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Novel Results on Positive Solutions for Nonlinear Caputo-Hadamard Fractional Volterra-Fredholm Integro Differential Equations

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Abstract. In this paper, we establish the existence and uniqueness of positive solutions for fractional Volterra-Fredholm integro-differential equation. This equation incorporates Caputo–Hadamard fractional derivatives and is defined with initial conditions. Our proof methodology relies on the Schauder fixed point theorem, the Banach contraction principle, upper and lower solution concepts, and their applications. To illustrate the significance of our theoretical findings, we also present a compelling example.

Keywords: fractional Volterra–Fredholm integro-differential equation, positive solutions, fixed point method

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1. Introduction and preliminaries

Fractional calculus introduces the extension of derivative and integral concepts to non-integer orders, representing a relatively recent area of exploration. Noteworthy contributions in this domain have been made by researchers like Kilbas et al. [17] and Podlubny [23], among others. The investigation of equations involving fractional differentiation and integration holds particular significance due to their broad applicability in various scientific and technological fields, spanning both natural and engineering domains. It's worth mentioning that many researchers have focused

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on studying the positivity properties of solutions for these equations, as evidenced by numerous references [1–6, 8, 9, 13–16] and the citations therein.

In recent times, the examination of Hadamard Fractional Differential Equations (Hadamard FDEs) has gained considerable significance. Notably, there has been substantial progress in the understanding of Hadamard derivatives in the context of differential equations. For a comprehensive exploration of the Hadamard fractional derivative, please refer to the following sources: [7, 10, 12, 18, 19].

In [20], we study the solutions of the nonlinear fractional differential equation involving the Caputo–Hadamard operator.

This paper study of the existence and uniqueness of positive solutions of the fractional Caputo–Hadamard nonlinear Volterra–Friedholm integrol-differential equations,

$$ {}^{C}\mathfrak{D}_1^{\mathfrak{w}}\mathfrak{u}(\mathfrak{r}) = \kappa(\mathfrak{r},\mathfrak{u}(\mathfrak{r})) + \int_1^{\mathfrak{r}} \mathbb{k}_0(\mathfrak{r},\varpi,\mathfrak{u}(\varpi))d\varpi + \int_1^{\xi} \mathbb{k}_1(\mathfrak{r},\varpi,\mathfrak{u}(\varpi))d\varpi + \mathfrak{D}_1^{\mathfrak{w}-1}\hbar(\mathfrak{r},\mathfrak{u}(\mathfrak{r})), \quad \mathfrak{r}\in\psi. \tag{1} $$

The Initial Conditions

$$ \mathfrak{u}(1) = \lambda_0, \quad \mathfrak{u}'(1) = \lambda_1 > 0, \quad \psi = [1,\xi] \tag{2} $$

where $1 < \mathfrak{w} \leqslant 2$, $\kappa : \psi \times \mathfrak{N} \times \mathfrak{N} \times \mathfrak{N} \to \mathfrak{N}$, and two additional functions $\hbar : \psi \times \mathfrak{N} \to \mathfrak{N}$ and $\zeta : \psi \to \mathfrak{N}$ are introduced as continuous functions. It is important to note that $\hbar$ exhibits a non-decreasing behavior on the set $\mathfrak{u}$ and $\lambda_1 \geqslant \hbar(1,\lambda_0)$.

This paper is organized as follows. In Sect. 2., we introduce fundamental definitions and results. In Sect. 3., we present the existence and uniqueness of positive solution for problem (1)–(2). In Sect. 4., we provide an example to illustrate our results.

2. Auxiliary results

Before presenting our primary results, we offer the essential definitions, preliminary details and assumptions that will be employed in our subsequent discourse. For see [11, 21, 22, 24–30].

Consider the set ψ defined as $\psi = (1,\xi]$. Let $\mathbb{C}(\psi)$ represent the Banach space comprising all continuous functions defined on ψ, equipped with the norm defined as:

$$ \|\mathfrak{u}\| = \sup\{|\mathfrak{u}(\mathfrak{r})| : \mathfrak{r}\in\psi\}. $$

Furthermore, let $\mathfrak{B}$ be a nonempty closed subset of $\mathbb{C}(\psi)$, which can be defined as:

$$ \mathfrak{B} = \{\mathfrak{u}(\mathfrak{r}) \in \mathbb{C}(\psi) : \mathfrak{u}(\mathfrak{r}) \geqslant 0, \quad \forall\mathfrak{r}\in\psi\}. $$

Definition 2.1 ([13]). *The Hadamard fractional integral of order* $\mathfrak{w}$ *, is defined as*

$$ \mathfrak{I}_1^{\mathfrak{w}}\mathfrak{u}(\mathfrak{r}) = \frac{1}{\Gamma(\mathfrak{w})}\int_1^{\mathfrak{r}} \left(\ln\frac{\mathfrak{r}}{\varpi}\right)^{\mathfrak{w}-1}\mathfrak{u}(\varpi)\frac{d\varpi}{\varpi}, \ \mathfrak{w} > 0. \tag{3} $$

Definition 2.2 ([27]). *The definition of the Caputo–Hadamard fractional derivative of order* $\mathfrak{w}$ *is given, where* $\mathfrak{u} : [1,\infty) \longrightarrow \mathfrak{N}$.

$$ \mathfrak{D}_1^{\mathfrak{w}}\mathfrak{u}(\mathfrak{r}) = \frac{1}{\Gamma(\upsilon-\mathfrak{w})}\int_1^{\mathfrak{r}} \left(\ln\frac{\mathfrak{r}}{\varpi}\right)^{\upsilon-\mathfrak{w}-1}\left(\mathfrak{r}\frac{d}{d\mathfrak{r}}\right)^{\upsilon}\mathfrak{u}(\varpi)\frac{d\varpi}{\varpi}, \ \upsilon-1 < \mathfrak{w} < \upsilon. \tag{4} $$

Lemma 2.3 ([27]). *Let* $\upsilon - 1 < \mathfrak{w} \leqslant \upsilon$, $\upsilon \in \mathbb{N}$ *and* $\mathfrak{u} \in \mathbb{C}^{\upsilon}(\psi)$. *Then the Caputo–Hadamard fractional differential equation*

$$(\mathfrak{I}^{\mathfrak{w}}\mathfrak{D}^{\mathfrak{w}}\mathfrak{u})(\mathfrak{r}) = \mathfrak{u}(\mathfrak{r})$$

$$(\mathfrak{I}_1^{\mathfrak{w}}\mathfrak{D}_1^{\mathfrak{w}}\mathfrak{u})(\mathfrak{r})) = \mathfrak{u}(\mathfrak{r}) - \sum_{j=0}^{\upsilon-1} \frac{\mathfrak{u}^j(1)}{\Gamma(j+1)}(\ln \mathfrak{r})^j.$$

Theorem 2.4 (Schauder's [28]). *Consider a non-empty, closed, and convex subset* Ω *within a Banach space denoted as* $\mathfrak{s}$. *Let* $\aleph : \Omega \to \Omega$ *be a continuous, compact operator. In such a scenario, it can be asserted that* $\aleph$ *possesses a fixed point within the set* Ω.

Theorem 2.5 ("Banach's fixed point theorem" [28]). *Let* Ω *be a non-empty complete metric space and* $\kappa : \Omega \to \Omega$, *is contraction mapping. Then, there exists a unique point* $\varpi \in \Omega$ *such that* $\Phi(\varpi) = \varpi$.

Definition 2.6. *Let* $\mathfrak{a}, \mathfrak{a}_0 \in \mathfrak{N}^+$, *and* $\mathfrak{a}_0 > \mathfrak{a}$ *For any* $\mathfrak{u} \in [\mathfrak{a}, \mathfrak{a}_0]$, *we define the upper-control function* $\mathbb{U}(\mathfrak{r}, \mathfrak{u}) = \sup_{\mathfrak{d} \leqslant \tau \leqslant \mathfrak{u}} \kappa(\mathfrak{r}, \tau)$ *and lower-control function* $\mathbb{L}(\mathfrak{r}, \mathfrak{u}) = \inf_{\mathfrak{u} \leqslant \tau \leqslant \mathfrak{d}_0} \kappa(\mathfrak{r}, \tau)$ *Obviously,* $\mathbb{U}(\mathfrak{r}, \mathfrak{u})$ *and* $\mathbb{L}(\mathfrak{r}, \mathfrak{u})$ *are monotonous non-decreasing on* $[\mathfrak{a}, \mathfrak{a}_0]$ *and*

$$\mathbb{L}(\mathfrak{r}, \mathfrak{u}) \leqslant \kappa(\mathfrak{r}, \mathfrak{u}) \leqslant \mathbb{U}(\mathfrak{r}, \mathfrak{u})$$

3. Principal findings

In this section, we will present the results pertaining to the existence and uniqueness of Eq. (1), subject to the condition (2). Prior to delving into the proof of our primary findings, we will introduce the following set of hypotheses:

(Δ_1) Let $\mathfrak{u}^*, \mathfrak{u}_* \in \mathfrak{B}$ such that $\mathfrak{a} \leqslant \mathfrak{u}_*(\mathfrak{r}) \leqslant \mathfrak{u}_*(\mathfrak{r}) \leqslant \mathfrak{a}_0$ such that

$$\mathfrak{D}^{\mathfrak{w}}\mathfrak{u}^*(\mathfrak{r}) - \int_1^{\mathfrak{r}} \mathbb{k}_0(\mathfrak{r}, \varpi, \mathfrak{u}^*(\varpi))d\varpi - \int_1^{\xi} \mathbb{k}_1(\mathfrak{r}, \varpi, \mathfrak{u}^*(\varpi))d\varpi - \mathfrak{D}^{\mathfrak{w}-1}\hbar(\mathfrak{r}, \mathfrak{u}^*(\mathfrak{r})) \geqslant \mathbb{U}(\mathfrak{r}, \mathfrak{u}^*(\mathfrak{r}))$$

$$\mathfrak{D}^{\mathfrak{w}}\mathfrak{u}_*(\mathfrak{r}) - \int_1^{\mathfrak{r}} \mathbb{k}_0(\mathfrak{r}, \varpi, \mathfrak{u}_*(\varpi))d\varpi - \int_1^{\xi} \mathbb{k}_1(\mathfrak{r}, \varpi, \mathfrak{u}_*(\varpi))d\varpi - \mathfrak{D}^{\mathfrak{w}-1}\hbar(\mathfrak{r}, \mathfrak{u}_*(\mathfrak{r})) \leqslant \mathbb{L}(\mathfrak{r}, \mathfrak{u}_*(\mathfrak{r}))$$

(Δ_2) There exist positive constants $\delta_\kappa, \delta_{\mathbb{k}_0}, \delta_{\mathbb{k}_1}$ and $\delta_\hbar$ such that

$$\|\kappa(\mathfrak{r}, \mathfrak{u}(\mathfrak{r})) - \kappa(\mathfrak{r}, \mathfrak{u}_0(\mathfrak{r}))\| \leqslant \delta_\kappa \|\mathfrak{u} - \mathfrak{u}_0\|, \ \mathfrak{r} \in \psi, \ \mathfrak{u}, \ \mathfrak{u}_0 \in \mathfrak{N}$$

$$\|\hbar(\mathfrak{r}, \mathfrak{u}(\mathfrak{r})) - \hbar(\mathfrak{r}, \mathfrak{u}_0(\mathfrak{r}))\| \leqslant \delta_\hbar \|\mathfrak{u} - \mathfrak{u}_0\|, \ \mathfrak{r} \in \psi, \ \mathfrak{u}, \ \mathfrak{u}_0 \in \mathfrak{N}$$

$$\|\mathbb{k}_0(\mathfrak{r}, \varpi, \mathfrak{u}(\varpi)) - \mathbb{k}_0(\mathfrak{r}, \varpi, \mathfrak{u}_0(\varpi))\| \leqslant \delta_{\mathbb{k}_0} \|\mathfrak{u} - \mathfrak{u}_0\|$$

$$\|\mathbb{k}_1(\mathfrak{r}, \varpi, \mathfrak{u}(\varpi)) - \mathbb{k}_1(\mathfrak{r}, \varpi, \mathfrak{u}_0(\varpi))\| \leqslant \delta_{\mathbb{k}_1} \|\mathfrak{u} - \mathfrak{u}_0\|, \quad (\mathfrak{r}, \varpi) \in \mathcal{G}, \quad \mathfrak{u}, \ \mathfrak{u}_0 \in \mathfrak{N}$$

where $\mathcal{G} = \{(\mathfrak{r}, \varpi) : \ 0 \leqslant \varpi \leqslant \mathfrak{r} \leqslant \xi\}$.

The functions $\mathfrak{u}^*$ and $\mathfrak{u}_*$ are respectively called the pair of upper and lower solutions for the problem (1)–(2).

Lemma 3.1. *If* $\mathfrak{u} \in \mathbb{C}(\psi)$, $\mathfrak{u}^{(2)}$ *and* $\dfrac{\partial \hbar}{\partial \mathfrak{r}}$ *exists, then* $\mathfrak{u}$ *is a solution to problem* (1)–(2) *if and only if*

$$\begin{aligned} \mathfrak{u}(\mathfrak{r}) \ = \ & \frac{1}{\Gamma(\mathfrak{w})} \int_1^{\mathfrak{r}} \big(\ln \frac{\mathfrak{r}}{\mathfrak{q}}\big)^{\mathfrak{w}-1} \Big[\kappa(\mathfrak{q}, \mathfrak{u}(\mathfrak{q})) + \int_1^{\mathfrak{q}} \mathbb{k}_0(\mathfrak{q}, \varpi, \mathfrak{u}(\varpi))d\varpi + \int_1^{\xi} \mathbb{k}_1(\mathfrak{q}, \varpi, \mathfrak{u}(\varpi))d\varpi\Big] \frac{d\mathfrak{q}}{\mathfrak{q}} + \\ + \ & \lambda_0 + (\lambda_1 - \hbar(1, \lambda_0)) \ln \mathfrak{r} + \int_1^{\mathfrak{r}} \hbar(\mathfrak{q}, \mathfrak{u}(\mathfrak{q})) \frac{d\mathfrak{q}}{\mathfrak{q}} \end{aligned} \tag{5}$$

Proof. Let $\mathfrak{u}(\mathfrak{r})$ be a solution of (1)–(2). First we write this equation as

$$\mathfrak{I}_1^{\mathfrak{w}}\mathfrak{D}_1^{\mathfrak{w}}\mathfrak{u}(\mathfrak{r}) = \mathfrak{I}_1^{\mathfrak{w}}\Big[\kappa(\mathfrak{r},\mathfrak{u}(\mathfrak{r})) + \int_1^{\mathfrak{r}} \mathbb{k}_0(\mathfrak{r},\varpi,\mathfrak{u}(\varpi))d\varpi + \int_1^{\xi} \mathbb{k}_1(\mathfrak{r},\varpi,\mathfrak{u}(\varpi))d\varpi + \mathfrak{D}^{\mathfrak{w}-1}\hbar(\mathfrak{r},\mathfrak{u}(\mathfrak{r}))\Big]$$

In view of Lemma 2.3, we get

$$\begin{aligned}
\mathfrak{u}(\mathfrak{r}) &= \mathfrak{u}(1) + \mathfrak{u}'(1)\ln\mathfrak{r} + \mathfrak{I}_1^{\mathfrak{w}}\mathfrak{D}^{\mathfrak{w}-1}\hbar(\mathfrak{r},\mathfrak{u}(\mathfrak{r})) + \\
&\quad + \mathfrak{I}_1^{\mathfrak{w}}\Big[\kappa(\mathfrak{r},\mathfrak{u}(\mathfrak{r})) + \int_1^{\mathfrak{r}} \mathbb{k}_0(\mathfrak{r},\varpi,\mathfrak{u}(\varpi))d\varpi + \int_1^{\xi} \mathbb{k}_1(\mathfrak{r},\varpi,\mathfrak{u}(\varpi))d\varpi\Big] = \\
&\quad = \mathfrak{I}^1\mathfrak{I}_1^{\mathfrak{w}-1}\mathfrak{D}^{\mathfrak{w}-1}\hbar(\mathfrak{r},\mathfrak{u}(\mathfrak{r})) + \\
&\quad + \mathfrak{I}_1^{\mathfrak{w}}\Big[\kappa(\mathfrak{r},\mathfrak{u}(\mathfrak{r})) + \int_1^{\mathfrak{r}} \mathbb{k}_0(\mathfrak{r},\varpi,\mathfrak{u}(\varpi))d\varpi + \int_1^{\xi} \mathbb{k}_1(\mathfrak{r},\varpi,\mathfrak{u}(\varpi))d\varpi\Big] + \\
&\quad + \mathfrak{I}^1\big(\hbar(\mathfrak{r},\mathfrak{u}(\mathfrak{r}) - \hbar(1,\mathfrak{u}(1))\big) + \\
&\quad + \mathfrak{I}_1^{\mathfrak{w}}\Big[\kappa(\mathfrak{r},\mathfrak{u}(\mathfrak{r})) + \int_1^{\mathfrak{r}} \mathbb{k}_0(\mathfrak{r},\varpi,\mathfrak{u}(\varpi))d\varpi + \int_1^{\xi} \mathbb{k}_1(\mathfrak{r},\varpi,\mathfrak{u}(\varpi))d\varpi\Big] = \\
&\quad = \mathfrak{u}(1) + \mathfrak{u}'(1)\ln\mathfrak{r} - \hbar(1,\mathfrak{u}(1))\ln\mathfrak{r} + \int_1^{\mathfrak{r}} \hbar(\mathfrak{q},\mathfrak{u}(\mathfrak{q}))\frac{d\mathfrak{q}}{\mathfrak{q}} + \\
&\quad + \frac{1}{\Gamma(\mathfrak{w})}\int_1^{\mathfrak{r}} \Big(\ln\frac{\mathfrak{r}}{\mathfrak{q}}\Big)^{\mathfrak{w}-1}\Big[\kappa(\mathfrak{q},\mathfrak{u}(\mathfrak{q})) + \int_1^{\mathfrak{q}} \mathbb{k}_0(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))d\varpi + \int_1^{\xi} \mathbb{k}_1(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))d\varpi\Big]\frac{d\mathfrak{q}}{\mathfrak{q}} = \\
&\quad = \frac{1}{\Gamma(\mathfrak{w})}\int_1^{\mathfrak{r}} \Big(\ln\frac{\mathfrak{r}}{\mathfrak{q}}\Big)^{\mathfrak{w}-1}\Big[\kappa(\mathfrak{q},\mathfrak{u}(\mathfrak{q})) + \int_1^{\mathfrak{q}} \mathbb{k}_0(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))d\varpi + \int_1^{\xi} \mathbb{k}_1(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))d\varpi\Big]\frac{d\mathfrak{q}}{\mathfrak{q}} + \\
&\quad + \lambda_0 + (\lambda_1 - \hbar(1,\lambda_0))\ln\mathfrak{r} + \int_1^{\mathfrak{r}} \hbar(\mathfrak{q},\mathfrak{u}(\mathfrak{q}))\frac{d\mathfrak{q}}{\mathfrak{q}}
\end{aligned}$$

Conversely, suppose $\mathfrak{u}$ satisfies (5), then applying ${}^{C}\mathfrak{D}^{\mathfrak{w}}$ to both sides of(5), we obtain

$$\begin{aligned}
\mathfrak{D}_1^{\mathfrak{w}}\mathfrak{u}(\mathfrak{r}) &= \mathfrak{D}_1^{\mathfrak{w}}\Bigg(\frac{1}{\Gamma(\mathfrak{w})}\int_1^{\mathfrak{r}} \Big(\ln\frac{\mathfrak{r}}{\mathfrak{q}}\Big)^{\mathfrak{w}-1}\Big[\kappa(\mathfrak{q},\mathfrak{u}(\mathfrak{q})) + \int_1^{\mathfrak{q}} \mathbb{k}_0(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))d\varpi + \int_1^{\xi} \mathbb{k}_1(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))d\varpi\Big]\frac{d\mathfrak{q}}{\mathfrak{q}} + \\
&\quad + \int_1^{\mathfrak{r}} \hbar(\mathfrak{q},\mathfrak{u}(\mathfrak{q}))\frac{d\mathfrak{q}}{\mathfrak{q}} + \lambda_0 + (\lambda_1 - \hbar(1,\lambda_0))\ln\mathfrak{r}\Bigg) = \\
&\quad = \mathfrak{D}_1^{\mathfrak{w}}\mathfrak{I}_1^{\mathfrak{w}}\Big[\kappa(\mathfrak{q},\mathfrak{u}(\mathfrak{q})) + \int_1^{\mathfrak{q}} \mathbb{k}_0(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))d\varpi + \int_1^{\xi} \mathbb{k}_1(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))d\varpi\Big] + \mathfrak{D}_1^{\mathfrak{w}}\mathfrak{I}_1^{\mathfrak{w}}\hbar(\mathfrak{q},\mathfrak{u}(\mathfrak{q}))\frac{d\mathfrak{q}}{\mathfrak{q}} + \\
&\quad + \mathfrak{D}_1^{\mathfrak{w}}\Big(\lambda_0 + (\lambda_1 - \hbar(1,\lambda_0))\ln\mathfrak{r}\Big) = \\
&\quad = \kappa(\mathfrak{r},\mathfrak{u}(\mathfrak{r})) + \int_1^{\mathfrak{r}} \mathbb{k}_0(\mathfrak{r},\varpi,\mathfrak{u}(\varpi))d\varpi + \int_1^{\xi} \mathbb{k}_1(\mathfrak{r},\varpi,\mathfrak{u}(\varpi))d\varpi + \mathfrak{D}_1^{\mathfrak{w}-1}\hbar(\mathfrak{r},\mathfrak{u}(\mathfrak{r}))
\end{aligned}$$

Moreover, the initial conditions $\mathfrak{u}(1) = \lambda_0$, and $\mathfrak{u}'(1) = \lambda_1$ hold. This completes the proof □

To transform (5) for compatibility with Schauder's fixed point theorem, we introduce the operator $\aleph : \Omega \longrightarrow \Omega$ as follows:

$$\begin{aligned}
(\aleph\mathfrak{u})(\mathfrak{r}) &= \frac{1}{\Gamma(\mathfrak{w})}\int_1^{\mathfrak{r}} \Big(\ln\frac{\mathfrak{r}}{\mathfrak{q}}\Big)^{\mathfrak{w}-1}\Big[\kappa(\mathfrak{q},\mathfrak{u}(\mathfrak{q})) + \int_1^{\mathfrak{q}} \mathbb{k}_0(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))d\varpi + \int_1^{\xi} \mathbb{k}_1(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))d\varpi\Big]\frac{d\mathfrak{q}}{\mathfrak{q}} + \\
&\quad + \int_1^{\mathfrak{r}} \hbar(\mathfrak{q},\mathfrak{u}(\mathfrak{q}))\frac{d\mathfrak{q}}{\mathfrak{q}} + \lambda_0 + (\lambda_1 - \hbar(1,\lambda_0))\ln\mathfrak{r} \qquad (6)
\end{aligned}$$

Theorem 3.2. *Assuming that conditions* $(\Delta_1)-(\Delta_2)$ *are satisfied, it can be deduced that there is at least one positive solution to the problem described by equations* (1)–(2).

Proof. Consider the set Φ defined as follows: $\Phi=\{\mathfrak{u}\in\mathfrak{B}:\mathfrak{u}_*(\mathfrak{r})\leqslant\mathfrak{u}(\mathfrak{r})\leqslant\mathfrak{u}^*(\mathfrak{r}),\ \mathfrak{r}\in\psi\}$. This set is equipped with the norm $|\mathfrak{u}|=\{\max_{\mathfrak{r}\in\psi}|\mathfrak{u}(\mathfrak{r})|:|\mathfrak{u}|\leqslant\ell\}$. As a result, we have that $\aleph$ represents a convex, bounded, and closed subset of the Banach space $\mathbb{C}(\psi)$. It's worth noting that the continuity of the operator $\aleph$ can be inferred from the continuity of the functions $\kappa,\hbar_0,\hbar_1$, and $\hbar$. Additionally, if $\mathfrak{u}\in\Phi$, it implies the existence of positive constants $\varUpsilon_\kappa,\varUpsilon_{\Bbbk_0},\varUpsilon_{\Bbbk_1}$, and $\varUpsilon_\hbar$. such that

$$\max\{\kappa(\mathfrak{r},\mathfrak{u}(\mathfrak{r})):\mathfrak{u}(\mathfrak{r})\leqslant\ell\}\leqslant\varUpsilon_\kappa$$
$$\max\{\hbar(\mathfrak{r},\mathfrak{u}(\mathfrak{r})):\mathfrak{u}(\mathfrak{r})\leqslant\ell\}\leqslant\varUpsilon_\hbar$$
$$\max\{\Bbbk_0(\mathfrak{r},\varpi,\mathfrak{u}(\varpi)):\mathfrak{r},\ \varpi\in\psi,\mathfrak{u}(\varpi)\leqslant\ell\}\leqslant\varUpsilon_{\Bbbk_0}$$
$$\max\{\Bbbk_1(\mathfrak{r},\varpi,\mathfrak{u}(\varpi)):\mathfrak{r},\ \varpi\in\psi,\mathfrak{u}(\varpi)\leqslant\ell\}\leqslant\varUpsilon_{\Bbbk_1}$$

Then

$$|(\aleph\mathfrak{u})(\mathfrak{r})|=\frac{1}{\Gamma(\mathfrak{w})}\int_1^{\mathfrak{r}}\Big(\ln\frac{\mathfrak{r}}{\mathfrak{q}}\Big)^{\mathfrak{w}-1}\Big[|\kappa(\mathfrak{q},\mathfrak{u}(\mathfrak{q}))|+\int_1^{\mathfrak{q}}|\Bbbk_0(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))d\varpi|+\int_1^{\xi}|\Bbbk_1(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))d\varpi|\Big]\frac{d\mathfrak{q}}{\mathfrak{q}}+$$
$$+\int_1^{\mathfrak{r}}|\hbar(\mathfrak{q},\mathfrak{u}(\mathfrak{q}))|\frac{d\mathfrak{q}}{\mathfrak{q}}+|\lambda_0+(\lambda_1-\hbar(1,\lambda_0))\ln\mathfrak{r}|\leqslant$$
$$\leqslant\frac{\varUpsilon_\kappa(\ln\xi)^{\mathfrak{w}}}{\Gamma(\mathfrak{w}+1)}+\frac{(\varUpsilon_{\Bbbk_0}+\varUpsilon_{\Bbbk_1})(\ln\xi)^{\mathfrak{w}+1}}{\Gamma(\mathfrak{w}+2)}+\lambda_0+\big(\lambda_1+\zeta^*+\varUpsilon_\hbar\big)\ln\xi$$

where $\zeta^*=|\hbar(1,\lambda_0)|$, thus

$$\|(\aleph\mathfrak{u})(\mathfrak{r})\|\leqslant\frac{\varUpsilon_\kappa(\ln\xi)^{\mathfrak{w}}}{\Gamma(\mathfrak{w}+1)}+\frac{(\varUpsilon_{\Bbbk_0}+\varUpsilon_{\Bbbk_1})(\ln\xi)^{\mathfrak{w}+1}}{\Gamma(\mathfrak{w}+2)}+\lambda_0+\big(\lambda_1+\zeta^*+\varUpsilon_\hbar\big)\ln\xi$$

Consequently, the set $\aleph(\Phi)$ is uniformly bounded.
Now, we proceed to establish the equicontinuity of $\aleph(\Phi)$. For each $\mathfrak{u}\in\Phi$. Then for $\mathfrak{r}_1,\mathfrak{r}_2\in[1.\xi]$ with $\mathfrak{r}_1<\mathfrak{r}_2$, we have

$$|(\aleph\mathfrak{u})(\mathfrak{r}_1)-(\aleph\mathfrak{u})(\mathfrak{r}_2)|=$$
$$=\frac{1}{\Gamma(\mathfrak{w})}\int_1^{\mathfrak{r}_1}\Big[\Big(\ln\frac{\mathfrak{r}_1}{\mathfrak{q}}\Big)^{\mathfrak{w}-1}-\Big(\ln\frac{\mathfrak{r}_2}{\mathfrak{q}}\Big)^{\mathfrak{w}-1}\Big]|\kappa(\mathfrak{q},\mathfrak{u}(\mathfrak{q}))|\frac{d\mathfrak{q}}{\mathfrak{q}}+\frac{1}{\Gamma(\mathfrak{w})}\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\Big(\ln\frac{\mathfrak{r}_2}{\mathfrak{q}}\Big)^{\mathfrak{w}-1}|\kappa(\mathfrak{q},\mathfrak{u}(\mathfrak{q}))|\frac{d\mathfrak{q}}{\mathfrak{q}}+$$
$$+\frac{1}{\Gamma(\mathfrak{w})}\int_1^{\mathfrak{r}_1}\Big[\Big(\ln\frac{\mathfrak{r}_1}{\mathfrak{q}}\Big)^{\mathfrak{w}-1}-\Big(\ln\frac{\mathfrak{r}_2}{\mathfrak{q}}\Big)^{\mathfrak{w}-1}\Big]\Big[\int_1^{\mathfrak{q}}|\Bbbk_0(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))|d\varpi+\int_1^{\xi}|\Bbbk_1(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))|d\varpi\Big]\frac{d\mathfrak{q}}{\mathfrak{q}}+$$
$$+\frac{1}{\Gamma(\mathfrak{w})}\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\Big(\ln\frac{\mathfrak{r}_2}{\mathfrak{q}}\Big)^{\mathfrak{w}-1}\Big[\int_1^{\mathfrak{q}}|\Bbbk_0(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))|d\varpi+\int_1^{\xi}|\Bbbk_1(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))|d\varpi\Big]\frac{d\mathfrak{q}}{\mathfrak{q}}+$$
$$+\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}|\hbar(\mathfrak{q},\mathfrak{u}(\mathfrak{q}))|\frac{d\mathfrak{q}}{\mathfrak{q}}+(\lambda_1+\zeta^*)(\ln\mathfrak{r}_2-\ln\mathfrak{r}_1)\leqslant$$
$$\leqslant\frac{\varUpsilon_\kappa}{\Gamma(\mathfrak{w}+1)}\Big[2\Big[\ln\frac{\mathfrak{r}_2}{\mathfrak{r}_1}\Big]^{\mathfrak{w}}+\big[\ln\mathfrak{r}_1\big]^{\mathfrak{w}}-\big[\ln\mathfrak{r}_2\big]^{\mathfrak{w}}\Big]+\frac{(\varUpsilon_{\Bbbk_0}+\varUpsilon_{\Bbbk_1})}{\Gamma(\mathfrak{w}+2)}\Big[2\Big[\ln\frac{\mathfrak{r}_2}{\mathfrak{r}_1}\Big]^{\mathfrak{w}+1}+\Big[\ln\mathfrak{r}_1\Big]^{\mathfrak{w}+1}-$$
$$-\big[\ln\mathfrak{r}_2\big]^{\mathfrak{w}+1}\Big]+\varUpsilon_\hbar\Big[\ln\frac{\mathfrak{r}_2}{\mathfrak{r}_1}\Big]+(\lambda_1+\zeta^*)\Big[\ln\frac{\mathfrak{r}_2}{\mathfrak{r}_1}\Big]\leqslant$$
$$\leqslant\frac{2\varUpsilon_\kappa}{\Gamma(\mathfrak{w}+1)}\Big[\ln\frac{\mathfrak{r}_2}{\mathfrak{r}_1}\Big]^{\mathfrak{w}}+\frac{2(\varUpsilon_{\Bbbk_0}+\varUpsilon_{\Bbbk_1})}{\Gamma(\mathfrak{w}+2)}\Big[\ln\frac{\mathfrak{r}_2}{\mathfrak{r}_1}\Big]^{\mathfrak{w}+1}+(\lambda_1+\zeta^*+\varUpsilon_\hbar)\Big[\ln\frac{\mathfrak{r}_2}{\mathfrak{r}_1}\Big]$$
$$\longrightarrow\ 0\quad\text{as }\mathfrak{r}_1\longrightarrow\mathfrak{r}_2$$

The convergence is independent of of $\mathfrak{u}$ within Φ, indicating that $\aleph(\Phi)$ is uniformly equicontinuous. By invoking the Arzela-Ascoli theorem, we can conclude that $\aleph : \Phi \longrightarrow \mathfrak{B}$ is a compact operator. To apply the Schauder fixed point theorem, the only remaining requirement is to demonstrate that $\aleph(\Phi) \subset \Phi$. For any $\mathfrak{u} \in \Phi$, then $\mathfrak{u}_*(\mathfrak{r}) \leqslant \mathfrak{u}(\mathfrak{r}) \leqslant \mathfrak{u}^*(\mathfrak{r})$ and by (Δ_1), we have

$$\begin{aligned}
(\aleph\mathfrak{u})(\mathfrak{r}) = &\frac{1}{\Gamma(\mathfrak{w})}\int_1^{\mathfrak{r}}\left(\ln\frac{\mathfrak{r}}{\mathfrak{q}}\right)^{\mathfrak{w}-1}\left[\kappa(\mathfrak{q},\mathfrak{u}(\mathfrak{q})) + \int_1^{\mathfrak{q}}\Bbbk_0(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))d\varpi + \int_1^{\xi}\Bbbk_1(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))d\varpi\right]\frac{d\mathfrak{q}}{\mathfrak{q}} + \\
&+ \int_1^{\mathfrak{r}}\hbar(\mathfrak{q},\mathfrak{u}(\mathfrak{q}))\frac{d\mathfrak{q}}{\mathfrak{q}} + \lambda_0 + (\lambda_1 - \hbar(1,\lambda_0))\ln\mathfrak{r} \leqslant \\
&\leqslant \frac{1}{\Gamma(\mathfrak{w})}\int_1^{\mathfrak{r}}\left(\ln\frac{\mathfrak{r}}{\mathfrak{q}}\right)^{\mathfrak{w}-1}\left[\mathbb{U}(\mathfrak{q},\mathfrak{u}(\mathfrak{q})) + \int_1^{\mathfrak{q}}\Bbbk_0(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))d\varpi + \int_1^{\xi}\Bbbk_1(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))d\varpi\right]\frac{d\mathfrak{q}}{\mathfrak{q}} + \\
&+ \int_1^{\mathfrak{r}}\hbar(\mathfrak{q},\mathfrak{u}(\mathfrak{q}))\frac{d\mathfrak{q}}{\mathfrak{q}} + \lambda_0 + (\lambda_1 - \hbar(1,\lambda_0))\ln\mathfrak{r} \leqslant \\
&\leqslant \frac{1}{\Gamma(\mathfrak{w})}\int_1^{\mathfrak{r}}\left(\ln\frac{\mathfrak{r}}{\mathfrak{q}}\right)^{\mathfrak{w}-1}\left[\mathbb{U}(\mathfrak{q},\mathfrak{u}^*(\mathfrak{q})) + \int_1^{\mathfrak{q}}\Bbbk_0(\mathfrak{q},\varpi,\mathfrak{u}^*(\varpi))d\varpi + \int_1^{\xi}\Bbbk_1(\mathfrak{q},\varpi,\mathfrak{u}^*(\varpi))d\varpi\right]\frac{d\mathfrak{q}}{\mathfrak{q}} + \\
&+ \lambda_0 + (\lambda_1 - \hbar(1,\lambda_0))\ln\mathfrak{r} + \int_1^{\mathfrak{r}}\hbar(\mathfrak{q},\mathfrak{u}^*(\mathfrak{q}))\frac{d\mathfrak{q}}{\mathfrak{q}} \leqslant \\
&\leqslant \mathfrak{u}^*(\mathfrak{r})
\end{aligned}$$

and

$$\begin{aligned}
(\aleph\mathfrak{u})(\mathfrak{r}) = &\frac{1}{\Gamma(\mathfrak{w})}\int_1^{\mathfrak{r}}\left(\ln\frac{\mathfrak{r}}{\mathfrak{q}}\right)^{\mathfrak{w}-1}\left[\kappa(\mathfrak{q},\mathfrak{u}(\mathfrak{q})) + \int_1^{\mathfrak{q}}\Bbbk_0(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))d\varpi + \int_1^{\xi}\Bbbk_1(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))d\varpi\right]\frac{d\mathfrak{q}}{\mathfrak{q}} + \\
&+ \int_1^{\mathfrak{r}}\hbar(\mathfrak{q},\mathfrak{u}(\mathfrak{q}))\frac{d\mathfrak{q}}{\mathfrak{q}} + \lambda_0 + (\lambda_1 - \hbar(1,\lambda_0))\ln\mathfrak{r} \geqslant \\
&\geqslant \frac{1}{\Gamma(\mathfrak{w})}\int_1^{\mathfrak{r}}\left(\ln\frac{\mathfrak{r}}{\mathfrak{q}}\right)^{\mathfrak{w}-1}\left[\mathbb{L}(\mathfrak{q},\mathfrak{u}(\mathfrak{q})) + \int_1^{\mathfrak{q}}\Bbbk_0(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))d\varpi + \int_1^{\xi}\Bbbk_1(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))d\varpi\right]\frac{d\mathfrak{q}}{\mathfrak{q}} + \\
&+ \int_1^{\mathfrak{r}}\hbar(\mathfrak{q},\mathfrak{u}(\mathfrak{q}))\frac{d\mathfrak{q}}{\mathfrak{q}} + \lambda_0 + (\lambda_1 - \hbar(1,\lambda_0))\ln\mathfrak{r} \geqslant \\
&\geqslant \frac{1}{\Gamma(\mathfrak{w})}\int_1^{\mathfrak{r}}\left(\ln\frac{\mathfrak{r}}{\mathfrak{q}}\right)^{\mathfrak{w}-1}\left[\mathbb{L}(\mathfrak{q},\mathfrak{u}_*(\mathfrak{q})) + \int_1^{\mathfrak{q}}\Bbbk_0(\mathfrak{q},\varpi,\mathfrak{u}_*(\varpi))d\varpi + \int_1^{\xi}\Bbbk_1(\mathfrak{q},\varpi,\mathfrak{u}_*(\varpi))d\varpi\right]\frac{d\mathfrak{q}}{\mathfrak{q}} + \\
&+ \lambda_0 + (\lambda_1 - \hbar(1,\lambda_0))\ln\mathfrak{r} + \int_1^{\mathfrak{r}}\hbar(\mathfrak{q},\mathfrak{u}_*(\mathfrak{q}))\frac{d\mathfrak{q}}{\mathfrak{q}} \geqslant \\
&\geqslant \mathfrak{u}_*(\mathfrak{r})
\end{aligned}$$

As a result, we have $\mathfrak{u}(\mathfrak{r}) \leqslant (\aleph\mathfrak{u})(\mathfrak{r}) \leqslant \mathfrak{u}^(\mathfrak{r})$, which means that $\aleph(\Phi) \subset \Phi$. In accordance with the Schauder fixed point theorem, the operator $\aleph$ possesses at least one fixed point, denoted as $\mathfrak{u} \in \Phi$. Consequently, problem (1)–(2) has at least one positive solution. □

Theorem 3.3. *Assume that* (Δ_2) *is satisfied and*

$$\Delta = \left[\frac{\delta_\kappa(\ln\xi)^{\mathfrak{w}}}{\Gamma(1+\mathfrak{w})} + \frac{(\delta_{\Bbbk_0} + \delta_{\Bbbk_1})(\ln\xi)^{1+\mathfrak{w}}}{\Gamma(2+\mathfrak{w})} + \delta_\hbar(\ln\xi)\right] < 1 \tag{7}$$

Then problem (1)–(2) *has a unique positive solution.*

Proof. It follows from Theorem 3.2 that problems(1)–(2) have at least one positive solution. Therefore, all that remains is for us to demonstrate that the operator defined in (6) is a contraction in Φ. In actuality, we have for each $\mathfrak{u}, \mathfrak{u}_0 \in \Phi$,

$$|(\aleph\mathfrak{u})(\mathfrak{r}) - (\aleph\mathfrak{u}_0)(\mathfrak{r})| = \frac{1}{\Gamma(\mathfrak{w})}\int_1^{\mathfrak{r}}\left(\ln\frac{\mathfrak{r}}{\mathfrak{q}}\right)^{\mathfrak{w}-1}\left|\kappa(\mathfrak{q},\mathfrak{u}(\mathfrak{q})) - \kappa(\mathfrak{q},\mathfrak{u}_0(\mathfrak{q}))\right|\frac{d\mathfrak{q}}{\mathfrak{q}} +$$

$$+ \quad \frac{1}{\Gamma(\mathfrak{w})}\int_1^{\mathfrak{r}}\left(\ln\frac{\mathfrak{r}}{\mathfrak{q}}\right)^{\mathfrak{w}-1}\left(\int_1^{\mathfrak{q}}\left|\mathbb{k}_0(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))-\mathbb{k}_0(\mathfrak{q},\varpi,\mathfrak{u}_0(\varpi))\right|d\varpi+\right.$$
$$+ \quad \left.\int_1^{\xi}\left|\mathbb{k}_1(\mathfrak{q},\varpi,\mathfrak{u}(\varpi))-\mathbb{k}_1(\mathfrak{q},\varpi,\mathfrak{u}_0(\varpi))\right|d\varpi\right)\frac{d\mathfrak{q}}{\mathfrak{q}}+$$
$$+ \quad \int_1^{\mathfrak{r}}\left|\hbar(\mathfrak{q},\mathfrak{u}(\mathfrak{q}))-\hbar(\mathfrak{q},\mathfrak{u}_0(\mathfrak{q}))\right|\frac{d\mathfrak{q}}{\mathfrak{q}}\leqslant$$
$$\leqslant \quad \left[\frac{\delta_\kappa(\ln\xi)^{\mathfrak{w}}}{\Gamma(\mathfrak{w}+1)}+\frac{(\delta_{\mathbb{k}_0}+\delta_{\mathbb{k}_1})(\ln\xi)^{\mathfrak{w}+1}}{\Gamma(\mathfrak{w}+2)}+\delta_\hbar(\ln\xi)\right]\|\mathfrak{u}-\mathfrak{u}_0\|$$

The contraction $\aleph$ is derived from (7). According to Theorem 2.5, it asserts the existence of a unique fixed point for the equation $\aleph$, which corresponds to the sole positive solution of the equations (1)–(2). With this, we conclude the proof. □

4. An application

As an application of our result, With an integral boundary condition, we examine the fractional Volterra-Fredholm integro-differential equation as follows:

$$\mathfrak{D}_1^{\frac{4}{3}}\mathfrak{u}(\mathfrak{r})-\mathfrak{D}_1^{\frac{1}{3}}\left[\frac{\mathfrak{u}(\mathfrak{r})}{3e^{\mathfrak{r}-1}}\right]=\frac{\cos(\mathfrak{r})}{9+e^{\mathfrak{r}^2-1}}\left(\frac{\mathfrak{u}(\mathfrak{r})}{|\mathfrak{u}|+1}\right)+\frac{1}{5}\int_1^{\mathfrak{r}}e^{-2(\varpi^2-\mathfrak{r}^2)}\mathfrak{u}(\varpi)d\varpi+\int_1^{e}\frac{e^{-\varpi^2\mathfrak{r}}}{20}\mathfrak{u}(\varpi)d\varpi \tag{8}$$
$$\mathfrak{u}(1)=1,\quad \mathfrak{u}'(1)=1$$

Since κ is continuous positive functions, $\mathbb{k}_0,\mathbb{k}_1$ and $\hbar$ are non-decreasing on $\mathbb{k}$. For $\mathbb{k},\mathbb{k}_0\in\mathfrak{N}^+$ and $\mathfrak{r}\in(1,e]$ we have:

$$|\kappa(\mathfrak{r},\mathfrak{u})-\kappa(\mathfrak{r},\mathfrak{u}_0)|=\left|\frac{\cos(\mathfrak{r})}{9+e^{\mathfrak{r}^2-1}}\left(\frac{|\mathfrak{u}|}{|\mathfrak{u}|+1}-\frac{\mathfrak{u}_0}{|\mathfrak{u}_0|+1}\right)\right|\leqslant$$
$$\leqslant\frac{1}{9+e^{\mathfrak{r}^2-1}}\left(\frac{|\mathfrak{u}-\mathfrak{u}_0|}{(|\mathfrak{u}|+1)(|\mathfrak{u}_0|+1)}\right)\leqslant$$
$$\leqslant\frac{1}{10}|\mathfrak{u}-\mathfrak{u}_0|$$

$$|\hbar(\mathfrak{r},\mathfrak{u})-\hbar(\mathfrak{r},\mathfrak{u}_0)|=\left|\frac{|\mathfrak{u}|}{3e^{\mathfrak{r}-1}}-\frac{|\mathfrak{u}_0|}{3e^{\mathfrak{r}-1}}\right|\leqslant\frac{1}{3e^{\mathfrak{r}-1}}|\mathfrak{u}-\mathfrak{u}_0|\leqslant$$
$$\leqslant\frac{1}{3}|\mathfrak{u}-\mathfrak{u}_0|$$

$$|\mathbb{k}_0(\mathfrak{r},\varpi,\mathfrak{u}(\varpi))-\mathbb{k}_0(\mathfrak{r},\varpi,\mathfrak{u}_0(\varpi))|=\left|\frac{|\mathfrak{u}|}{5e^{(\varpi^2-\mathfrak{r}^2)}}\frac{|\mathfrak{u}_0|}{5e^{2(\varpi^2-\mathfrak{r}^2)}}\right|\leqslant\frac{1}{5e^{2(\varpi^2-\mathfrak{r}^2)}}|\mathfrak{u}-\mathfrak{u}_0|\leqslant$$
$$\leqslant\frac{1}{10}|\mathfrak{u}-\mathfrak{u}_0|$$

and

$$|\mathbb{k}_1(\mathfrak{r},\varpi,\mathfrak{u}(\varpi))-\mathbb{k}_1(\mathfrak{r},\varpi,\mathfrak{u}_0(\varpi))|=\left|\frac{|\mathfrak{u}|}{5e^{(\varpi^2-\mathfrak{r}^2)}}\frac{|\mathfrak{u}_0|}{10e^{(\varpi^2\mathfrak{r})}}\right|\leqslant\frac{1}{10e^{(\varpi^2\mathfrak{r})}}|\mathfrak{u}-\mathfrak{u}_0|\leqslant$$
$$\leqslant\frac{1}{20}|\mathfrak{u}-\mathfrak{u}_0|$$

Currently, the conditions $(\Delta_1) - (\Delta_3)$ is fulfilled, given that

$$\mathfrak{w} = \frac{4}{3}, \quad \xi = e, \quad \delta_\kappa = \frac{1}{10}, \quad \delta_\hbar = \frac{1}{3}, \quad \delta_{\mathbb{k}_0} = \frac{1}{10}, \quad \delta_{\mathbb{k}_1} = \frac{1}{20}.$$

subsequently, through a series of calculations, it is determined that.

$$\Delta = \left[\frac{\delta_\kappa (\ln\xi)^{\mathfrak{w}}}{\Gamma(1+\mathfrak{w})} + \frac{(\delta_{\mathbb{k}_0} + \delta_{\mathbb{k}_1})(\ln\xi)^{1+\mathfrak{w}}}{\Gamma(2+\mathfrak{w})} + \delta_\hbar(\ln\xi)\right] \cong 0.314 < 1$$

Then by Theorem 3.3, the equation (8) has a unique positive solution.

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Новые результаты по положительным решениям для нелинейных дробных производных Капуто-Адамара интегро-дифференциальных уравнений Вольтерра-Фредгольма

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Аннотация. В этой статье мы устанавливаем существование и единственность положительных решений для дробного интегро-дифференциального уравнения Вольтерра-Фредгольма. Это уравнение включает дробные производные Капуто–Адамара и определяется начальными условиями. Наша методология доказательства опирается на теорему Шаудера о неподвижной точке, принцип сокращения Банаха, концепции верхнего и нижнего решения и их приложения. Чтобы проиллюстрировать значимость наших теоретических выводов, мы также приводим убедительный пример.

Ключевые слова: дробное интегро-дифференциальное уравнение Вольтерра–Фредгольма, положительные решения, метод неподвижной точки