the ring of integers $\mathbb{Z}$ is generated by three involutions and two of these involutions commute.*

Theorem 1 answers the question formulated by Ya. N. Nuzhin [1, question 15.67] for the group $G_2(\mathbb{Z})$: *What adjoint Chevalley groups over the ring of integers are generated by three involutions, two of which commute?*

This problem has not been solved. We just know that groups $SL_n(\mathbb{Z})$, $n \geqslant 14$ are generated by three involutions, two of which commute [2]. Groups $PSL_n(\mathbb{Z})$ are generated by three involutions, two of which commute when $n \geqslant 5$ [3]. Note also that adjoint Chevalley group $B_2(\mathbb{Z})$ is not generated by three involutions, two of which commute. It follows from the fact that group $PSp_4(3)$ is not generated by three involutions, two of which commute [4].

1. Notation and preliminary results

Let Φ be a reduced indecomposable root system. Let us denote adjoint Chevalley group over a field K by $\Phi(K)$. This group is generated by root subgroups $X_r = \{x_r(t) \mid t \in K\}$, $r \in \Phi$. Let us denote special linear group by $SL_2(K)$ and subgroup generated by the set M by $\langle M\rangle$.

Lemma 1 ([5, Theorem 6.3.1., p.88]). *There is a homomorphism from $SL_2(K)$ onto subgroup $\langle X_r, X_{-r}\rangle$ of $\Phi(K)$ such that*

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \longrightarrow x_r(t),$$

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \longrightarrow x_{-r}(t).$$

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Then K^* is the multiplicative group of the field K. Let us assume that

$n_r(t) = x_r(t)x_{-r}(-t^{-1})x_r(t)$,
$h_r(t) = n_r(t)n_r(-1)$,
$n_r = n_r(1)$, $r \in \Phi$, $t \in K^*$.

With conjugations the diagonal elements act on root elements as follows:

$$h_r(t)x_s(u)h_r(t)^{-1} = x_s(t^{A_{rs}}u), \tag{1}$$

where $A_{rs} = 2(r,s)/(r,r)$ and (x,y) is the scalar product of vectors x, y.

Let H be a diagonal subgroup of a group $\Phi(K)$ generated by elements $h_r(t)$, $r \in \Phi, t \in K^*$. Let N be a monomial subgroup of the group $\Phi(K)$ generated by H and elements n_r, $r \in \Phi$ and let W be a Weyl group of type Φ.

Lemma 2. *The Chevalley group $\Phi(R)$ over an euclidian ring R is generated by the root elements $x_r(1)$, $r \in \pm\Pi$, where Π is a fundamental root subsystem of the root system Φ.*

Proof. Let us assume that $G = \langle x_r(1) \mid r \in \pm\Pi\rangle$ then $n_r \in G$ for $r \in \pm\Pi$. Fundamental reflections w_r, $r \in \Pi$ are images of elements n_r, $r \in \Pi$ under homomorphism from N into W. Elements w_r, $r \in \Pi$ generate the group W [5, Proposition 2.1.8, p.17]. Hence $n_s \in G$ for all $s \in \Phi$. Since

$$n_s x_r(t) n_s^{-1} = x_{w_s(r)}(\pm t)$$

and group W acts transitivly on the roots with the same length then $x_r(1) \in G$ for all $r \in \Phi$. By consequence 3 from [6, c. 107] group $\Phi(K)$ is generated by root elements $x_r(1)$, $r \in \Phi$, hence $G = \Phi(K)$. □

Next we need 7-dimensional matrix representation of the Chevalley group $G_2(K)$ [7]. Let us fix a fundamental system of roots of $\{a, b\}$ of type G_2. Then the root elements have the following representation

$$x_a(t) = e + t(e_{67} + 2e_{45} - e_{34} - e_{12}) - t^2 e_{35},$$

$$x_{-a}(t) = e + t(e_{76} + e_{54} - 2e_{43} - e_{21}) - t^2 e_{53},$$

$$x_{a+b}(t) = e + t(e_{13} - e_{24} + 2e_{46} - e_{57}) - t^2 e_{26},$$

$$x_{-a-b}(t) = e + t(e_{31} - 2e_{42} + e_{64} - e_{75}) - t^2 e_{62},$$

$$x_{2a+b}(t) = e + t(e_{47} + e_{36} - e_{25} - 2e_{14}) - t^2 e_{17},$$

$$x_{-2a-b}(t) = e + t(e_{74} + e_{63} - e_{52} - 2e_{41}) - t^2 e_{71},$$

$$x_b(t) = e + t(e_{56} - e_{23}),$$

$$x_{-b}(t) = e + t(e_{65} - e_{32}),$$

$$x_{3a+b}(t) = e + t(e_{15} - e_{37}),$$

$$x_{-3a-b}(t) = e + t(e_{51} - e_{73}),$$

$$x_{3a+2b}(t) = e + t(e_{27} - e_{16}),$$

$$x_{-3a-2b}(t) = e + t(e_{72} - e_{61}),$$

where e is the indentity matrix, matrices e_{ij} have entries equal to 1 at (i,j) and other entries equal to 0.

2. Proof of Theorem 1

As in previous section $\{a, b\}$ is a fundamental root system of type G_2, where a is the short root.

Let us denote

$$\begin{aligned}\alpha &= x_a(1)h_b(-1),\\ \beta &= x_{-b}(1)h_a(-1),\\ \gamma &= n_a n_{3a+2b} h_b(-1).\end{aligned}$$

Our goal is to show that $\{\alpha, \beta, \gamma\}$ are three involutions that generate group $G_2(\mathbb{Z})$ with $\alpha\beta = \beta\alpha$.

Let us show that α, β and γ are involutions. Applying equality (1), we obtain

$$\alpha^2 = x_a(1)h_b(-1)x_a(1)h_b(-1) = x_a(1)x_a((-1)^{A_{ba}}) = 1,$$

because $A_{ba} = \dfrac{2(b,a)}{(b,b)} = \dfrac{-2\sqrt{3}|b||a|}{2|b||b|} = 1.$

Similarly we have

$$\beta^2 = x_{-b}(1)h_a(-1)x_{-b}(1)h_a(-1) = x_{-b}(1)x_{-b}((-1)^{A_{a,-b}}) = 1,$$

because $A_{a,-b} = \dfrac{2(a,-b)}{(a,a)} = \dfrac{2\sqrt{3}|a||b|}{2|a||a|} = 3.$

Elements n_a and n_{3a+2b} are commute because $a \pm (3a + 2b)$ is not a root. Hence we have

$$\begin{aligned}\gamma^2 = \; & n_a n_{3a+2b} n_a n_{3a+2b} =\\ & = n_a n_a n_{3a+2b} n_{3a+2b} =\\ & = h_a(-1)h_{3a+2b}(-1) = 1.\end{aligned}$$

Now we show that the equality above is true. Diagonal elements $h_a(-1)$ and $h_{3a+2b}(-1)$ act equally by conjugations (1) on generating elements $x_a(1)$ and $x_b(1)$ of group $G_2(\mathbb{Z})$. Note also that elements $h_a(-1)$ and $h_{3a+2b}(-1)$ from matrix representation of group $G_2(K)$ over field K [7] are represented by matrix $diag(-1, -1, 1, 1, 1, -1, -1)$.

Then we show that $\alpha\beta = \beta\alpha$. For this we just need to show that $(\alpha\beta)^2 = 1$. Simple manipulations give us the following result

$$\begin{aligned}(\alpha\beta)^2 = \; & x_a(1)h_b(-1)x_{-b}(1)h_a(-1)x_a(1)h_b(-1)x_{-b}(1)h_a(-1) =\\ & = h_b(1)h_a(1)x_a(-1)x_{-b}(-1)x_a(1)x_{-b}(1)h_b(-1)h_a(-1) =\\ & = h_b(1)h_a(1)x_a(-1)x_a(1)x_{-b}(-1)x_{-b}(1)h_b(-1)h_a(-1) =\\ & = h_b(1)h_a(1)h_b(-1)h_a(-1) =\\ & = h_b(1)h_b(-1)h_a(1)h_a(-1) = 1.\end{aligned}$$

Let us denote $M = \langle \alpha, \beta, \gamma \rangle$. We show that $M = G_2(\mathbb{Z})$. We have the following relation

$$\begin{aligned}\alpha^\gamma = \; & n_a n_{3a+2b} h_b(-1) x_a(1) h_b(-1) n_a n_{3a+2b} h_b(-1) =\\ = \; & n_a n_{3a+2b} x_a((-1)^{\frac{2(b,a)}{(b,b)}}) n_a n_{3a+2b} h_b(-1) =\\ = \; & n_a n_{3a+2b} x_a((-1)^{\frac{2\sqrt{3}|a||b|}{2|a||a|}}) n_a n_{3a+2b} h_b(-1) =\\ = \; & n_a n_{3a+2b} x_a((-1)^1) n_a n_{3a+2b} h_b(-1) =\\ = \; & n_{3a+2b} n_a x_a(-1) n_a n_{3a+2b} h_b(-1).\end{aligned} \tag{2}$$

In matrix representation of the group $G_2(\mathbb{Z})$ we have

$$n_a = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$n_{3a+2b} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$x_a(-1) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Following some manipulations we obtain

$$n_{3a+2b}n_a x_a(-1) n_a n_{3a+2b} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} = x_{-a}(1).$$

Thus, in view of (2) we have

$$\alpha^\gamma = x_{-a}(1)h_b(-1).$$

Let us introduce

$$\theta = \alpha\alpha^\gamma = x_a(1)x_{-a}(-1).$$

We show that $\theta^3 = h_a(-1)$. Since mapping ψ from Lemma 1 is isomorphism for group $G_2(\mathbb{Z})$ we can use matrix representation. Then manipulations with matrices of the second order give the following equalities

$$\theta^3 = (x_a(1)x_{-a}(-1))^3 = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\right)^3 =$$
$$= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = h_a(-1).$$

Therefore

$$h_a(-1) \in M. \tag{3}$$

Then

$$\beta^\gamma = x_b(1)h_a(-1). \tag{4}$$

It follows from (4) that $x_b(\pm 1) = \beta^\gamma h_a(-1)$. After applying (3), we get inclusion

$$x_b(\pm 1) \in M.$$

By definition $n_b = x_b(1)x_{-b}(-1)x_b(1)$. Then $x_{-b}(\pm 1) = (x_b(1))^\gamma$ and $n_b \in M$. Therefore

$$n_b^2 = h_b(-1) \in M. \tag{5}$$

From relation (5) and equality $x_a(1) = \alpha h_b(-1)$ we get inclusion

$$x_a(1) \in M.$$

The ring of integers $\mathbb{Z}$ is euclidean ring then by Lemma 2 and inclusions

$$x_{\pm a}(1),\ x_{\pm b}(1) \in M,$$

we obtain $M = G_2(\mathbb{Z})$.

Therefore, group $G_2(\mathbb{Z})$ is generated by three involutions α, β and γ. First two involutions commute. □

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Порождающие тройки инволюций группы Шевалле типа G_2 над кольцом целых чисел

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В работе доказано, что группа $G_2(\mathbb{Z})$ порождается тремя инволюциями, две из которых перестановочны.

Ключевые слова: кольцо целых чисел, порождающие тройки инволюций, группа Шевалле