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Some Remarks and Corrections of Recent Results from the Framework of S-metric Spaces

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Abstract. The content of this paper consists of results on Wardowski's F-contraction within S-metric spaces. Namely, in it we present corrections to some recent results by using only the property F1 of strict increasing of the function F. In our results, we combine β-admissible functions with F-contractions. Finally, we give an example that shows that F-contraction in the framework of S-metric spaces is a true generalization of Banach's contraction principle in the same framework.

Keywords: S-metric space, b-metric space, fixed point, F-contraction, β-admissible.

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1. Introduction and preliminaries

After Banach's result in 1922, many researchers in mathematics tried to somehow generalize his famous result. They did this either by breaking the metric space axioms or by generalizing the right-hand side "$\lambda d(x,y)$" of the Banach's contractive condition $d(fx,fy) \leqslant \lambda d(x,y)$ of the mapping f from the metric space (X,d) to itself, where the scalar $\lambda \in [0,1)$. Using the first case, new classes of spaces called generalized metric spaces such as partial metric spaces, metric-like spaces, b-metric spaces, partial b-metric and b-metric like spaces were created. In the last 20-30 years, these kinds of spaces have been studied a lot. According to the second case, many contractive conditions arose within metric spaces, such as Kannan, Chatterjea, Reich, Hardy-Rogers, Boyd–Wong, Meir–Keeler, Ćirić and many others. For more details on the various types of contractions see [3]. In addition to the aforementioned generalizations of the famous Banach result from 1922, in 2012 the Polish mathematician D. Wardowski [26] introduced a new type of contraction called F-contraction. And by using it, he gives a result that in the true sense generalizes Banach's contraction principle from 1922. After that, many works appeared on F-contractions that were applied to almost all the mentioned general metric spaces. Wardowski

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introduces a strictly increasing function F defined on $(0,+\infty)$ with values in $\mathbb{R}$ and denotes this property by **F1**. In addition to it, it adds two more properties **F2** (For each sequence $\{c_n\}$ of positive numbers, the following holds: $\lim_{n\to+\infty} c_n = 0$ if and only if $\lim_{n\to+\infty} F(c_n) = -\infty$) and **F3** (There exist $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$). The set of all functions F that map $(0,+\infty)$ to $\mathbb{R}$ and that satisfy **F1**, **F2** and **F3** in the literature is denoted by $\mathcal{F}$. For these and more details, the reader can see the papers [9,10,25,26] and references therein. Particularly useful is the paper [9] in which all known results are proved using only the property **F1.** In works such as [2,13,14,18] some authors consider Wardowski's approach but within the framework of S-metric spaces. In doing so, they use all three properties of mapping F and the so-called β-admissible mapping T from the given space (X,S) to itself. We will substantially improve these results in our discussion by using only the strictly increasing mapping F. We will also take an β-admissible mapping which is of transitive type (Definition 1.3., 1.4. and Lemma 1.3.). Also, see [19]. In both previous cases, the "distance" between two points was considered as a function with two variables, i.e., we had a mapping from X^2 to $[0,+\infty)$ i.e. metric d, partial metric p, b-metric b etc. Later, some researchers instead of two-variable functions went to three-variable functions, i.e., the mappings from X^3 to $[0,+\infty)$ and thus arrived at the following four classes of spaces: G-metric, G_b-metric spaces, S-metric and S_b-metric, The aim of this paper is to give an overview of the results on the last two classes of space as well as to present some new observations about them. We begin with definitions of basic terms in the class of S-metric spaces.

Definition 1.1 ([20]). *Let X be a non-empty set and denote by S the mapping from X^3 to $[0,+\infty)$ that satisfies the following axioms:*

(S1): $S(x,y,z) = 0$ if and only if $x = y = z$;

(S2): $S(x,y,z) \leqslant S(x,x,a) + S(y,y,a) + S(z,z,a)$ for all x,y,z,a from X.

Then the pair (X,S) is called an S-metric space and the mapping S is called a S-metric on X.

Some examples of S-metric spaces:

Example 1.1. *Let $\|.\|$ a norm on the vector space V, then $S(x,y,z) = \|y+z-2x\| + \|y-z\|$ is an $S-$metric on V.*

Example 1.2. *Let $\|.\|$ a norm on the vector space V, then $S(x,y,z) = \|x-z\| + \|y-z\|$ is an $S-$metric on vector space V.*

Properties such as convergence of a sequence, Cauchyness of a sequence, complete of the space and continuity of a function, all within S-metric spaces are given by the following definition:

Definition 1.2 ([20]). *Let (X,S) be an S-metric space.*

(1) A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n,x_n,x) \to 0$ as $n \to +\infty$.

(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if $S(x_n,x_n,x_m) \to 0$ as $n,m \to +\infty$.

(3) The S-metric space (X,S) is said to be complete if every Cauchy sequence is convergent.

(4) A mapping $T : X \to X$ is said to be S-continuous if $\{Tx_n\}$ is S-convergent to Tx, where $\{x_n\}$ is an S-convergent sequence converging to x.

For still details reader can see the following papers: [1,2,4–8,11–14,17,18,20–23,27].

Similar to metric and G-metric spaces, open and closed balls are defined and the corresponding topology is based on them. For details, see the papers on S-metric spaces in the reference list. Here, the sequence converges on the S-metric if and only if it converges on that resulting topology. It is well known that such equivalence does not hold for b-metric, G_b-metric and S_b-metric spaces.

This is because the open sphere defined in them does not have to be open in the generated topology.

In this paper, we will discuss several recent results established by several authors and published recently in [2,13,14,18]. All these results connect F-contractions to β-admissible mappings of both within S-metric spaces. Using the connection of S-metric and b-metric spaces that is given and explained in the following Proposition, we will in one of the next papers provide a substantial correction of the result from [5]. Now we state the position on the relationship between S-metric and b-metric spaces:

Proposition 1.1. *Let (X, S) be an S-metric space. Then with $b(x, y) = S(x, x, y)$ a b-metric on the set X is given.*

The following applies:

a) *(X, S) is complete S-metric space if and only if (X, b) is a complete b-metric space;*

b) *A Sequence x_n converges in S-metric space (X, S) if and only if it converges in b-metric space (X, b);*

c) *The same applies when the sequence x_n is a Cauchy sequence. Namely, it is Cauchy in (X, S) if and only if it is Cauchy in the b-metric space (X, b);*

d) *The mapping T from X to itself is continuous in (X, S) if and only if it is continuous in (X, b);*

e) *Since S is a continuous function with three variables, then the newly defined b-metric b is also such, i.e., a continuous function with two variables.*

Remark 1.1. *For a proof of the mentioned properties, see the recent interesting paper [21]. Thus, the coefficient s in the obtained b-metric space is equal to $\frac{3}{2}$. Let us also mention one error from the work of G. S. Saluja [18]: If (X, S) is a given S-metric space, then with $d_G(x, y) = S(x, x, y) + S(y, y, x)$ a metric on the set X is defined. According to Proposition 1.1, it is false. Indeed, if the above equality were possible, then we would have that the metric d_G is equal to the $2 \cdot b$ from Proposition 1.1. But since the coefficient s of the b-metric b is equal to $\frac{3}{2} > 1$, it means that b is not a metric. It follows from the assertion of the author in [18] that $b = \frac{1}{2} d_G$, i.e., that b is a metric, because obviously $\frac{1}{2} d_G$ is a metric.*

In the continuation of the work, we significantly improve the results from several works ([2, 13, 14, 18]). We will only use the property **F1**, i.e. strict increasing of mapping F. In addition to the two lemmas that will be listed, we will use the following important property of the strictly increasing function F from $(0, +\infty)$ to $\mathbb{R}$. It reads: For each strictly increasing function F from $(0, +\infty)$ to $\mathbb{R}$ the following applies: $F(a - 0) \leqslant F(a) \leqslant F(a + 0)$ where $F(a - 0)$ and $F(a + 0)$ are respectively the left and right limits of the function F at point a. Note that the following also applies: either $F(0 + 0) = -\infty$ or $F(0 + 0) = A$ where A is a real number.

In some works on F-contractions in S-metric spaces, it is assumed that the contractive condition

$$\tau + F\left(S\left(Tx, Ty, Tz\right)\right) \leqslant F\left(S\left(x, y, z\right)\right)$$

holds whenever $S(Tx, Ty, Tz) > 0$. In this case, in the proofs, instead of x, i.e., y and z, the authors take $x = y = x_{n-1}$ and $z = x_n$, respectively. Namely, in some papers they assume that $\tau + F(Tx, Tx, Ty)) \leqslant F(S(x, x, y))$ is fulfilled whenever $S(Tx, Tx, Ty) > 0$.

As in the case of considering F-contractions within metric spaces ([9,24]), the following two Lemmas occupy an important place.

Lemma 1.1. *Let (X, S) be an S-metric space and $\{x_n\}$ be a Picard's sequence in it. If $S\left(x_{n+1}, x_{n+1}, x_n\right) < S\left(x_n, x_n, x_{n-1}\right)$ for all $n \in \mathbb{N}$ then $x_n \neq x_m$ whenever $n \neq m$.*

Proof. Suppose the opposite, i.e., let $x_n = x_m$ for some n, m from $\mathbb{N}$ with $n < m$. Due to the fact that $x_{n+1} = Tx_n = Tx_m = x_{m+1}$ we have

$$S(x_{n+1}, x_{n+1}, x_n) = S(x_{m+1}, x_{m+1}, x_m) < S(x_m, x_m, x_{m-1}) < \dots$$

$$\dots < S(x_{n+2}, x_{n+2}, x_{n+1}) < S(x_{n+1}, x_{n+1}, x_n) < \dots,$$

which is a contradiction. □

Lemma 1.2 ([22]). *Let (X, S) be an S-metric space and let $\{x_n\}$ be a Picard's sequence in it such that*

$$\lim_{n\to+\infty} S(x_{n+1}, x_{n+1}, x_n) = 0.$$

If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that the following sequences tend to ε^+ when $k \to +\infty$:

$$\{S(x_{m_k}, x_{m_k}, x_{n_k})\}, \{S(x_{m_k}, x_{m_k}, x_{n_k+1})\}, \{S(x_{m_k-1}, x_{m_k-1}, x_{n_k})\},$$

$$\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k+1})\}, \{S(x_{m_k+1}, x_{m_k+1}, x_{n_k+1})\}, \dots$$

For the all details of the proof see [22] as well as [16].

The following two concepts and the result that connects them will be useful to us in the continuation of our work.

Definition 1.3 ([27]). *Let $T : X \to X$ and $\beta : X^3 \to [0, +\infty)$. Then we say that T is β-admissible if for all $x, y, z \in X$ we have*

$$\beta(x, y, z) \geqslant 1 \textit{ implies } \beta(Tx, Ty, Tz) \geqslant 1.$$

Definition 1.4 ([27]). *Let $\beta : X^3 \to [0, +\infty)$. We say that β is transitive if*

$$\beta(x, y, y) \geqslant 1 \textit{ and } \beta(y, z, z) \geqslant 1 \textit{ implies } \beta(x, z, z) \geqslant 1,$$

for all $x, y, z \in X$.

The next result is the key to our further consideration and correction of already published results on F-contractions in S-metric spaces.

Lemma 1.3 ([27]). *Let $T : X \to X$ and $\beta : X^3 \to [0, +\infty)$ be β-admissible and transitive, respectively. Assume that there exists $x_0 \in X$ such that*
$\beta(x_0, Tx_0, Tx_0) \geqslant 1$. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$. Then
$\beta(x_m, x_n, x_n) \geqslant 1$, for all $m, n \in \mathbb{N}$ with $m < n$.

We end this part of the paper with a list of all known $F - \beta$ contractive conditions, where F is a mapping from $(0, +\infty)$ to $\mathbb{R}$ with all three known properties, while β is a mapping from X^3 to $[0, +\infty)$ $\beta-$admissible or β-admissible and transitive. We list all those contractive conditions:

1) $S(Tx, Ty, Tz) > 0$ implies $\tau + \beta(x, y, z) \cdot F(S(Tx, Ty, Tz)) \leqslant F(S(x, y, z))$;
2) $S(Tx, Ty, Tz) > 0$ implies $\tau + F(\beta(x, y, z) \cdot S(Tx, Ty, Tz)) \leqslant F(S(x, y, z))$;
3) $S(Tx, Tx, Ty) > 0$ implies $\tau + \beta(x, x, y) \cdot (F(S(Tx, Tx, Ty) \leqslant F(S(x, x, y))$;
4) $S(Tx, Tx, Ty) > 0$ implies $\tau + F(\beta(x, x, y) \cdot S(Tx, Tx, Ty)) \leqslant F(S(x, x, y))$.

If in 3)–4) on the right sides if $S(x, x, y)$ is replaced with

$$M(x, x, y) = \max\{S(x, x, y), S(x, x, Tx), S(y, y, Ty), (S(x, x, Tx) + S(x, x, Ty) + S(y, y, Tx))\},$$

then we have the so-called $\beta - F-$weak contractions. In some papers from the reference list, theorems about the existence of a fixed point for $\beta - F$ contractions have been proven, but only if the mapping F satisfies all 3 listed properties. In the continuation of the work, we will significantly improve such results in the sense that we add the property of transitivity to the mapping β and remove the properties **F2** and **F3** from the mapping F.

Remark 1.2. *According to Proposition 1.1, we have that relations 3) and 4) give an F-contraction within b-metric spaces.*

We now state one of the first results stated and proved in ([14], Theorem 1).

Theorem 1.1. *Let (X, S) be a complete S-metric space and $T : X \to X$ be a $\beta - F-$weak contraction satisfying the following conditions:*
(T1) T is β-admissible,
(T2) there exists $u_0 \in X$ such that $\beta(u_0, u_0, Tu_0) \geqslant 1$,
(T3) T is S-continuous.
Then T has a fixed point.

Remark 1.3. *For several results of this type the reader can see ([14], Theorem 2, [18], Theorem 3.2, [23], Theorem 5, [15], Theorems 2, 3 and 4).*

2. Main results

Our first new general result is given by the following Theorem:

Theorem 2.1. *If (X, S) is a complete S-metric spaces and if there exists $\tau > 0$ such that*

$$\tau + F(S(Tx, Ty, Tz)) \leqslant F(S(x, y, z)),$$

whenever $S(Tx, Ty, Tz) > 0$, where $F : [0, +\infty) \to \mathbb{R}$ is a strictly increasing function, then the mapping T has a unique fixed point in X. Moreover, if x is an arbitrary point in X, then the Picard's sequence $T^n x$ converges to this fixed point.

Proof. If u and v are two different fixed points of the mapping T, then
$S(u, u, v) = S(Tu, Tu, Tv) > 0$, because $Tu \neq Tv$. In this case, according to the given contractive condition we get $\tau + F(S(u, u, v)) \leqslant F(S(u, u, v))$ which is a contradiction with $\tau > 0$. So, if T has a fixed point it is unique. Also, from the given contractive condition the continuity of the mapping T follows. Indeed, the given contractive condition gives $S(Tx, Ty, Tz) < S(x, y, z)$ which means that the mapping $T : X \to X$ is S-continuous.

Now, we will prove the existence of the fixed point for the mapping T. Let x_0 be an arbitrary point in X and let $x_n = T^n x_0$ be the corresponding Picard's sequence. If $x_{n-1} = x_n$ for some $n \in \mathbb{N}$, then obviously x_{n-1} is a unique fixed point of the mapping T. And hence Theorem is proved. Let us now in the following prove that x_{n-1} is different from x_n for each $n \in \mathbb{N}$. Putting in the given contractive condition $x = y = x_{n-1}$ and $z = x_n$, we get that
$S(Tx, Ty, Tz) = S(x_n, x_n, x_{n+1})$ which further means that:

$$\tau + F(S(x_n, x_n, x_{n+1})) \leqslant F(S(x_{n-1}, x_{n-1}, x_n)).$$

Since $\tau > 0$ and F is a strictly increasing function, we get that the sequence
$\{S(x_n, x_n, x_{n+1})\}$ is strictly decreasing and therefore its limit $S^* \geqslant 0$ exists when $n \to +\infty$. Let us prove that $S^* = 0$. Indeed, by switching to the limit when $n \to +\infty$ in the last relation

and using the property of the function F about left and right limits we get $\tau + F(S^* + 0) \leqslant F(S^* + 0)$, which is a contradiction since $\tau > 0$. Note that according to Lemma 1.1 obtained that all members of the sequence x_n are mutually different, i.e., that $x_n \neq x_m$ whenever $n \neq m$. We need this when we prove the Cauchyness of the sequence $\{x_n\}$ because we will need the area of definition of the function F.

In order to prove that $\{x_n\}$ is a Cauchy sequence, we apply Lemma 1.2, i.e. We put $x = y = x_{n_k}$, $z = x_{m_k}$ in the given contractive condition, and so we get the following

$$\tau + F\left(S\left(x_{n_k+1}, x_{n_k+1}, x_{m_k+1}\right)\right) \leqslant F\left(S\left(x_{n_k}, x_{n_k}, x_{n_k}\right)\right).$$

If, in the last relation, we let k tends to $+\infty$ and use the property on the left and right limits of the function F, we get: $\tau + F(\varepsilon^+) \leqslant F(\varepsilon^+)$, which is obviously a contradiction with $\tau > 0$. Therefore, the sequence $\{x_n\}$ is a Cauchy sequence and since (X, S) is a complete S-metric space, it converges to some point, say u from X. By definition, this is written as $S(x_n, x_n, u) \to 0$ when $n \to +\infty$. From the continuity of the mapping T we have that $Tx_n \to Tu$ as $n \to +\infty$, which is written as $S(Tx_n, Tx_n, Tu) \to 0$ when $n \to +\infty$ or in the equivalent form $S(x_{n+1}, x_{n+1}, Tu) \to 0$ when $n \to +\infty$. The latter means that the sequence $\{x_{n+1}\} \to Tu$ when $n \to +\infty$. Due to the uniqueness of the limit within S-metric spaces, we have that $Tu = u$ is a unique fixed point of the F-contraction T on S-metric space. The proof of the Theorem is completed. □

The following theorem is a mild generalization of Theorem 2.1 and its proof is the same as the proof of Theorem 2.1.

Theorem 2.2. *If (X, S) is a complete S-metric spaces and if there exists $\tau > 0$ such that*

$$\tau + F(S(Tx, Tx, Ty)) \leqslant F(S(x, x, y)),$$

whenever $S(Tx, Tx, Ty) > 0$, where $F : [0, +\infty) \to \mathbb{R}$ is a strictly increasing function, then the mapping T has a unique fixed point in X. Moreover, if x is an arbitrary point in X, then the Picard's sequence $T^n x$ converges to this fixed point.

Corollary 2.1. *By putting $F = \ln$ in the Theorems 2.1 and 2.2 we get one type of Banach contraction principle within S-metric spaces. It reads*

$$0 < S(Tx, Ty, Tz) \leqslant e^{-\tau} S(x, y, z) \text{ and } 0 < S(Tx, Tx, Ty) \leqslant e^{-\tau} S(x, x, y),$$

respectively, where $\tau > 0$.

The previous Corollary shows that by choosing the function F, it is obtained that the F-contraction within S-metric spaces is also a Banach's contraction in the same framework. While the following example shows that there is an F-contraction that is not a Banach's contraction, i.e., that Wardowski's approach is also a true generalization of the Banach's contraction in this class of spaces within S-metric spaces. It is inspired by Wardowsi's example from the paper [26].

Example 2.1. *Consider the sequence $\{x_n\}_{n\in\mathbb{N}}$ as follows: $x_n = \dfrac{n(n+1)}{2}$. Let $X = \{x_n : n \in \mathbb{N}\}$ and $S(x, y, z) = |x - z| + |y - z|$. Then (X, S) is a complete S-metric space. Define the mapping $T : X. \to X$ by the formulae: $T(x_n) = x_{n-1}$ for $n > 1$ and $T(x_1) = x_1 = 1$, that is, x_1 is a fixed point of T. The mapping T with the F-contractive condition as in Theorem 2.1. is not an F-contraction with $F = ln$ (which means that T is not the*

Banach's contraction). Indeed, we check it. Since, $x_n \neq x_m$ whenever $n \neq m$, we have for that case $S(Tx_n, Tx_n, Tx_1) > 0$. Therefore, for $n > 1$ we get

$$\lim_{n\to+\infty} \frac{S(Tx_n, Tx_n, Tx_1)}{S(x_n, x_n, x_1)} = \lim_{n\to+\infty} \frac{|x_{n-1}-1| + |x_{n-1}-1|}{|x_n - x_1| + |x_n - x_1|} =$$

$$= \lim_{n\to+\infty} \frac{|x_{n-1}-1|}{|x_n-1|} = \lim_{n\to+\infty} \frac{\frac{(n-1)n}{2}-1}{\frac{n(n+1)}{2}-1} = 1.$$

This means that the condition

$$\tau + \ln(S(Tx_n, Tx_n, Tx_1)) \leqslant \ln(S(x_n, x_n, x_1)),$$

that is, the condition

$$\ln \frac{S(Tx_n, Tx_n, Tx_1)}{S(x_n, x_n, x_1)} \leqslant e^{-\tau}$$

is not possible for sufficient large n and any positive τ

Assuming now that $F(\alpha) = \alpha + \ln\alpha$, we obtain, according to Theorem 2.1 that Wardowski F-contraction in the framework of S-metric spaces is a true generalization of Banach contraction principle in the same framework. And in this case we have that $x_n \neq x_m$ whenever $n \neq m$, that is, $S(Tx_n, Tx_n, Tx_m) > 0$. Therefore, if $\tau = e^{-1}$ we check the following relation

$$e^{-1} + S(Tx_n, Tx_n, Tx_m) + \ln(S(Tx_n, Tx_n, Tx_m)) \leqslant S(x_n, x_n, x_m) + \ln(S(x_n, x_n, x_m)),$$

or equivalently,

$$\ln \frac{S(Tx_n, Tx_n, Tx_m)}{S(x_n, x_n, x_m)} \leqslant S(x_n, x_n, x_m) - S(Tx_n, Tx_n, Tx_m) - e^{-1},$$

that is,

$$\frac{S(Tx_n, Tx_n, Tx_m)}{S(x_n, x_n, x_m)} e^{S(Tx_n,Tx_n,Tx_m)-S(x_n,x_n,x_m)} \leqslant e^{-1}.$$

Since $Tx_n = x_{n-1}, Tx_m = x_{m-1}$ whenever both $n > m > 1$ and $Tx_1 = x_1$ the last inequality become

$$\frac{|x_{n-1} - x_{m-1}|}{|x_n - x_m|} e^{|x_{n-1}-x_{m-1}|-|x_n-x_m|} \leqslant e^{-1}.$$

We further get

$$\frac{\frac{(n-1)n}{2} - \frac{(m-1)m}{2}}{\frac{n(n+1)}{2} - \frac{m(m+1)}{2}} e^{\left|\frac{(n-1)n-(m-1)m}{2}\right| - \left|\frac{n(n+1)-m(m+1)}{2}\right|} =$$

$$= \frac{n+m-1}{n+m+1} e^{-(n-m)} \leqslant e^{-1},$$

which is true, because $\frac{n+m-1}{n+m+1} < 1$ as well as from $n > m$ we get $n - m \geqslant 1$, i.e., $e^{-(n-m)} \leqslant e^{-1}$.

About corrected results. Just as in Theorems 2.1 and 2.2, using Lemmas 1.1 and 1.2, we managed to get rid of the application of the properties **F2** and **F3** in the proofs, so in the rest of this paper we will describe (state) the steps of the proof of Theorem 1.1 using only the property **F1**. The price that for what we pay for is adding transitivity to the mapping $\beta : X^3 \to [0, +\infty)$. The same procedure applies to the correction of the evidence of the results mentioned in **Remark 1.1.** It is well known that in the presence of a β-admissible mapping, a possible fixed point is not necessarily unique. That is why we only approach proving its existence. The first step in this proof would be to prove, starting from a given point x_0 with properties **(T2)** and **(T1)**, the existence of a sequence $\{x_n\}$ such that $\beta(x_n, x_n, x_{n+1}) \geqslant 1$ for each n from $\mathbb{N}$.

Using this obtained relation from the given contractive condition, it is obtained for the sequence x_n that the sequence $S(x_n, x_n, x_{n+1})$ is strictly decreasing. From there, according to the property about the left and right limits of the strictly increasing function F, it is obtained that $S(x_n, x_n, x_{n+1})$ tends to zero when n tends to $+\infty$. According to Lemma 1.1, it follows from the strictly decreasing sequence $S(x_n, x_n, x_{n+1})$ that all members of the sequence x_n are mutually different. Therefore, the conditions have been met to apply Lemma 1.2. to prove that the sequence $\{x_n\}$ is Cauchy. Of course, to eliminate the sequence $\beta(x_n, x_n, x_{n+1})$ the Lemma 1.3 is used. And then, in the last step, by letting k tends to $+\infty$, as in the proof of Theorems 2.1 and 2.2, we get a contradiction with $\tau > 0$. The rest of the proof is further as in Theorems 2.1. This method can therefore be used to improve (shorten) all the proofs of the results mentioned in the works from the list of references. To conclude: if any β-admissible function does not participate in the contractive condition, the fixed point is unique and it is sufficient only for the function **F** to assume the property **F1**. If a β-admissible function is present, with the addition that it is transitive, it is again sufficient to assume only the property **F1**, but then the fixed point does not have to be unique. □

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Некоторые замечания и исправления недавних результатов из теории S-метрических пространств

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Аннотация. Содержание этой статьи состоит из результатов по F-сжатию Вардовского в S-метрических пространствах. В ней мы представляем поправки к некоторым недавним результатам, используя только свойство F1 строгого возрастания функции F. В наших результатах мы объединяем β-допустимые функции с F-сжатиями. Наконец, мы приводим пример, показывающий, что F-сжатие в рамках S-метрических пространств является истинным обобщением принципа сжатия Банаха в тех же рамках.

Ключевые слова: S-метрическое пространство, b-метрическое пространство, неподвижная точка, F-сокращение, β-допустимо.