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A Classical Aspect of the Dirac Equation in the Context of Conformable Fractional Derivative

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Abstract. In this article, in the context of the conformable fractional derivative (CFD) and employing Ehrenfest's theorem, we investigate the classical limit of the Dirac equation within conformable fractional quantum mechanics. This leads to obtaining deformed classical equations. Here, we assess the effectiveness of Ehrenfest's theorem in deriving the classical limit considering CFD. Also, we examine the correspondence principle under the influence of CFD. Additionally, we obtain the conformable fractional continuity equation.

Keywords: conformable fractional Dirac equation, conformable fractional continuity equation, Ehrenfest's theorem, classical limit, correspondence principle, conformable quantum mechanics.

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1. Introduction and preliminaries

The Dirac equation is a relativistic quantum mechanical equation that specifically describes massive particles with spin-$\frac{1}{2}$, such as electrons. It is a fundamental equation in quantum mechanics, providing a framework for understanding the behavior of these particles within the realm of relativistic effects. The classical limit of the Dirac equation can be investigated by neglecting the influences of quantum mechanics. In doing so, we can describe the system's conduct using classical physics, providing insights into the classical aspects of the system. In the classical limit, phenomena inherent to quantum mechanics, such as interference, superposition and entanglement, are expected to diminish at the macroscopic scale, however, this demise is not easy to explain. In this scenario, the quantum system adheres to the classical laws of physic. The classical limit is commonly defined in terms of the limit of a vanishing Planck's constant, i.e., $\hbar \to 0$ as scaled with the system's action. In this context, Hamilton's principle adopts its classical expression, and all operators commute. In the following, we present some scenarios and approaches that help explain the exploration of the classical limit of the Dirac equation. So, one can initiate the exploration by examining the solutions of the equation under conditions of large distances and durations, or under the conditions of large energies and momenta. Within these limits, the effects of quantum mechanics become negligible [1]. Put differently, the classical limit emerges if the system possesses a big quantum number, undergoes significant interactions with its surroundings, or if its de Broglie wavelength becomes significantly smaller compared to other relevant length measurements. A frequent example illustrating the classical limit of a quantum system is the Bohr correspondence principle [2], which asserts that in the limit of large quantum numbers, a quantum system exhibits conduct similar to the corresponding classical

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system. Also, the Ehrenfest's theorem is considerably used when exploring the classical limit of quantum mechanical systems [3]. This theorem establishes a connection between the evolution of expected values of observables and classical equations of motion. It serves as an effective tool for understanding the conduct of such systems. Through its application, we observe the way quantum mechanical influences dissipate, giving way to classical dynamics [4]. In the context of the Dirac equation, this theorem remains used to explore its classical limit, there, the quantum influences will be very small, leading simplify the Dirac equation to its classical counterpart. In this work, we aim to investigate whether it can be asserted that Ehrenfest's theorem is applicable to the classical limit of the Dirac equation within conformable fractional quantum mechanics (CFQM) and under some conditions.

Extensive research in the literature [5–14] has delved into the alignment between quantum and classical aspects. We also emphasize that other concepts may overlap with the concept of the classical limit, such as the semiclassical and non-relativistic limits. Note that the semiclassical limit of a quantum mechanical system, can be attained if external potentials vary slowly, like in the case of the electrostatic potential [15]. On the other hand, the non-relativistic limit of a relativistic quantum mechanical system as the Dirac equation [16, 17], is the limit where the speed of the particle is much less than the speed of light, i.e., $v \ll c$ or low energy in front of the rest energy, consequently, this limit permits to neglect the relativistic influences. However, the non-relativistic and classical limits are related but distinct concepts, they address different aspects of the system's behavior. It is important to highlight that in many physical situations, the classical limit and the non-relativistic limit can align, leading to similar descriptions of the system's conduct.

On the other side, the concept of a fractional derivative has been receiving a lot of attention in recent years, [18–20]. The fractional derivative, dating back to as early as calculus itself, traces its origins to 1695 when L'Hospital posed inquiries to Leibniz about $\frac{d^n f}{dx^n}$ when n equals $\frac{1}{2}$. However, Leibniz responded that this would be "an apparent paradox, from which one day useful consequences will be drawn" [21]. Since then, researchers have endeavored to elucidate the concept of fractional derivatives, predominantly employing integral formulations. Various definitions have emerged over time, including those by Riemann–Liouville, Caputo, Riesz, Weyl, Grьnwald, Riesz–Caputo, Chen, and Hadamard, [22–24]. Among these, the Riemann–Liouville and Caputo formulations stand out as the most prevalent. For further insight into diverse mathematical aspects of fractional calculus, refer to the seminal works in [25, 26]. The fractional derivative has played an essential role across various domains including physics, chemistry, biology, and engineering. See, for example, [27–29].

However, a few years ago, Khalil et al. [30] introduced a new concept of derivative known as the conformable fractional derivative (CFD). Since then, extensive studies have been conducted on the development of CFD calculus, exploring its properties. For example, conformable fractional versions of fundamental mathematical tools such as the chain rule, exponential functions, Gronwall's inequality, integration by parts, Taylor power series expansions, Laplace or Fourier transforms, linear differential systems [31, 32], divergence theorem [33], spherical harmonics [34], Nikiforov–Uvarov Method [35] and $\mathcal{PT}$ symmetry [36] have been proposed and discussed. Also, applications in various physical contexts [37–50].

Our objective in this study is to investigate some aspects of the Dirac equation within the CFQM including classical limit by using the Ehrenfest's theorem and continuity equation. This work came as a continuation of some works on the classical and non-relativistic limits we did before [8, 12, 16, 17, 51]. As an example, in [12], we studied the classical limit and Ehrenfest's theorem of the noncommutative Dirac equation in the context of minimal uncertainty in momentum. There, we explored the comparison between the classical and non-relativistic limits. Furthermore, in [8], we have investigated Ehrenfest's theorem from the Dirac equation in a noncommutative phase-space.

The rest of the paper is outlined as follows. In Section 2, the CFQM is briefly reviewed. In Section 3, the CL of the Dirac equation in the context of CFD using the Ehrenfest's theorem is explored, where in sub-Section 3.1, a conformable fractional Dirac equation in presence of an electromagnetic field is found. In sub-Section 3.2, based on the Ehrenfest's theorem, conformable fractional classical equations are obtained. Also, in Section 4, a conformable fractional four-vector current density is obtained. Then, in Section 5, the correspondence principle is examined in the context of CFD by using quantum and classical versions of the harmonic oscillator. We present the conclusion and remarks in Section 6.

2. Brief overview on the conformable fractional quantum mechanics

Let shortly review the postulates and basic formulas of the conformable fractional quantum mechanics [30] we use in this work. So, for a smooth function in x, the conformable fractional derivative is expressed as follows:

$$D_x^\alpha f(x) = \lim_{\epsilon\to 0} \frac{f\left(x+\epsilon |x|^{1-\alpha}\right) - f(x)}{\epsilon} = |x|^{1-\alpha}\, \partial_x f(x), \tag{1}$$

where $0 < \alpha \leqslant 1$ is assumed. Note that D^α is the conformable fractional derivative operator. At x = 0, the fractional derivative is $D_x^\alpha f(0) = \lim_{x\to 0} D_x^\alpha f(x)$. But, the conformable partial derivative of f in x_i is defined by [33]:

$$\frac{\partial^\alpha}{\partial x_i^\alpha} f(x_1,\ldots,x_m) = \lim_{\epsilon\to 0} \frac{f\left(x_1,\ldots,x_{i-1},x_i+\epsilon |x_i|^{1-\alpha},\ldots,x_m\right) - f(x_1,\ldots,x_m)}{\epsilon}. \tag{2}$$

The application of the CFD in quantum mechanics is given in some literatures [37–39, 46], leading to CFQM. However, its postulates, basics and essential properties have been well constructed [30, 39, 41]. The CFD satisfies the following properties:

The linearity:

$$D_x^\alpha \{af(x) + bg(x)\} = aD_x^\alpha f(x) + bD_x^\alpha g(x), \tag{3}$$

the Leibniz rule:

$$D_x^\alpha \{f(x)\, g(x)\} = [D_x^\alpha f(x)]\, g(x) + f(x)\, D_x^\alpha g(x), \tag{4}$$

the chain rule:

$$D_x^\alpha f(g(x)) = \frac{df}{du}(D_x^\alpha u), \tag{5}$$

where f, g be α-differentiable functions. All of the classical derivative rules, such as sum, product, division, etc. are same as the conformable derivative. Also, the inner product in Hilbert space related to CFQM is given as follows:

$$\langle f|\, g\rangle = \int_{-\infty}^{\infty} g^*(x)\, f(x)\, |x|^{\alpha-1}\, dx. \tag{6}$$

The definition of the expectation value of a physical operator $\mathcal{O}$ in relation to the state ψ is as follows:

$$\langle\psi|\, \mathcal{O}\psi\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\, \mathcal{O}\psi(x,t)\, |x|^{\alpha-1}\, dx, \tag{7}$$

but $\mathcal{O}$ to be a Hermitian operator, one may obey

$$\langle\psi|\, \mathcal{O}\psi\rangle = \langle\mathcal{O}\psi|\, \psi\rangle. \tag{8}$$

The fractional integral is defined as

$$\mathcal{J}^{\alpha}_{\alpha|x} f(x) = \int_{\alpha}^{x} |\mathcal{W}|^{\alpha-1} f(\mathcal{W})\, d\mathcal{W}, \tag{9}$$

where $f(x)$ is any continuous function. Furthermore, the relation between the CFD and the fractional integral is as follows:

$$\begin{cases} D^{\alpha}_{x} \mathcal{J}^{\alpha}_{\alpha|x} f(x) = f(x), \\ \mathcal{J}^{\alpha}_{\alpha|x} D^{\alpha}_{x} f(x) = f(x) - f(a). \end{cases} \tag{10}$$

Note that the coordinate realization of α-position $\hat{x}_{\alpha}$, and α-momentum $\hat{p}_{\alpha}$ are

$$\hat{x}_{\alpha} = x, \quad \hat{p}_{\alpha} = -i\hbar^{\alpha}_{\alpha} D^{\alpha}_{x}, \tag{11}$$

here one can merely check that both position and momentum operators are Hermitian. Then, from the de Broglio relation $p = \frac{\hbar}{\lambda}$ and Planck relation $E = \frac{\hbar}{T}$, we have [39] $\hat{x}_{\alpha}\psi = x\psi$, $\hat{p}_{\alpha}\psi = p^{\alpha}\psi$, which yield the following the x-representation:

$$\hat{x}_{\alpha} = x, \quad \hat{p}_{\alpha} = -i\hbar^{\alpha}_{\alpha} D^{\alpha}_{x} \text{ and } \hat{\mathscr{H}}_{\alpha} = -i\hbar_{\alpha} D^{\alpha}_{t}, \tag{12}$$

with

$$D^{\alpha}_{x} = |x|^{1-\alpha} \frac{\partial}{\partial x}, \text{ and } D^{\alpha}_{t} = |t|^{1-\alpha} \frac{\partial}{\partial t}, \tag{13}$$

where $\hbar_{\alpha} = \frac{h}{(2\pi)^{\frac{1}{\alpha}}}$, and $\hat{\mathscr{H}}_{\alpha}$ is a α-Hamiltonian operator. Note that the α-position operator has dimension of length while the α-momentum operator has dimension of momentum$^{\alpha}$ and α-Hamiltonian operator has dimension of energy$^{\alpha}$. In CFQM, the commutator of the α-position operator and α-momentum operator is

$$[\hat{x}_{\alpha}, \hat{p}_{\alpha}] = i\hbar^{\alpha}_{\alpha} |\hat{x}|^{1-\alpha}. \tag{14}$$

...Also, in Section 4, a conformable fractional four-vector current density is obtained. Then, the correspondence principle is examined in the context of CFD by comparing the quantum and classical versions of the harmonic oscillator. Finally, we present the conclusion and remarks in Section 6."

3. Classical limit of the conformable fractional Dirac equation

In this section, we extend the Dirac equation to the CFQM and subsequently employ it to investigate some classical aspects.

3.1. 3D conformable fractional Dirac equation

The conformable fractional form of the Dirac equation is given as follows [39,46]:

$$\left\{ i\hbar^{\alpha} \gamma^{\mu} \partial^{\alpha}_{\mu} - m^{\alpha} c^{\alpha} \right\} \psi(x^{\mu}) = 0, \tag{15}$$

where $0 < \alpha \leqslant 1$, which we call the fractional and $\gamma^{\mu} = \left(\gamma^{0}, \gamma^{k}\right))$ are the Dirac matrices. Now, by multiplying equation (14) from the left by γ^{0} and separating the time and the spatial parts, one can have

$$\left\{ i\hbar^{\alpha} \left(\gamma^{0}\right)^{2} \partial^{\alpha}_{0} - i\hbar^{\alpha} \gamma^{0} \gamma^{k} \partial^{\alpha}_{k} - \gamma^{0} m^{\alpha} c^{\alpha} \right\} \psi(\overrightarrow{x}, t) = 0, \tag{16}$$

with $\left(\gamma^0\right)^2 = 1$. Then, supposing that

$$\psi\left(\overrightarrow{x}, t\right) = \psi\left(\overrightarrow{x}\right) e^{-\frac{i}{\hbar^\alpha} E^\alpha \frac{t^\alpha}{\alpha}}, \tag{17}$$

with $i\hbar^\alpha \frac{1}{c}\partial_0^\alpha = i\hbar^\alpha \frac{1}{c} D_t^\alpha$, the time-independent Dirac equation in interaction with an electromagnetic four-potential $A^\mu = \left(\Phi_\alpha, \overrightarrow{A}_\alpha\right)$ (in SI units) reads

$$\left\{c^\alpha \overrightarrow{\alpha} \cdot \left(\hat{\overrightarrow{p}}_\alpha - e\overrightarrow{A}_\alpha\left(\overrightarrow{x}\right)\right) + e\Phi_\alpha + \beta m^\alpha c^{2\alpha}\right\} \psi\left(\overrightarrow{x}\right) = E^\alpha \psi\left(\overrightarrow{x}\right), \tag{18}$$

where $\psi\left(\overrightarrow{x}, t\right) = \begin{pmatrix} \phi\left(\overrightarrow{x}, t\right) & \chi\left(\overrightarrow{x}, t\right) \end{pmatrix}^T$ is the bispinor in the Dirac representation. The Dirac matrices $\overrightarrow{\alpha} = \gamma^0 \overrightarrow{\gamma}$ and $\beta = \gamma^0$ satisfy the following anticommutation relations

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij},\ \{\alpha_i, \beta\} = 0, \quad \alpha_i^2 = \beta^2. \tag{19}$$

Then in more elegant simple form, we have

$$\left\{c^\alpha \overrightarrow{\alpha} \cdot \hat{\overrightarrow{\Pi}}_\alpha + e\Phi_\alpha + \beta m^\alpha c^{2\alpha}\right\} \psi\left(\overrightarrow{x}\right) = E^\alpha \psi\left(\overrightarrow{x}\right), \tag{20}$$

where the minimal substitution $\hat{\overrightarrow{p}}_\alpha - e\overrightarrow{A}_\alpha\left(\overrightarrow{x}\right) = \hat{\overrightarrow{\Pi}}_\alpha$. Next, we move to employ the obtained conformable fractional Dirac equation (20) to explore the classical limit through Ehrenfest's theorem.

3.2. Ehrenfest's theorem in the context of CFQM

Ehrenfest's theorem, originating from the Dirac equation, establishes that the time evolution of expected values of observables in quantum mechanics aligns with classical equations of motion. Essentially, it suggests that the average conduct of a quantum system corresponds to classical physics. Additionally, it is noteworthy that this theorem applies to all quantum systems. However, in the present context, we are computing the time derivatives of position and kinetic momentum operators for Dirac particles interacting with an electromagnetic field in the context of CFQM. However, the equation of motion for an arbitrary operator $\hat{\mathcal{F}}$ is expressed as follows:

$$\frac{d\hat{\mathcal{F}}}{dt} = \frac{\partial \hat{\mathcal{F}}}{\partial t} + \frac{i}{\hbar}\left[\hat{\mathscr{H}}, \hat{\mathcal{F}}\right], \tag{21}$$

where $\hat{\mathscr{H}}$ is the Hamiltonian operator. Now, let start with the operator of position

$$\frac{d\hat{\overrightarrow{x}}}{dt} = \frac{\partial \hat{\overrightarrow{x}}}{\partial t} + \frac{i}{\hbar}\left[\hat{\mathscr{H}}_\alpha, \hat{\overrightarrow{x}}\right] = \frac{i}{\hbar}\left[\hat{\mathscr{H}}_\alpha, \hat{\overrightarrow{x}}\right], \tag{22}$$

and the Hamiltonian operator from equation (20) is given as:

$$\hat{\mathscr{H}}_\alpha = c^\alpha \overrightarrow{\alpha} \cdot \hat{\overrightarrow{\Pi}}_\alpha + e\Phi_\alpha + \beta m^\alpha c^{2\alpha}, \tag{23}$$

subsequently, the commutator expressed in equation (22) is as follows:

$$\begin{aligned}\left[\hat{\mathscr{H}}_\alpha, \hat{\overrightarrow{x}}\right] = c^\alpha \left[\hat{\overrightarrow{\alpha}} \cdot \hat{\overrightarrow{p}}_\alpha, \hat{\overrightarrow{x}}\right] - ec^\alpha \left[\hat{\overrightarrow{\alpha}} \cdot \overrightarrow{A}_\alpha, \hat{\overrightarrow{x}}\right] + e\left[\Phi_\alpha, \hat{\overrightarrow{x}}\right] \\ + m^\alpha c^{2\alpha}\left[\hat{\beta}, \hat{\overrightarrow{x}}\right],\end{aligned} \tag{24}$$

The position operator $\hat{x}$ is diagonal with respect to the spinor indices, i.e., $\hat{\overrightarrow{x}}\psi = \overrightarrow{x}\psi$ and contains no differentiation, thus $\left[\hat{\beta}, \hat{\overrightarrow{x}}\right] = \left[\hat{\overrightarrow{\alpha}}, \hat{\overrightarrow{x}}\right] = 0$, then for three arbitrary vectors $\overrightarrow{A}_1$, $\overrightarrow{A}_2$ and $\overrightarrow{A}_3$ we use the identity

$$\left[\overrightarrow{A}_1\overrightarrow{A}_2, \overrightarrow{A}_3\right] = \left[\overrightarrow{A}_1, \overrightarrow{A}_3\right]\overrightarrow{A}_2 + \overrightarrow{A}_1\left[\overrightarrow{A}_2, \overrightarrow{A}_3\right]. \tag{25}$$

Then, we obtain

$$\left[\hat{\overrightarrow{\alpha}} \cdot \hat{\overrightarrow{p}}_\alpha, \hat{\overrightarrow{x}}\right] = -i\hbar_\alpha^\alpha \left|\hat{x}\right|^{1-\alpha} \hat{\overrightarrow{\alpha}}, \tag{26}$$

also

$$\left[\overrightarrow{A}_\alpha, \hat{\overrightarrow{x}}\right] = \left[\Phi_\alpha, \hat{\overrightarrow{x}}\right] = 0, \tag{27}$$

because both $\overrightarrow{A}_\alpha$, Φ_α are functions of x.

Therefore, we obtain

$$\frac{d\hat{\overrightarrow{x}}}{dt} = \frac{1}{\hbar}\hbar_\alpha^\alpha c^\alpha \left|\hat{x}\right|^{1-\alpha} \hat{\overrightarrow{\alpha}}. \tag{28}$$

Let us subsequently examine how the operator (28) acts on the Dirac spinor. By focusing on individual components ψ, we obtain

$$\frac{d\hat{x}}{dt}\psi = \pm\frac{1}{\hbar}\hbar_\alpha^\alpha c^\alpha \left|\hat{x}\right|^{1-\alpha} \psi, \tag{29}$$

where the eigenvalues of $\hat{\overrightarrow{\alpha}}$ are ± 1. This result has no classical analogue, as the Dirac particle, despite the effects considered, continues to move at the speed of light c^α.

So, to better understand the behaviour of a particle velocity obtained from the classical limit of Dirac equation within the framework of CFQM, we plot equation (29)

Fig. 1 represents the variation of the velocity with respect to the fractional parameter α for different values of x. But, Fig. 2 represents the variation of the velocity with respect to x for different values of the fractional parameter α. However, the velocity as a function of the fractional parameter behaves as a Gaussian where it is symmetric around the central value of the distribution ($\alpha = 0.6$). The curve peaks at the mean and decreases symmetrically on either side, i.e., zero and unity. Also, in Fig. 2, the effect of the fractional parameter on the variation of the velocity is shown, which appears to be considerable at $\alpha = 0.6$.

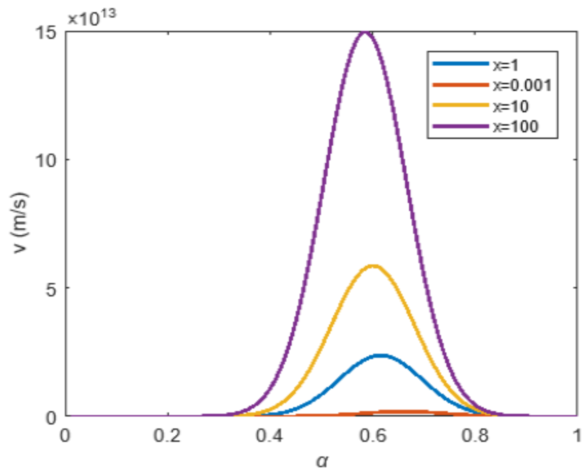

Fig. 1. Velocity versus α for x=0.001,1,10,100

Now, the equation of motion for the kinetic momentum operator $\hat{\overrightarrow{\Pi}} = \hat{\overrightarrow{p}} - \frac{e}{c}\overrightarrow{A}$ is

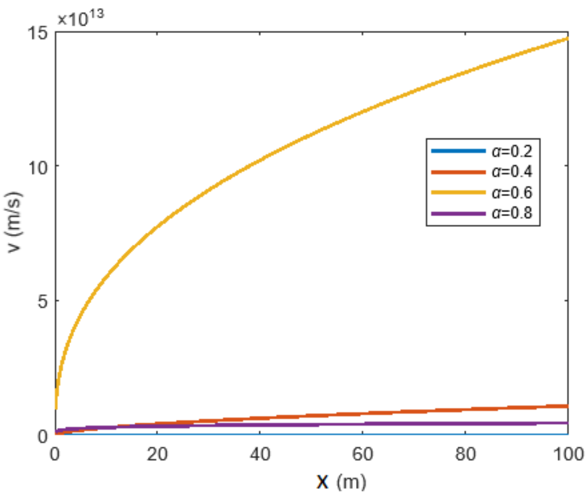

Fig. 2. Velocity versus x for $\alpha = 0.1,\, 0.4,\, 0.6,\, 0.8$

$$\frac{d\overrightarrow{\hat{\Pi}}}{dt} = \frac{\partial\overrightarrow{\hat{\Pi}}}{\partial t} + \frac{i}{\hbar}\left[\hat{\mathscr{H}}_{\alpha}, \overrightarrow{\hat{\Pi}}\right] = -e\frac{\partial\overrightarrow{A}}{\partial t} + \frac{i}{\hbar}\left[\hat{\mathscr{H}}_{\alpha}, \overrightarrow{\hat{\Pi}}\right], \tag{30}$$

consequently, the commutator is given by

$$\left[\hat{\mathscr{H}}_{\alpha}, \overrightarrow{\hat{\Pi}}\right] = \left[\hat{\mathscr{H}}_{\alpha}, \overrightarrow{\hat{p}}\right] - e\left[\hat{\mathscr{H}}_{\alpha}, \overrightarrow{A}\right]. \tag{31}$$

First, we compute the initial commutator in equation (31)

$$\begin{aligned}\left[\hat{\mathscr{H}}_{\alpha}, \overrightarrow{\hat{p}}\right] = c^{\alpha}\left[\overrightarrow{\hat{\alpha}} \cdot \overrightarrow{\hat{p}}_{\alpha}, \overrightarrow{\hat{p}}\right] - c^{\alpha}e\left[\overrightarrow{\hat{\alpha}} \cdot \overrightarrow{A}_{\alpha}, \overrightarrow{\hat{p}}\right] + e\left[\varPhi_{\alpha}, \overrightarrow{\hat{p}}\right] \\ +m^{\alpha}c^{2\alpha}\left[\hat{\beta}, \overrightarrow{\hat{p}}\right],\end{aligned} \tag{32}$$

with $\left[\hat{\beta}, \overrightarrow{\hat{p}}\right] = \left[\overrightarrow{\hat{\alpha}}, \overrightarrow{\hat{p}}\right] = 0$ as $\hat{\beta}$ and $\overrightarrow{\hat{\alpha}}$ are independent of the spatial coordinates. Furthermore, we obtain

$$\left[\varPhi_{\alpha}, \overrightarrow{\hat{p}}\right] = i\hbar\left[\overrightarrow{\nabla}, \varPhi_{\alpha}\right] = i\hbar\left(\overrightarrow{\nabla}\varPhi_{\alpha} - \varPhi_{\alpha}\overrightarrow{\nabla}\right), \tag{33}$$

then making use of equation (33), we have

$$\left[\varPhi_{\alpha}, \overrightarrow{\hat{p}}\right]\psi = i\hbar\left(\overrightarrow{\nabla}\varPhi_{\alpha} - \varPhi_{\alpha}\overrightarrow{\nabla}\right)\psi = i\hbar\left(\overrightarrow{\nabla}\varPhi_{\alpha}\right)\psi, \tag{34}$$

and

$$\left[\overrightarrow{\hat{\alpha}} \cdot \overrightarrow{\hat{p}}_{\alpha}, \overrightarrow{\hat{p}}\right] = \frac{1}{\hbar}\hbar_{\alpha}^{\alpha}\left[|\hat{x}|^{1-\alpha}, \overrightarrow{\hat{p}}\right]\overrightarrow{\hat{\alpha}} \cdot \overrightarrow{\hat{p}}. \tag{35}$$

Also

$$\left[\overrightarrow{\hat{\alpha}} \cdot \overrightarrow{A}_{\alpha}, \overrightarrow{\hat{p}}\right] = -i\hbar\sum_{i,j}\hat{\alpha}_i\left[(A_{\alpha})_i, \nabla_j\right]e_j, \tag{36}$$

then, taking into account the effect of equation (36) on ψ, we find

$$\left[\overrightarrow{\hat{\alpha}} \cdot \overrightarrow{A}_{\alpha}, \overrightarrow{\hat{p}}\right]\psi = i\hbar\sum_{i,j}\hat{\alpha}_i\left(\nabla_j(A_{\alpha})_i\psi - (A_{\alpha})_i\nabla_j\psi\right)e_j = i\hbar\sum_{i,j}\hat{\alpha}_i\left(\nabla_j(A_{\alpha})_i\right)e_j\psi. \tag{37}$$

Now, we move on to the second commutator in equation (31), thus, we have

$$\left[\mathscr{H}_\alpha, \overrightarrow{A}\right] = c^\alpha \left[\hat{\overrightarrow{\alpha}} \cdot \hat{\overrightarrow{p}}_\alpha, \overrightarrow{A}\right] - c^\alpha e \left[\hat{\overrightarrow{\alpha}} \cdot \overrightarrow{A}_\alpha, \overrightarrow{A}\right] + e\left[\Phi_\alpha, \overrightarrow{A}\right] + m^\alpha c^{2\alpha}\left[\hat{\beta}, \overrightarrow{A}\right]. \tag{38}$$

Subsequently, we proceed to calculate each commutator in equation (38), starting with

$$\left[\hat{\overrightarrow{\alpha}} \cdot \hat{\overrightarrow{p}}_\alpha, \overrightarrow{A}\right] = -i\hbar_\alpha^\alpha \sum_{i,j} \hat{\alpha}_i \left[|\hat{x}|^{1-\alpha} \nabla_i, A_j\right] e_j, \tag{39}$$

and its act on ψ yields

$$\left[\hat{\overrightarrow{\alpha}} \cdot \hat{\overrightarrow{p}}_\alpha, \overrightarrow{A}\right]\psi = -i\hbar_\alpha^\alpha |\hat{x}|^{1-\alpha} \sum_{i,j} \hat{\alpha}_i \left(\nabla_i A_j\right) e_j \psi. \tag{40}$$

Note that in equations (34, 40), the gradient acts on $\overrightarrow{A}$ only. Then, we proceed with

$$\left[\hat{\beta}, \overrightarrow{A}\right] = \left[\hat{\overrightarrow{\alpha}}, \overrightarrow{A}\right] = 0, \tag{41}$$

and

$$\left[\Phi_\alpha, \overrightarrow{A}\right] = \left[\overrightarrow{A}_\alpha, \overrightarrow{A}\right] = 0. \tag{42}$$

Now, in total, we have

$$\frac{d\hat{\overrightarrow{\Pi}}}{dt} = -e\left\{\frac{\partial \overrightarrow{A}}{\partial t} + \left(\overrightarrow{\nabla}\Phi_\alpha\right)\right\} + ec^\alpha \sum_{i,j} \left(\hat{\alpha}_i\right)\left(\nabla_j \left(A_\alpha\right)_i - \frac{1}{\hbar}\hbar_\alpha^\alpha |\hat{x}|^{1-\alpha} \nabla_i A_j\right) e_j + \frac{1}{\hbar}\hbar_\alpha^\alpha c^\alpha \left[|\hat{x}|^{1-\alpha}, \overrightarrow{\nabla}\right] \hat{\overrightarrow{\alpha}} \cdot \hat{\overrightarrow{p}}. \tag{43}$$

Then, with

$$-\overrightarrow{E}_\alpha = \frac{\partial \overrightarrow{A}}{\partial t} + \overrightarrow{\nabla}\Phi_\alpha, \tag{44}$$

and if $A_\alpha = \frac{1}{\hbar}\hbar_\alpha^\alpha |\hat{x}|^{1-\alpha} \overrightarrow{A}$, one has

$$\frac{d\hat{\overrightarrow{\Pi}}}{dt} = e\overrightarrow{E}_\alpha + e\frac{1}{\hbar}\hbar_\alpha^\alpha c^\alpha |\hat{x}|^{1-\alpha} \sum_{i,j} \left(\hat{\alpha}_i\right)\left(\nabla_j A_i - \nabla_i A_j\right) e_j + \frac{1}{\hbar}\hbar_\alpha^\alpha c^\alpha \left[|\hat{x}|^{1-\alpha}, \overrightarrow{\nabla}\right] \hat{\overrightarrow{\alpha}} \cdot \hat{\overrightarrow{p}}. \tag{45}$$

By applying equation (28) and performing some simplifications, we obtain

$$\frac{1}{\hbar}\hbar_\alpha^\alpha c^\alpha |\hat{x}|^{1-\alpha} \sum_{i,j} \left(\hat{\alpha}_i\right)\left(\nabla_j A_i - \nabla_i A_j\right) e_j = \sum_{i,j} \left(v_i\right)\left(\nabla_j A_i - \nabla_i A_j\right) e_j = \overrightarrow{v} \times \mathrm{curl}\overrightarrow{A}. \tag{46}$$

Subsequently, we have

$$\frac{d\hat{\overrightarrow{\Pi}}}{dt} = e\left(\overrightarrow{E}_\alpha + \overrightarrow{v} \times \overrightarrow{B}\right) + \frac{1}{\hbar}\hbar_\alpha^\alpha c^\alpha \left[|\hat{x}|^{1-\alpha}, \overrightarrow{\nabla}\right] \hat{\overrightarrow{\alpha}} \cdot \hat{\overrightarrow{p}}, \tag{47}$$

with $\mathrm{curl}\overrightarrow{A} = \overrightarrow{B}$. As evident, equation (47) represents a α-Lorentz force in the classical case, describing the force exerted by the electromagnetic field on an electron with an electric charge e.

Similar to the velocity in equation (28), the impact of CFQM on the Lorentz force is prominently featured in equation (47). In the limit of $\alpha \to 1$, we obtain

$$\frac{d\overrightarrow{\hat{\Pi}}}{dt} = e\left(\overrightarrow{E} + \overrightarrow{v} \times \overrightarrow{B}\right), \tag{48}$$

which corresponds to the classical form of the Lorentz force.

Let us now turn to a discussion of our results. It is evident that $\overrightarrow{\hat{x}}$ does not adhere to classical equations of motion. However, a classical equation of motion can be determined for the operator $\overrightarrow{\hat{\Pi}}$. Interestingly, equation (47) formally resembles its classical counterpart, but it is important to note that expectation values from equation (48) are not practical due to the Zitterbewegung, with a reduction in velocity. At best, projecting the even contributions from (48) yields result relevant to a classical single-particle description. Equation (47) highlights how CFD alters the Lorentz force, introducing deformations as a result. Likewise, equation (28) shows that CFD also affects velocity. Notably, the application of CFQM are found to impact Ehrenfest's theorem.

4. Conformable fractional continuity equation

We define $\overline{\psi} \equiv \psi^\dagger\gamma^0$, where $\psi^\dagger$ is the complex conjugate of the row vector corresponding to the column vector ψ. Then, by taking the adjoint of equation (16):

$$\overline{\psi}\left\{-i\hbar^\alpha\gamma^0\partial_0^\alpha + i\hbar^\alpha\left(\gamma^k\right)^\dagger \overleftarrow{\partial}^\alpha - m^\alpha c^\alpha\right\} = 0, \tag{49}$$

with $\overleftarrow{\partial}^\alpha = \partial_k^\alpha$ acts from the right on $\overline{\psi}$, also, from the Hermicity of γ^μ, we have

$$\left(\gamma^\mu\right)^\dagger = \gamma^0\gamma^\mu\gamma^0, \text{ and } \left(\alpha^k\right)^\dagger = \alpha^k. \tag{50}$$

Also, the Dirac equation and its adjoint can be obtained through the variation of the action using the conformable fractional Lagrangian density, which is expressed as follows

$$\mathcal{L}_\alpha = -i\hbar^\alpha c^\alpha\overline{\psi}\gamma^\mu\partial_\mu^\alpha\psi - m^\alpha c^{2\alpha}\overline{\psi}\psi. \tag{51}$$

Then, if one performs variation with respect to ψ, the result is the adjoint Dirac equation. Conversely, varying it with respect to $\overline{\psi}$ yields the Dirac equation. However, from equations (16), (49), we can obtain the following the α-continuity equation

$$\overline{\psi}\gamma^k\overleftarrow{\partial}^\alpha\psi - \overline{\psi}\gamma^0\partial_0^\alpha\psi + \overline{\psi}\gamma^0\partial_0^\alpha\psi - \overline{\psi}\gamma^k\partial_k^\alpha\psi = 0. \tag{52}$$

In a more elegant concise form, equation (52) becomes

$$D_\mu^\alpha\mathcal{J}_\alpha^\mu = D_t^\alpha\rho_\alpha + D_x^\alpha\mathcal{J}_\alpha^k = 0, \tag{53}$$

where the four-vector α-current density (the α-probability density and the α-probability flux) is given as follows

$$\mathcal{J}_\alpha^\mu : \begin{cases} \rho_\alpha = \overline{\psi}\gamma^0\psi = \psi^\dagger\psi = \left|\psi\right|^2, \\ \mathcal{J}_\alpha^k = \overline{\psi}\gamma^k\psi = \psi^\dagger\alpha^k\psi. \end{cases} \tag{54}$$

One can see from equation (54) that CFQM does not affect the four-vector current. However, the continuity equation itself is affected.

5. Correspondence principle in CFQM

The correspondence principle asserts that the behavior of systems described by the theory of quantum mechanics mirrors classical physics when quantum numbers become large. For example, at high energies, quantum calculations must align with classical counterparts. Here, we attempt examine how large quantum numbers can give rise to classical conduct. Consider the following conformable fractional quantum harmonic oscillator [39]:

$$E_n = \hbar\omega\left(2n + l + \frac{3}{2}\right)^{\frac{1}{\alpha}}, \tag{55}$$

n, l are quantum numbers, with ω is the angular frequency of the oscillator. The energy is usually described by the single quantum number $N = 2n + l$. $(N = 0, 1, 2, \dots)$ But, the classical fractional harmonic oscillator in three dimensions is

$$E = \frac{1}{2} m^{\alpha} \omega^{2\alpha} \frac{x^{\alpha}}{\alpha}. \tag{56}$$

Thus, from equations (55) and (56), the quantum number has the value

$$N = \frac{x^{\alpha^2}}{\hbar^{\alpha} 2^{\alpha} \alpha^{\alpha}} m^{\alpha^2} \omega^{\left(2\alpha^2 - \alpha\right)} - \frac{3}{2}. \tag{57}$$

Now, for typical values m=1 Kg, $\omega = 1\,\mathrm{rad.s}^{-1}$, and x = 1m, one can get

$$N = \frac{1}{\hbar^{\alpha} 2^{\alpha} \alpha^{\alpha}} - \frac{3}{2}. \tag{58}$$

Then, for a better understanding the correspondence limit between quantum and classical harmonic oscillator, we plot equation (58).

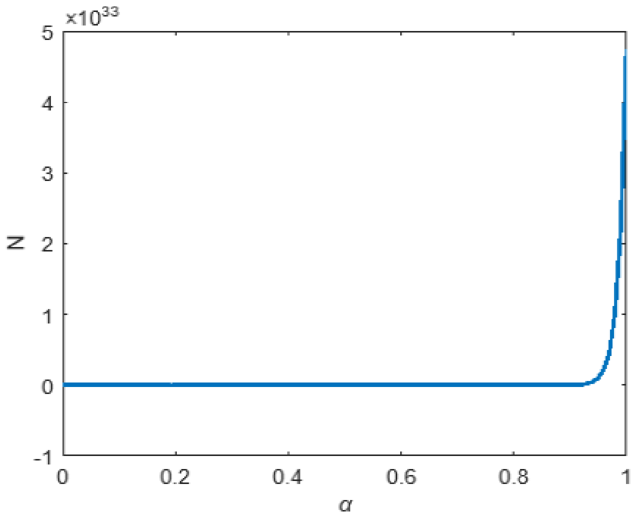

Fig. 3. N versus α (plot of equation (58))

In Fig. 3, we can see that only when the fractional parameter approaches unity, the system is in the correspondence limit, and for $\alpha = 1$,

$$N = \frac{1}{2\hbar} - \frac{3}{2} = 0.474126 \times 10^{34}, \tag{59}$$

which is a very large number; this, in turn confirms that the system is in the correspondence limit.

6. Conclusion and remarks

In this research, we have analytically investigated the classical limit of the Dirac equation interacting with electromagnetic potential within the CFQM, employing Ehrenfest's theorem. Our analysis successfully reveals the influence of the CFD on the classical limit, resulting in α-deformed classical equations. Our results confirm the feasibility of achieving the classical limit within the framework of CFDM. Once again, Ehrenfest's theorem proves effective in deriving classical limit of the Dirac equation, irrespective of the effects present in the relativistic system. Therefore, we emphasize the importance of such type of theorems. Moreover, it is shown that the inclusion of CFD does not alter the current density four-vector. Additionally, CFD appears to affect the correspondence limit between the quantum system and its classical counterpart.

Clearly, our findings serve as a valuable resource for further investigations, including nonrelativistic and semiclassical limits, as well as scenarios involving other types of fractional derivatives. Expanding this study to encompass more general cases, such as particles with arbitrary higher spins, would be a promising avenue for future research. Notably, in the limit as $\alpha \to 1$, the α-deformed Dirac and the obtained classical equations reduce to those of ordinary QM, confirming that our findings are consistent with and reducible to those found and discussed in the literature.

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Conflicts of Interest

The author declares no conflict of interest.

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Классический аспект уравнения Дирака в контексте согласованной дробной производной

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Аннотация. В этой статье в контексте конформной дробной производной (CFD) и с использованием теоремы Эренфеста мы исследуем классический предел уравнения Дирака в рамках конформной дробной квантовой механики. Это приводит к получению деформированных классических уравнений. Здесь мы оцениваем эффективность теоремы Эренфеста при выводе классического предела с учетом CFD. Также мы исследуем принцип соответствия под влиянием CFD. Кроме того, мы получаем конформное дробное уравнение непрерывности.

Ключевые слова: согласованное дробное уравнение Дирака, согласованное дробное уравнение непрерывности, теорема Эренфеста, классический предел, принцип соответствия, согласованная квантовая механика.