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Weighted Analogue of LBB Conditions for Solving the Stokes Problem with Model Boundary Conditions in a Domain with Singularity

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Abstract. In the paper the concept of a R_ν-generalized solution for the Stokes problem with model boundary conditions in a domain with a corner singularity is defined. Weighted analogue of the Ladyzhenskaya-Babuska-Brezzi conditions in a domain with a reentrant corner is proven.

Keywords: corner singularity, Stokes problem with model boundary conditions, R_ν-generalized solution.

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Introduction

The article considers the Stokes system. The solution of such a system is the main problem in computational fluid dynamics. The system with homogeneous Dirichlet boundary conditions for the velocity field has been studied most from both theoretical and practical points of view. A detailed analysis of the problem is presented in [1, 2]. Using the Schur complement operator (see, for example, [3]), we actually reduce the problem to separately finding the velocity field $\mathbf{u}$ and pressure p. Moreover, in order to find the pressure function, it is not necessary to know its values on the boundary of a domain and require additional smoothness of the solution (see, for example, [3]). In the presented article, fundamentally different boundary conditions are considered, namely $\mathbf{u} \cdot \mathbf{n} = 0$ and $\operatorname{curl} \mathbf{u} = 0$, where $\mathbf{u} \cdot \mathbf{n} = u_1 n_1 + u_2 n_2, \mathbf{n}$ is the outer unit normal vector to the boundary, and $\operatorname{curl} \mathbf{u} = \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2}$. Such boundary conditions will be called model conditions. They are of particular interest from a practical point of view, associated with the Schur complement operator. More details on this can be found in [4]. On the other hand, it is fundamental to consider the Stokes system in a polygonal non-convex domain Ω with a reentrant corner ω on the boundary, i.e. a corner greater than π. In this case, a problem with a corner singularity is considered. Moreover, as is known (see, for example, [5]), a generalized solution of such a problem in the velocity-pressure variables $(\mathbf{u}, p)$ does not belong to the Sobolev spaces $\mathbf{W}_2^2(\Omega)$ and $W_2^1(\Omega)$, respectively. Therefore, by the principle of consistent estimates, using any classical approximate approach (see [6]), its approximate solution converges to an exact one at

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a rate no faster than $\mathcal{O}(h^\alpha)$, where h is the mesh step, and α is significantly less than one and decreases with increasing the value of a corner ω. At the same time, an appropriate convergence rate is of the order $\mathcal{O}(h)$, as is in the case of a convex domain Ω.

The article proposes to define the solution of the Stokes problem with model boundary conditions as an R_ν-generalized one in sets of weighted spaces. In this case, we will look for a solution in sets of more general spaces than the weighted Sobolev spaces $W^k_{2,\beta}(\Omega)$, $\beta > 0$. Note that the resulting variational formulation is not symmetric, unlike the classical one for determining a generalized solution of the problem [4]. This will further add difficulties to the proof of the existence and uniqueness R_ν-generalized solution of the Stokes problem with the proposed model boundary conditions.

The first time that a solution as an R_ν-generalized one was defined for elliptic problems with Dirichlet boundary conditions in [7]. The construction of a weighted finite element method for finding an approximate R_ν-generalized solution made it possible to obtain the convergence rate of such a solution to an exact one, which does not depend on the value of a reentrant corner ω. It is equal to $\mathcal{O}(h)$ for various differential problems with Dirichlet boundary conditions [8–12]. Moreover, the result is achieved without refinement of the mesh in the vicinity of the singularity point. In the presented article we study function properties in sets of weighted spaces. We will establish a weighted analogue of the Ladyzhenskaya-Babushka-Brezzi conditions for the Stokes problem with the considered model boundary conditions.

1. Formulation of Stokes problem with model boundary conditions. Definition of an R_ν-generalized solution

Let a domain Ω be a bounded non-convex polygon with the boundary $\partial\Omega$, containing a reentrant corner $\omega, \omega \in (\pi, 2\pi)$ at the origin $\mathcal{O} = (0,0)$, $\bar{\Omega} = \Omega \cup \partial\Omega$.

Let $\mathbf{x} = (x_1, x_2)$ be an element of R^2, $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$, $d\mathbf{x} = dx_1 dx_2$. The Stokes problem is that, for given functions $\mathbf{f} = (f_1, f_2)$ and g in Ω: find the velocity field $\mathbf{u} = (u_1, u_2)$ and pressure p, which satisfy the system of differential equations and boundary conditions

$$-\triangle\mathbf{u} + \nabla p = \mathbf{f}, \qquad \operatorname{div} \mathbf{u} = g \qquad \text{in} \qquad \Omega, \tag{1}$$

$$\mathbf{u}\cdot\mathbf{n} = 0 \qquad \text{on} \qquad \partial\Omega, \tag{2}$$

$$\operatorname{curl} \mathbf{u} = 0 \qquad \text{on} \qquad \partial\Omega, \tag{3}$$

where $\mathbf{u}\cdot\mathbf{n} = u_1 n_1 + u_2 n_2$, $\operatorname{curl} \mathbf{u} = \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2}$ и $\mathbf{n} = (n_1, n_2)$ be the outer unit normal vector to $\partial\Omega$.

Let us define necessary spaces and sets of weight functions. We denote by $L_{2,\alpha}(\Omega)$ the weighted space of functions $v(\mathbf{x})$ with limited norm

$$\|v\|_{L_{2,\alpha}(\Omega)} = \left(\int_\Omega \rho^{2\alpha}(\mathbf{x})\, v^2(\mathbf{x}) d\mathbf{x}\right)^{1/2}, \qquad \alpha > 0.$$

We will highlight in bold spaces and sets of functions $\mathbf{v} = (v_1, v_2)$. Here $\mathbf{v} \in \mathbf{L}_{2,\alpha}(\Omega)$, if the quantity $\|\mathbf{v}\|_{\mathbf{L}_{2,\alpha}(\Omega)} = \left(\|v_1\|^2_{L_{2,\alpha}(\Omega)} + \|v_2\|^2_{L_{2,\alpha}(\Omega)}\right)^{1/2}$ is limited.

Let $\mathbf{H}_\alpha(\mathrm{curl})(\Omega)$ be the space of functions $\mathbf{v}(\mathbf{x})$ such that $\mathbf{v} \in \mathbf{L}_{2,\alpha}(\Omega)$ and $\operatorname{curl} \mathbf{v} \in L_{2,\alpha}(\Omega)$ with bounded norm

$$\|\mathbf{v}\|_{\mathbf{H}_\alpha(\mathrm{curl})(\Omega)} = \left(\|\mathbf{v}\|^2_{\mathbf{L}_{2,\alpha}(\Omega)} + \|\operatorname{curl} \mathbf{v}\|^2_{L_{2,\alpha}(\Omega)}\right)^{1/2}.$$

Denote by $\mathbf{H}_\alpha(\mathrm{div})(\Omega)$ the space of functions $\mathbf{v}(\mathbf{x})$ such that $\mathbf{v} \in \mathbf{L}_{2,\alpha}(\Omega)$ and div $\mathbf{v} \in L_{2,\alpha}(\Omega)$ with limited norm

$$\|\mathbf{v}\|_{\mathbf{H}_\alpha(\mathrm{div})(\Omega)} = \Big(\|\mathbf{v}\|^2_{\mathbf{L}_{2,\alpha}(\Omega)} + \|\mathrm{div}\ \mathbf{v}\|^2_{L_{2,\alpha}(\Omega)}\Big)^{1/2}.$$

Let $\overset{\circ}{\mathbf{H}}_\alpha(\mathrm{div})(\Omega)$ be the subspace of $\mathbf{H}_\alpha(\mathrm{div})(\Omega)$ such that $\{\mathbf{v} \in \mathbf{H}_\alpha(\mathrm{div})(\Omega) : \mathbf{v}\cdot\mathbf{n} = 0$ на $\partial\Omega\}$ with bounded norm of the space $\mathbf{H}_\alpha(\mathrm{div})(\Omega)$. Next, we denote by $\mathbf{U}_\alpha(\Omega)$ the intersection of spaces $\overset{\circ}{\mathbf{H}}_\alpha(\mathrm{div})(\Omega)$ and $\mathbf{H}_\alpha(\mathrm{curl})(\Omega)$ of functions $\mathbf{v}(\mathbf{x})$ with limited norm

$$\|\mathbf{v}\|_{\mathbf{U}_\alpha(\Omega)} = \Big(\|\mathbf{v}\|^2_{\mathbf{L}_{2,\alpha}(\Omega)} + \|\mathrm{div}\ \mathbf{v}\|^2_{L_{2,\alpha}(\Omega)} + \|\mathrm{curl}\ \mathbf{v}\|^2_{L_{2,\alpha}(\Omega)}\Big)^{1/2}.$$

Let $W^1_{2,\alpha}(\Omega)$ be the weighted space of functions $v(\mathbf{x})$ with bounded norm

$$\|v\|_{W^1_{2,\alpha}(\Omega)} = \Big(\|v\|^2_{L_{2,\alpha}(\Omega)} + \sum_{|l|=1} \int_\Omega \rho^{2\alpha}(\mathbf{x})|D^l v(\mathbf{x})|^2 d\mathbf{x}\Big)^{1/2},$$

where $D^l v(\mathbf{x}) = \dfrac{\partial^{|l|} v(\mathbf{x})}{\partial x_1^{l_1} \partial x_2^{l_2}}$, $l = (l_1, l_2)$, $|l| = l_1 + l_2$, l_i are non-negative integers $i \in \{1, 2\}$. Denote by $\overset{\circ}{W}{}^1_{2,\alpha}(\Omega)$ the subspace of functions $v(\mathbf{x})$ from $W^1_{2,\alpha}(\Omega)$ such that $v = 0$ on $\partial\Omega$ with limited norm $W^1_{2,\alpha}(\Omega)$. Similarly, we introduce spaces $\mathbf{W}^1_{2,\alpha}(\Omega)$ and $\overset{\circ}{\mathbf{W}}{}^1_{2,\alpha}(\Omega)$ of functions $\mathbf{v} = (v_1, v_2)$ such that $v_i \in W^1_{2,\alpha}(\Omega, \delta)$ and $v_i \in \overset{\circ}{W}{}^1_{2,\alpha}(\Omega, \delta)$, respectively, with bounded norm $\|\mathbf{v}\|_{\mathbf{W}^1_{2,\alpha}(\Omega)} = \Big(\|v_1\|^2_{W^1_{2,\alpha}(\Omega)} + \|v_2\|^2_{W^1_{2,\alpha}(\Omega)}\Big)^{1/2}$.

By $\Omega_\delta = \{\mathbf{x} \in \bar{\Omega} : \|\mathbf{x}\| \leqslant \delta \ll 1, \delta > 0\}$ we denote the intersection of the circle with a radius δ centered at the origin $\mathcal{O}$ with $\bar{\Omega}$ and introduce the weight function $\rho(\mathbf{x})$ in $\bar{\Omega}$ as follows: $\rho(\mathbf{x}) = \begin{cases} \|\mathbf{x}\|, & \text{if } \mathbf{x} \in \Omega_\delta, \\ \delta\ , & \text{if } \mathbf{x} \in \bar{\Omega} \setminus \Omega_\delta. \end{cases}$

Let us define the following conditions for the function $v(\mathbf{x})$:

$$\|v\|_{L_{2,\alpha}(\Omega\setminus\Omega_\delta)} \geqslant C_1 > 0, \tag{4}$$

$$|v(\mathbf{x})| \leqslant C_2 \delta^{\alpha-\tau} \rho^{\tau-\alpha}(\mathbf{x}), \quad \mathbf{x} \in \Omega_\delta, \tag{5}$$

where C_2 is a positive constant, τ is a small positive parameter independent of δ, α and $v(\mathbf{x})$. Denote by $L_{2,\alpha}(\Omega, \delta)$ a set of functions $v(\mathbf{x})$ from the space $L_{2,\alpha}(\Omega)$ satisfying conditions (4) and (5) with limited norm $L_{2,\alpha}(\Omega)$. Let $L^0_{2,\alpha}(\Omega, \delta)$ be a subset of functions $v(\mathbf{x})$ from $L_{2,\alpha}(\Omega, \delta)$ such that $\int_\Omega \rho^\alpha(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} = 0$ with bounded norm $L_{2,\alpha}(\Omega)$.

Next, we define sets $\mathbf{H}_\alpha(\mathrm{curl})(\Omega, \delta)$, $\mathbf{H}_\alpha(\mathrm{div})(\Omega, \delta)$, $\overset{\circ}{\mathbf{H}}_\alpha(\mathrm{div})(\Omega, \delta)$ and $\mathbf{U}_\alpha(\Omega, \delta)$ of functions $\mathbf{v} = (v_1, v_2)$ from spaces $\mathbf{H}_\alpha(\mathrm{curl})(\Omega), \mathbf{H}_\alpha(\mathrm{div})(\Omega), \overset{\circ}{\mathbf{H}}_\alpha(\mathrm{div})(\Omega)$ and $\mathbf{U}_\alpha(\Omega)$, respectively, which components satisfy conditions (4) and (5) with limited norms of relevant spaces. Let $W^1_{2,\alpha}(\Omega, \delta)$ and $\overset{\circ}{W}{}^1_{2,\alpha}(\Omega, \delta)$ are sets of functions from spaces $W^1_{2,\alpha}(\Omega)$ and $\overset{\circ}{W}{}^1_{2,\alpha}(\Omega)$ respectively, satisfying conditions (4), (5) and $|D^1 v(\mathbf{x})| \leqslant C_2 \delta^{\alpha-\tau} \rho^{\tau-\alpha-1}(\mathbf{x})$, $\mathbf{x} \in \Omega_\delta$, with bounded norm of the space $W^1_{2,\alpha}(\Omega)$. Denote by $\mathbf{W}^1_{2,\alpha}(\Omega, \delta)$ and $\overset{\circ}{\mathbf{W}}{}^1_{2,\alpha}(\Omega, \delta)$ sets of functions $\mathbf{v} = (v_1, v_2)$ such that $v_i \in W^1_{2,\alpha}(\Omega, \delta)$ and $v_i \in \overset{\circ}{W}{}^1_{2,\alpha}(\Omega, \delta)$ respectively.

Let us prove the following assertion.

Lemma 1. *Let the function* $\mathbf{u} \in \mathbf{U}_\nu(\Omega, \delta)$ *and* *curl* $\mathbf{u} = 0$ *on* $\partial\Omega$, *then for an arbitrary function* $\mathbf{v} \in \mathbf{U}_\nu(\Omega, \delta)$ *an identity*

$$\int_\Omega \nabla\mathbf{u} : \nabla(\rho^{2\nu}\mathbf{v})d\mathbf{x} + I(\mathbf{u}, \mathbf{v}) = \int_\Omega \textit{curl}\ \mathbf{u}\ \textit{curl}\ (\rho^{2\nu}\mathbf{v})d\mathbf{x} + \int_\Omega \textit{div}\ \mathbf{u}\ \textit{div}\ (\rho^{2\nu}\mathbf{v})d\mathbf{x} \tag{6}$$

holds, where

$$I(\mathbf{u}, \mathbf{v}) := -\Bigg[\int\limits_{\partial\Omega} \rho^{2\nu}\frac{\partial u_1}{\partial x_1}n_1v_1ds + \int\limits_{\partial\Omega} \rho^{2\nu}\frac{\partial u_1}{\partial x_2}n_2v_1ds + \int\limits_{\partial\Omega} \rho^{2\nu}\frac{\partial u_2}{\partial x_1}n_1v_2ds + \int\limits_{\partial\Omega} \rho^{2\nu}\frac{\partial u_2}{\partial x_2}n_2v_2ds\Bigg]. \tag{7}$$

Proof. By definition

$$\int_\Omega \nabla\mathbf{u} : \nabla(\rho^{2\nu}\mathbf{v})d\mathbf{x} = \int_\Omega \Big[\frac{\partial u_1}{\partial x_1}\frac{\partial(\rho^{2\nu}v_1)}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\frac{\partial(\rho^{2\nu}v_1)}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\frac{\partial(\rho^{2\nu}v_2)}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial(\rho^{2\nu}v_2)}{\partial x_2}\Big]d\mathbf{x}, \tag{8}$$

$$\begin{aligned} \int_\Omega \text{curl}\ \mathbf{u}\ \text{curl}\ (\rho^{2\nu}\mathbf{v})d\mathbf{x} = \int_\Omega \Big[&\frac{\partial u_2}{\partial x_1}\frac{\partial(\rho^{2\nu}v_2)}{\partial x_1} - \frac{\partial u_2}{\partial x_1}\frac{\partial(\rho^{2\nu}v_1)}{\partial x_2} - \\ &- \frac{\partial u_1}{\partial x_2}\frac{\partial(\rho^{2\nu}v_2)}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\frac{\partial(\rho^{2\nu}v_1)}{\partial x_2}\Big]d\mathbf{x}, \end{aligned} \tag{9}$$

$$\begin{aligned} \int_\Omega \text{div}\ \mathbf{u}\ \text{div}\ (\rho^{2\nu}\mathbf{v})d\mathbf{x} = \int_\Omega \Big[&\frac{\partial u_1}{\partial x_1}\frac{\partial(\rho^{2\nu}v_1)}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\frac{\partial(\rho^{2\nu}v_2)}{\partial x_2} + \\ &+ \frac{\partial u_2}{\partial x_2}\frac{\partial(\rho^{2\nu}v_1)}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial(\rho^{2\nu}v_2)}{\partial x_2}\Big]d\mathbf{x}. \end{aligned} \tag{10}$$

The following equalities

$$-\int_\Omega \frac{\partial u_2}{\partial x_1}\frac{\partial(\rho^{2\nu}v_1)}{\partial x_2}d\mathbf{x} = -\int_\Omega \frac{\partial u_2}{\partial x_2}\frac{\partial(\rho^{2\nu}v_1)}{\partial x_1}d\mathbf{x} + \int\limits_{\partial\Omega} \rho^{2\nu}\frac{\partial u_2}{\partial x_2}n_1v_1ds - \int\limits_{\partial\Omega} \rho^{2\nu}\frac{\partial u_2}{\partial x_1}n_2v_1ds, \tag{11}$$

$$-\int_\Omega \frac{\partial u_1}{\partial x_2}\frac{\partial(\rho^{2\nu}v_2)}{\partial x_1}d\mathbf{x} = -\int_\Omega \frac{\partial u_1}{\partial x_1}\frac{\partial(\rho^{2\nu}v_2)}{\partial x_2}d\mathbf{x} + \int\limits_{\partial\Omega} \rho^{2\nu}\frac{\partial u_1}{\partial x_1}n_2v_2ds - \int\limits_{\partial\Omega} \rho^{2\nu}\frac{\partial u_1}{\partial x_2}n_1v_2ds \tag{12}$$

are valid. Using expansions (8)-(10), we have

$$\int_\Omega \nabla\mathbf{u} : \nabla(\rho^{2\nu}\mathbf{v})d\mathbf{x} = \int_\Omega \text{curl}\ \mathbf{u}\ \text{curl}\ (\rho^{2\nu}\mathbf{v})d\mathbf{x} + \int_\Omega \text{div}\ \mathbf{u}\ \text{div}\ (\rho^{2\nu}\mathbf{v})d\mathbf{x} - E(\mathbf{u}, \mathbf{v}), \tag{13}$$

where

$$E(\mathbf{u}, \mathbf{v}) := \int_\Omega \Big[\frac{\partial u_1}{\partial x_1}\frac{\partial(\rho^{2\nu}v_2)}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\frac{\partial(\rho^{2\nu}v_1)}{\partial x_1} - \frac{\partial u_2}{\partial x_1}\frac{\partial(\rho^{2\nu}v_1)}{\partial x_2} - \frac{\partial u_1}{\partial x_2}\frac{\partial(\rho^{2\nu}v_2)}{\partial x_1}\Big]d\mathbf{x}. \tag{14}$$

Applying equalities (11) and (12) to (14), we conclude

$$E(\mathbf{u}, \mathbf{v}) = \int\limits_{\partial\Omega} \rho^{2\nu}\Big[\frac{\partial u_2}{\partial x_2}n_1v_1 + \frac{\partial u_1}{\partial x_1}n_2v_2 - \frac{\partial u_2}{\partial x_1}n_2v_1 - \frac{\partial u_1}{\partial x_2}n_1v_2\Big]ds. \tag{15}$$

Using together (7), (15), conditions $\mathbf{v} \cdot \mathbf{n} = 0$ and $\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = 0$ on $\partial\Omega$, and an equality $\int\limits_{\partial\Omega} \rho^{2\nu}\Big(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\Big)n_2v_1ds = \int\limits_{\partial\Omega} \rho^{2\nu}\Big(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\Big)n_1v_2ds$, we get $I(\mathbf{u}, \mathbf{v}) = E(\mathbf{u}, \mathbf{v})$. Lemma 1 is proven. □

Let us introduce bilinear and linear forms

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \text{curl}\ \mathbf{u}\ \text{curl}\ (\rho^{2\nu}\mathbf{v})d\mathbf{x} + \int_\Omega \text{div}\ \mathbf{u}\ \text{div}\ (\rho^{2\nu}\mathbf{v})d\mathbf{x}, \quad b_1(\mathbf{v},s) = -\int_\Omega s\ \text{div}\ (\rho^{2\nu}\mathbf{v})d\mathbf{x},$$

$$b_2(\mathbf{u},q) = -\int_\Omega (\rho^{2\nu}\, q)\ \text{div}\ \mathbf{u}\, d\mathbf{x}, \quad l(\mathbf{v}) = \int_\Omega \mathbf{f}\cdot(\rho^{2\nu}\mathbf{v})d\mathbf{x}, \quad c(q) = \int_\Omega \rho^{2\nu} g\, q d\mathbf{x}.$$

We define the concept of an R_ν-generalized solution of the Stokes problem (1)–(3) with model boundary conditions in weighted sets.

Definition 1. *A pair of functions* $(\mathbf{u}_\nu, p_\nu) \in \mathbf{U}_\nu(\Omega,\delta)\times L^0_{2,\nu}(\Omega,\delta)$ *is called an* R_ν*-generalized solution of the Stokes problem (1)–(3), the function* $\mathbf{u}_\nu$ *satisfies the boundary conditions (2) and (3), if for all pairs of functions* $(\mathbf{v},q)\in \mathbf{U}_\nu(\Omega,\delta)\times L^0_{2,\nu}(\Omega,\delta)$ *the integral identities*

$$a(\mathbf{u}_\nu,\mathbf{v}) + b_1(\mathbf{v},p_\nu) = l(\mathbf{v}), \tag{16}$$

$$b_2(\mathbf{u}_\nu,q) = c(q) \tag{17}$$

hold, where $\mathbf{f}\in \mathbf{L}_{2,\gamma}(\Omega)$, $g\in L_{2,\beta}(\Omega)$, $0\leqslant \gamma,\beta\leqslant \nu$ *and* $\mathbf{u}_\nu = (u_{1,\nu},\, u_{2,\nu})$.

Remark 1. *Since the bilinear form* $b_2(\cdot,\cdot)$ *does not coincide with the bilinear form* $b_1(\cdot,\cdot)$*, therefore the variational formulation for an* R_ν*-generalized solution of the problem is not symmetric, in contrast to the standard variational formulation for a generalized solution of the problem [4].*

Remark 2. *The bilinear form* $a(\cdot,\cdot)$ *is not a symmetric one.*

2. Auxiliary statements

Let us formulate and prove necessary statements.

Lemma 2 ([13]). *Let* $\nu > 0$*. For an arbitrary function* $z\in L_{2,\nu}(\Omega)$*, satisfying conditions (4), (5), the following estimate*

$$\int_{\Omega_\delta} \rho^{2\nu-2} z^2 d\mathbf{x} \leqslant C_3^2\delta^{2\nu}\|z\|^2_{L_{2,\nu}(\Omega)} \tag{18}$$

is valid, where $C_3 = \dfrac{C_2}{C_1}\sqrt{\dfrac{\varphi_1-\varphi_0}{2\tau}}$, $(\varphi_1-\varphi_0)$ *is the magnitude of the change of a reentrant corner.*

Corollary 1. *Let conditions of Lemma 2 be satisfied, then*

$$\int_\Omega \Big[\sum_{i=1}^{2}\Big(\frac{\partial\rho^\nu}{\partial x_i}\Big)^2\Big] z^2 d\mathbf{x} \leqslant \nu^2 C_3^2\delta^{2\nu}\|z\|^2_{L_{2,\nu}(\Omega)}. \tag{19}$$

Proof. Thanks to the fact that $\sum\limits_{i=1}^{2}\Big(\dfrac{\partial\rho^\nu}{\partial x_i}\Big)^2 = \begin{cases}\nu^2\rho^{2\nu-2}, \mathbf{x}\in\Omega_\delta,\\ 0, \mathbf{x}\in\bar\Omega\setminus\Omega_\delta\end{cases}$ and Lemma 2 its inequality (18), directly implies an estimate (19) of Corollary 1.

Let us connect norms of functions $\mathbf{z}$ and $\rho^\nu\mathbf{z}$ from sets $\mathbf{U}_\nu(\Omega,\delta)$ and $\mathbf{U}_0(\Omega,\delta)$, respectively.

Lemma 3. *The function* $\mathbf{z} \in \mathbf{U}_\nu(\Omega, \delta)$ *if and only if* $\rho^\nu \mathbf{z} \in \mathbf{U}_0(\Omega, \delta)$ *and*

$$\|\rho^\nu \mathbf{z}\|_{\mathbf{U}_0(\Omega)} \leqslant C_4 \|\mathbf{z}\|_{\mathbf{U}_\nu(\Omega)}, \tag{20}$$

$$\|\mathbf{z}\|_{\mathbf{U}_\nu(\Omega)} \leqslant C_4 \|\rho^\nu \mathbf{z}\|_{\mathbf{U}_0(\Omega)}, \tag{21}$$

where $C_4 = \max\{\sqrt{2}, \sqrt{1 + 4\nu^2 C_3^2 \delta^{2\nu}}\}$.

Proof. 1. Let the function $\mathbf{z} \in \mathbf{U}_\nu(\Omega, \delta)$. We show that $\rho^\nu \mathbf{z} \in \mathbf{U}_0(\Omega, \delta)$ and an inequality (20) holds. We have decompositions

$$\operatorname{curl}(\rho^\nu \mathbf{z}) = \rho^\nu \operatorname{curl} \mathbf{z} + \Big[z_2 \frac{\partial \rho^\nu}{\partial x_1} - z_1 \frac{\partial \rho^\nu}{\partial x_2}\Big], \tag{22}$$

$$\operatorname{div}(\rho^\nu \mathbf{z}) = \rho^\nu \operatorname{div} \mathbf{z} + \Big[z_1 \frac{\partial \rho^\nu}{\partial x_1} + z_2 \frac{\partial \rho^\nu}{\partial x_2}\Big]. \tag{23}$$

Using expansions (22), (23), we conclude

$$\|\operatorname{curl}(\rho^\nu \mathbf{z})\|^2_{L_{2,0}(\Omega)} \leqslant 2\|\operatorname{curl} \mathbf{z}\|^2_{L_{2,\nu}(\Omega)} + 4\int_\Omega \Big(\frac{\partial \rho^\nu}{\partial x_1}\Big)^2 z_2^2 d\mathbf{x} + 4\int_\Omega \Big(\frac{\partial \rho^\nu}{\partial x_2}\Big)^2 z_1^2 d\mathbf{x}, \tag{24}$$

$$\|\operatorname{div}(\rho^\nu \mathbf{z})\|^2_{L_{2,0}(\Omega)} \leqslant 2\|\operatorname{div} \mathbf{z}\|^2_{L_{2,\nu}(\Omega)} + 4\int_\Omega \Big(\frac{\partial \rho^\nu}{\partial x_1}\Big)^2 z_1^2 d\mathbf{x} + 4\int_\Omega \Big(\frac{\partial \rho^\nu}{\partial x_2}\Big)^2 z_2^2 d\mathbf{x}. \tag{25}$$

Since $\|\rho^\nu \mathbf{z}\|_{\mathbf{L}_{2,0}(\Omega)} = \|\mathbf{z}\|_{\mathbf{L}_{2,\nu}(\Omega)}$, then applying relations (24), (25), and next Corollary 1 to Lemma 2 its estimate (19), we have a chain of inequalities

$$\|\rho^\nu \mathbf{z}\|^2_{\mathbf{U}_0(\Omega)} = \|\rho^\nu \mathbf{z}\|^2_{\mathbf{L}_{2,0}(\Omega)} + \|\operatorname{curl}(\rho^\nu \mathbf{z})\|^2_{L_{2,0}(\Omega)} + \|\operatorname{div}(\rho^\nu \mathbf{z})\|^2_{L_{2,0}(\Omega)} \leqslant \|\mathbf{z}\|^2_{\mathbf{L}_{2,\nu}(\Omega)} +$$

$$+2\|\operatorname{curl} \mathbf{z}\|^2_{L_{2,\nu}(\Omega)} + 2\|\operatorname{div} \mathbf{z}\|^2_{L_{2,\nu}(\Omega)} + 4\sum_{j=1}^{2} \int_\Omega \Big[\sum_{i=1}^{2} \Big(\frac{\partial \rho^\nu}{\partial x_i}\Big)^2\Big] z_j^2 d\mathbf{x} \leqslant \Big(1 + 4\nu^2 C_3^2 \delta^{2\nu}\Big) \|\mathbf{z}\|^2_{\mathbf{L}_{2,\nu}(\Omega)} +$$

$$+2\|\operatorname{curl} \mathbf{z}\|^2_{L_{2,\nu}(\Omega)} + 2\|\operatorname{div} \mathbf{z}\|^2_{L_{2,\nu}(\Omega)} \leqslant \max\{2, 1 + 4\nu^2 C_3^2 \delta^{2\nu}\} \|\mathbf{z}\|^2_{\mathbf{U}_\nu(\Omega)}.$$

An estimate (20) is proven and $\rho^\nu \mathbf{z} \in \mathbf{U}_0(\Omega, \delta)$.

2. Let the function $\rho^\nu \mathbf{z} \in \mathbf{U}_0(\Omega, \delta)$. We show that $\mathbf{z} \in \mathbf{U}_\nu(\Omega, \delta)$ and an inequality (21) holds. We express in (22) and (23) terms $\big(\rho^\nu \operatorname{curl} \mathbf{z}\big)$ and $\big(\rho^\nu \operatorname{div} \mathbf{z}\big)$, respectively, then

$$\|\operatorname{curl} \mathbf{z}\|^2_{L_{2,\nu}(\Omega)} \leqslant 2\|\operatorname{curl}(\rho^\nu \mathbf{z})\|^2_{L_{2,0}(\Omega)} + 4\int_\Omega \Big(\frac{\partial \rho^\nu}{\partial x_1}\Big)^2 z_2^2 d\mathbf{x} + 4\int_\Omega \Big(\frac{\partial \rho^\nu}{\partial x_2}\Big)^2 z_1^2 d\mathbf{x}, \tag{26}$$

$$\|\operatorname{div} \mathbf{z}\|^2_{L_{2,\nu}(\Omega)} \leqslant 2\|\operatorname{div}(\rho^\nu \mathbf{z})\|^2_{L_{2,0}(\Omega)} + 4\int_\Omega \Big(\frac{\partial \rho^\nu}{\partial x_1}\Big)^2 z_1^2 d\mathbf{x} + 4\int_\Omega \Big(\frac{\partial \rho^\nu}{\partial x_2}\Big)^2 z_2^2 d\mathbf{x}. \tag{27}$$

Since $\|\mathbf{z}\|_{\mathbf{L}_{2,\nu}(\Omega)} = \|\rho^\nu \mathbf{z}\|_{\mathbf{L}_{2,0}(\Omega)}$, then applying inequalities (26), (27), and next Corollary 1 to Lemma 2 its estimate (19), we have a chain of relations

$$\|\mathbf{z}\|^2_{\mathbf{U}_\nu(\Omega)} = \|\mathbf{z}\|^2_{\mathbf{L}_{2,\nu}(\Omega)} + \|\operatorname{curl} \mathbf{z}\|^2_{L_{2,\nu}(\Omega)} + \|\operatorname{div} \mathbf{z}\|^2_{L_{2,\nu}(\Omega)} \leqslant \|\rho^\nu \mathbf{z}\|^2_{\mathbf{L}_{2,0}(\Omega)} + 2\|\operatorname{curl}(\rho^\nu \mathbf{z})\|^2_{L_{2,0}(\Omega)} +$$

$$+2\|\operatorname{div}(\rho^\nu \mathbf{z})\|^2_{L_{2,0}(\Omega)} + 4\sum_{j=1}^{2} \int_\Omega \Big[\sum_{i=1}^{2} \Big(\frac{\partial \rho^\nu}{\partial x_i}\Big)^2\Big] z_j^2 d\mathbf{x} \leqslant \max\{2, 1 + 4\nu^2 C_3^2 \delta^{2\nu}\} \|\rho^\nu \mathbf{z}\|^2_{\mathbf{U}_0(\Omega)}.$$

An estimate (21) is obtained and $\mathbf{z} \in \mathbf{U}_\nu(\Omega, \delta)$. Lemma 3 is proven. □

Lemma 4. *Let $\nu > 0$, then there exists a value $\delta_0 = \delta_0(\nu) > 0$, such that for any $\delta \in (0, \delta_0]$ and an arbitrary function $\mathbf{z} \in \mathbf{U}_\nu(\Omega, \delta)$ an inequality*

$$\|\mathbf{z}\|^2_{\mathbf{L}_{2,\nu}(\Omega)} \leqslant 8C^2_\Omega(\|curl\ \mathbf{z}\|^2_{L_{2,\nu}(\Omega)} + \|div\ \mathbf{z}\|^2_{L_{2,\nu}(\Omega)}) \tag{28}$$

holds.

Proof. Using Lemma 3, if $z \in \mathbf{U}_\nu(\Omega, \delta)$, then $\rho^\nu \mathbf{z} \in \mathbf{U}_0(\Omega, \delta)$, and Lemma 3.6 [6]:

$$\|\rho^\nu \mathbf{z}\|_{\mathbf{L}_{2,0}(\Omega)} \leqslant C_\Omega(\|\mathrm{curl}(\rho^\nu \mathbf{z})\|_{L_{2,0}(\Omega)} + \|\mathrm{div}(\rho^\nu \mathbf{z})\|_{L_{2,0}(\Omega)}),$$

i. e.

$$\|\mathbf{z}\|^2_{\mathbf{L}_{2,\nu}(\Omega)} \leqslant 2C^2_\Omega(\|\mathrm{curl}(\rho^\nu \mathbf{z})\|^2_{L_{2,0}(\Omega)} + \|\mathrm{div}(\rho^\nu \mathbf{z})\|^2_{L_{2,0}(\Omega)}). \tag{29}$$

Applying estimates (24) and (25) for the first and second terms of the right-hand side of (29), respectively, and then Corollary 1 to Lemma 2 its inequality (19), we have

$$\|\mathbf{z}\|^2_{\mathbf{L}_{2,\nu}(\Omega)} \leqslant 4C^2_\Omega(\|\mathrm{curl}\ \mathbf{z}\|^2_{L_{2,\nu}(\Omega)} + \|\mathrm{div}\ \mathbf{z}\|^2_{L_{2,\nu}(\Omega)}) + 8\nu^2 C_3^2 C_\Omega^2 \delta^{2\nu} \|\mathbf{z}\|^2_{\mathbf{L}_{2,\nu}(\Omega)}$$

and

$$\left(1 - 8\nu^2 C_3^2 C_\Omega^2 \delta^{2\nu}\right)\|\mathbf{z}\|^2_{\mathbf{L}_{2,\nu}(\Omega)} \leqslant 4C^2_\Omega(\|\mathrm{curl}\ \mathbf{z}\|^2_{L_{2,\nu}(\Omega)} + \|\mathrm{div}\ \mathbf{z}\|^2_{L_{2,\nu}(\Omega)}). \tag{30}$$

For $\nu > 0$, there exists such a value $\delta_0 = \delta_0(\nu) > 0 : \nu^2 C_\Omega^2 C_3^2 \delta_0^{2\nu} = \frac{1}{16}$, that for each $\delta \in (0, \delta_0]$, according to (30), the following chain of relations

$$\frac{1}{2}\|\mathbf{z}\|^2_{\mathbf{L}_{2,\nu}(\Omega)} \leqslant \left(1 - 8\nu^2 C_3^2 C_\Omega^2 \delta^{2\nu}\right)\|\mathbf{z}\|^2_{\mathbf{L}_{2,\nu}(\Omega)} \leqslant 4C^2_\Omega(\|\mathrm{curl}\ \mathbf{z}\|^2_{L_{2,\nu}(\Omega)} + \|\mathrm{div}\ \mathbf{z}\|^2_{L_{2,\nu}(\Omega)})$$

is valid. Lemma 4 is proven.

By the definition of a norm in the space $\mathbf{U}_\nu(\Omega)$, the following statement is derived directly from Lemma 4.

Corollary 2. *Let conditions of Lemma 4 be satisfied, then*

$$\|\mathbf{z}\|^2_{\mathbf{U}_\nu(\Omega)} \leqslant (1 + 8C^2_\Omega)(\|\mathrm{curl}\ \mathbf{z}\|^2_{L_{2,\nu}(\Omega)} + \|\mathrm{div}\ \mathbf{z}\|^2_{L_{2,\nu}(\Omega)}). \tag{31}$$

3. Weighted analogue of LBB-conditions of forms $b_i(\mathbf{v}, s)$

Let us prove a weighted analogue of LBB-conditions of forms $b_i(\mathbf{v}, s)$ on sets of functions $\mathbf{v} \in \mathbf{U}_\nu(\Omega, \delta)$ and $s \in L^0_{2,\nu}(\Omega, \delta)$.

Theorem 1. *For each $\nu > 0$ there exists a value $\delta_1 = \delta_1(\nu) > 0$ ($\delta_1 \leqslant \delta_0, \delta_0$ from Lemma 4) such that for all $\delta \in (0, \delta_1]$ and an arbitrary function $s \in L^0_{2,\nu}(\Omega, \delta)$ the following inequalities*

$$0 < \beta_i \|s\|_{L_{2,\nu}(\Omega)} \leqslant \sup_{\mathbf{v} \in \mathbf{U}_\nu(\Omega,\delta)} \frac{b_i(\mathbf{v}, s)}{\|\mathbf{v}\|_{\mathbf{U}_\nu(\Omega)}}$$

hold, where $\beta_i = \dfrac{\gamma_i}{2\sqrt{1 + 8C^2_\Omega}}$, $i = 1, 2$.

Proof. In [14], it is proved that there exists a value $\delta_2 = \delta_2(\nu) > 0$, such that for all $\delta \in (0, \delta_2]$ and an arbitrary function $s \in L^0_{2,\nu}(\Omega, \delta)$ the following inequalities

$$0 < \gamma_i \|s\|_{L_{2,\nu}(\Omega)} \leqslant \sup_{\mathbf{v} \in \overset{\circ}{\mathbf{W}}{}^1_{2,\nu}(\Omega,\delta)} \frac{b_i(\mathbf{v}, s)}{\|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)}}, \quad \gamma_i > 0 \tag{32}$$

hold. If we use an inequality (31) of Corollary 2 and an estimate for an arbitrary function $\mathbf{v} \in \overset{\circ}{\mathbf{W}}{}^{1}_{2,\nu}(\Omega, \delta)$:

$$\|\text{div } \mathbf{v}\|_{L_{2,\nu}(\Omega)} + \|\text{curl } \mathbf{v}\|_{L_{2,\nu}(\Omega)} \leqslant 2\|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)},$$

then, due to the fact that $\overset{\circ}{\mathbf{W}}{}^{1}_{2,\nu}(\Omega, \delta) \subset \mathbf{U}_\nu(\Omega, \delta)$ and for all $\delta \in (0, \delta_1]$, where $\delta_1 = \min\{\delta_0, \delta_2\}$, from (32) we obtain a chain of inequalities

$$\gamma_i \|s\|_{L_{2,\nu}(\Omega)} \leqslant \sup_{\mathbf{v} \in \overset{\circ}{\mathbf{W}}{}^{1}_{2,\nu}(\Omega,\delta)} \frac{b_i(\mathbf{v}, s)}{\|\mathbf{v}\|_{\mathbf{W}^1_{2,\nu}(\Omega)}} \leqslant 2 \sup_{\mathbf{v} \in \overset{\circ}{\mathbf{W}}{}^{1}_{2,\nu}(\Omega,\delta)} \frac{b_i(\mathbf{v}, s)}{\|\text{div } \mathbf{v}\|_{L_{2,\nu}(\Omega)} + \|\text{curl } \mathbf{v}\|_{L_{2,\nu}(\Omega)}} \leqslant$$

$$\leqslant 2 \sup_{\mathbf{v} \in \mathbf{U}_\nu(\Omega,\delta)} \frac{b_i(\mathbf{v}, s)}{\|\text{div } \mathbf{v}\|_{L_{2,\nu}(\Omega)} + \|\text{curl } \mathbf{v}\|_{L_{2,\nu}(\Omega)}} \leqslant 2\sqrt{1 + 8C_\Omega^2} \sup_{\mathbf{v} \in \mathbf{U}_\nu(\Omega,\delta)} \frac{b_i(\mathbf{v}, s)}{\|\mathbf{v}\|_{\mathbf{U}_\nu(\Omega)}}.$$

An estimate of Theorem 1 is obtained. □

Conclusions

In the article the concept of an R_ν-generalized solution for the Stokes problem with model boundary conditions in a polygonal non-convex domain with a reentrant corner on the boundary in weighted sets is defined. In this case, the variational formulation of the problem is not symmetric. Weighted analogue of the Ladyzhenskaya–Babushka–Brezzi conditions in special norms of weighted spaces is established.

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Весовой аналог LBB-условий для решения задачи Стокса с модельными граничными условиями в области с сингулярностью

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Аннотация. В работе определено понятие R_ν-обобщённого решения задачи Стокса с модельными граничными условиями в области с угловой сингулярностью. Доказан весовой аналог условий Ладыженской-Бабушки-Брецци в области с входящим углом.

Ключевые слова: угловая особенность, задача Стокса с модельными граничными условиями, R_ν-обобщённое решение.