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On some Commutative and Idempotent Finite Groupoids Associated with Subnets of Multilayer Feedforward Neural Networks

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Abstract. The work studies commutative and idempotent finite groupoids that are associated with subnetworks of multilayer feedforward neural networks (hereinafter simply neural networks). Previously, the concept of a neural network subnet was introduced. This paper introduces the concept of a generalized subnetwork of a neural network. This concept generalizes the previously introduced concept. The resulting groupoids are called additive and multiplicative groupoids of generalized subnets of a given neural network. These groupoids model the union and intersection of generalized subnets of a neural network. The conditions that the neural network architecture must satisfy in order for the additive groupoid of generalized subnets to be associative are identified. The conditions that the neural network architecture must satisfy in order for the multiplicative groupoid of generalized subnets to be associative are obtained. Subgroupoids of the constructed groupoids are studied.

Keywords: groupoid, multilayer neural network of feedforward signal propagation, subnetwork of multilayer neural network of feedforward signal propagation, additive groupoid of generalized subnetworks, multiplicative groupoid of generalized subnetworks, generalized subnetwork.

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Introduction

In this work, only multilayer feedforward neural networks are considered (therefore, we will further call them simply neural networks or networks). Information about neural networks can be found in the works [1–4]. The work is a continuation of the works [1, 5, 6]. In the work [1] for each network $\mathcal{N}$, a commutative and idempotent groupoid is constructed $\mathrm{AGS}(\mathcal{N})$. This groupoid is called the additive groupoid of neural network subnets of the neural network $\mathcal{N}$. In the work [6] a multiplicative groupoid of subnets $\mathrm{MGS}(\mathcal{N})$ is constructed. The supports of the groupoids $\mathrm{AGS}(\mathcal{N})$ and $\mathrm{MGS}(\mathcal{N})$ coincide.

The connection between elements of groupoids $\mathrm{AGS}(\mathcal{N})$ and $\mathrm{MGS}(\mathcal{N})$ with neural network subnets $\mathcal{N}$ is discussed in [1,6]. Article [1] introduces the concept of a subnetwork of a multilayer feedforward neural network (see Definition 4 of [1]). Subnet data is obtained from the original network by disabling a certain set of neurons. After switching off the selected neurons, the synaptic

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connections that connect the excluded neurons to any other neurons disappear. The remaining neurons and synaptic connections have the same architectural parameters as in the original network. That is, the activation functions, threshold values, weights of synaptic connections for the neurons and synaptic connections remaining in the subnetwork do not change. Elements of a groupoid $\mathrm{AGS}(\mathcal{N})$ (hence, $\mathrm{MGS}(\mathcal{N})$) contain information about the neurons remaining after switching off. The operation in the groupoid $\mathrm{AGS}(\mathcal{N})$ allows you to model the merging (i.e. unioning) of two subnets into one network, whenever possible. The groupoid operation $\mathrm{MGS}(\mathcal{N})$ allows you to model the intersection of two subnets when possible.

Objectives of the work. Introduction of new groupoids that allow modeling of various processes associated with neural networks. Studying the properties of a neural network depending on the algebraic properties of groupoids built on this neural network.

Main results. This work expands the concept *multilayer feedforward neural network*. By virtue of Definition 3 of [1], a neural network must have at least two layers of neurons. The latter seemed justified in the context that it is networks with at least 2 layers that are of practical value. But this led to excessive formalism. Thus, some elements of the groupoid $\mathrm{AGS}(\mathcal{N})$ could be associated with subnets of the neural network $\mathcal{N}$, but many others could not. At the same time, in practice, situations arise when it is convenient to carry out various manipulations with layers of neurons. In other works, neural networks were composed of neurons (see, for example, [2]), which were associated with abstract automata.

Definition 3 of [1] in this work has been modified so that a neural network can have one layer of neurons (see Definition 1.1). A single neuron can now also be considered a neural network by Definition 1.1. In this work, the concept of a neural network subnet was expanded (see Definition 1.3). Now a neural network subnet can consist of neurons of one layer (see Definition 2.1). One neuron can now also be considered a subnetwork. In Definition 4 of [1] subnetworks were required to contain neurons of at least two layers.

The Definition 2.1 introduces the concept of a *generalized neural network subnet*. This concept allows us to consider generalized subnetworks in which a certain selected set of synaptic connections has been disconnected. The disconnection of a synaptic link is modeled by assigning a weight of zero to that synaptic link. The introduction of this feature is justified from a practical point of view. In practice, it can be convenient to remove weak synaptic connections from a trained neural network (that is, weight connections that are small enough and have little effect on the operation of the network). The latter leads to improved performance of the algorithm built on neural network principles.

The introduction of the concept of a generalized subnetwork of a neural network leads to the appearance of *additive groupoid of generalized subnets* and *multiplicative groupoid of generalized subnets*: $\widehat{\mathrm{AGS}}(\mathcal{N})$ and $\widehat{\mathrm{MGS}}(\mathcal{N})$ (see Definition 2.2). The elements of these groupoids now carry information about the neurons that remain after removing all other neurons, and about the synaptic connections that will be disconnected. Operations in these groupoids will continue to model the union and intersection of neural network subnets.

Let $n(\mathcal{N})$ denote the number of layers of neurons in the network $\mathcal{N}$. The main results of the work are formulated in the form of Theorems 3.1, 3.2 and 4.1. The groupoid $\widehat{\mathrm{AGS}}(\mathcal{N})$ is associative iff $n(\mathcal{N}) = 1$ or $n(\mathcal{N}) = 2$. The groupoid $\widehat{\mathrm{MGS}}(\mathcal{N})$ is associative iff in the neural network $\mathcal{N}$ only the first and last layers have more than one neuron. In particular, gruppoid $\widehat{\mathrm{MGS}}(\mathcal{N})$ is associative if $n(\mathcal{N}) = 1$ or $n(\mathcal{N}) = 2$; in these cases, there are no restrictions on the layers of neurons. Thus, we see that the associativity condition for the groupoids $\widehat{\mathrm{AGS}}(\mathcal{N})$ and $\widehat{\mathrm{MGS}}(\mathcal{N})$ imposes restrictions on architecture (i.e. structure) of the neural network $\mathcal{N}$.

Theorem 4.1 reveals the connection between the generalized subnetwork $\mathcal{N}'$ networks $\mathcal{N}$ and subgroupoids of groupoids $\widehat{\mathrm{AGS}}(\mathcal{N})$ and $\widehat{\mathrm{MGS}}(\mathcal{N})$.

Algebraic properties of groupoids $\widehat{\mathrm{AGS}}(\mathcal{N})$ and $\widehat{\mathrm{MGS}}(\mathcal{N})$ are closely related to the structure of the graph of the neural network $\mathcal{N}$ (this is confirmed by Theorems 3.1, 3.2).

1. Basic definitions

Further, $\mathbb{R}$ is the set of real numbers and $F(\mathbb{R}) := \mathrm{Hom}(\mathbb{R}, \mathbb{R})$ is the set of all mappings from $\mathbb{R}$ to $\mathbb{R}$.

Definition 1.1. Let the following objects be given:
1) the tuple $(M_1, \ldots, M_n)$ of length $n \geqslant 1$ of finite non-empty sets, where for $i \neq j$ the condition $M_i \cap M_j = \varnothing$ is satisfied;
2) the set $S := (M_1 \times M_2) \cup (M_2 \times M_3) \cup \cdots \cup (M_{n-1} \times M_n)$;
3) the mapping $f : S \to \mathbb{R}$;
4) the set $A := M_1 \cup \cdots \cup M_n$;
5) the mapping $g : A \to F(\mathbb{R})$;
6) the mapping $l : A \to \mathbb{R}$.

Then the tuple $\mathcal{N} = (M_1, \ldots, M_n, f, g, l)$ will be called *a multilayer feedforward neural network.*

Neural network operation. Each neural network $\mathcal{N} = (M_1, \ldots, M_n, f, g, l)$ and each two bijections

$$i : M_1 :\to \{1, \ldots, |M_1|\}, \quad o : M_n \to \{1, \ldots, |M_n|\}$$

corresponds to the mapping $F_{i,o,\mathcal{N}} : \mathbb{R}^{|M_1|} \to \mathbb{R}^{|M_n|}$, which implements the operation of a neural network as a computing circuit. The mapping $F_{i,o,\mathcal{N}}$ is defined using an artificial neuron (McCulloch–Pitts; see [2]) model. If compositions of neural networks are studied, then the bijections i and o must be written into the definition of definition 1.1 (see [7]).

Standard notations associated with neural networks. We will associate the following notations with each neural network $\mathcal{N} = (M_1, \ldots, M_n, f, g, l)$:

$$n(\mathcal{N}) = n, \quad A(\mathcal{N}) = \bigcup_{i=1}^{n} M_i, \quad Syn(\mathcal{N}) = \bigcup_{i=1}^{n-1} M_i \times M_{i+1}.$$

Thus, $n(\mathcal{N})$ is the number of all layers of the neural network, $A(\mathcal{N})$ is the set of all neurons, and $Syn(\mathcal{N})$ is the set all synaptic connections. We will call the tuple $(M_1, \ldots, M_n)$ *the main tuple of neurons* of the network $\mathcal{N}$.

A tuple of empty sets will be denoted by the symbol $\overline{\varnothing} := (\varnothing, \ldots, \varnothing)$ (the length of such a tuple will always be clear from the context). Let two tuples $\overline{X} = (X_1, \ldots, X_n)$ and $\overline{Y} = (Y_1, \ldots, Y_n)$ of finite sets be given. Then we will use the notation

$$\overline{X} \cup \overline{Y} := (X_1 \cup Y_1, \ldots, X_n \cup Y_n); \quad \overline{X} \cap \overline{Y} := (X_1 \cap Y_1, \ldots, X_n \cap Y_n);$$

$$\overline{X} \subseteq \overline{Y} \Leftrightarrow (X_1 \subseteq Y_1) \wedge (X_2 \subseteq Y_2) \wedge \cdots \wedge (X_n \subseteq Y_n).$$

Definition 1.2. Let $(X_1, \ldots, X_n)$ be some tuple composed of finite sets. We will say that the tuple is *continuous* if for any distinct $i, j\ in \{1, \ldots, n\}$ the following implication holds: if $X_i \neq \varnothing$ and $X_j \neq \varnothing$ and $i < j$, then for any $s \in \{i, i+1, \ldots, j-1, j\}$ the inequality $X_s \neq \varnothing$ holds. The tuple $\overline{\varnothing}$ is considered continuous by definition.

For a tuple of sets to be continuous, it must not contain an alternation of a non-empty set with an interval of empty sets, and then again with a non-empty set.

Let us introduce a definition of *subnet*, similar to Definition 4 from [1]. But it differs from it in that single-layer networks can now also be subnets.

Definition 1.3. Let the neural network be defined $\mathcal{N} = (M_1, \ldots, M_n, f, g, l)$ and a continuous tuple $(X_1, \ldots, X_n)$ is given such that the conditions are satisfied $(X_1, \ldots, X_n) \subseteq (M_1, \ldots, M_n)$ and $(X_1, \ldots, X_n) \neq \overline{\varnothing}$. We assume that $(Y_1, \ldots, Y_m)$ is a tuple obtained from a tuple $(X_1, \ldots, X_n)$ by deleting components equal to the empty set, where $m \leqslant n$. If f' is the restriction of the function f on the set $S' := (Y_1 \times Y_2) \cup (Y_2 \times Y_3) \cup \cdots \cup (Y_{m-1} \times Y_m)$ and g', l' are the restriction of the function g and the restriction of the function l on the set $A' := Y_1 \cup \cdots \cup Y_m$, then the object $\mathcal{N}' := (Y_1, \ldots, Y_m, f', g', l')$ will be called *subnet* of the network $\mathcal{N}$. We will say that the tuple $(X_1, \ldots, X_n)$ *induces* a subnetwork $\mathcal{N}'$. The tuple $(Y_1, \ldots, Y_m)$ is the main tuple of neurons of the subnetwork $\mathcal{N}'$. In general, the tuples $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_m)$ can be different.

Groupoids AGS($\mathcal{N}$) are introduced into [1], and groupoids MGS($\mathcal{N}$) are introduced into [6]. For the convenience of the reader, we give an explicit definition below.

Definition 1.4. Let a neural network $\mathcal{N} = (M_1, \ldots, M_n, f, g, l)$ be defined with a main tuple of neurons $\overline{M} = (M_1, \ldots, M_n)$. The set of all possible continuous tuples $\overline{X} \subseteq \overline{M}$ will be denoted by the symbol AGS($\mathcal{N}$). Further, $\overline{X}$ and $\overline{Y}$ are two arbitrary element from AGS($\mathcal{N}$). Let us define binary algebraic operations $(+)$ and $(*)$ on the set AGS($\mathcal{N}$):

$$\overline{X} + \overline{Y} := \begin{cases} \overline{X} \cup \overline{Y}, & \text{if } \overline{X} \cup \overline{Y} \in \mathrm{AGS}(\mathcal{N}), \\ \overline{\varnothing}, & \text{if } \overline{X} \cup \overline{Y} \notin \mathrm{AGS}(\mathcal{N}); \end{cases} \quad \overline{X} * \overline{Y} := \begin{cases} \overline{X} \cap \overline{Y}, & \text{if } \overline{X} \cap \overline{Y} \in \mathrm{AGS}(\mathcal{N}), \\ \overline{\varnothing}, & \text{if } \overline{X} \cap \overline{Y} \notin \mathrm{AGS}(\mathcal{N}). \end{cases}$$

Then the groupoid AGS($\mathcal{N}$) := (AGS($\mathcal{N}$), +) will be called *additive groupoid of neural network subnets* $\mathcal{N}$. The groupoid MGS($\mathcal{N}$) := (AGS($\mathcal{N}$), $*$) will be called *the multiplicative groupoid of neural network subnets* $\mathcal{N}$.

Remark 1.1. Each tuple $\overline{X} \neq \overline{\varnothing}$ of AGS($\mathcal{N}$) induces some subnetwork. Two different tuples from AGS($\mathcal{N}$) induce different subnets of the network $\mathcal{N}$ (this follows trivially from the definition of 1.3). Each subnet of the network $\mathcal{N}$ is induced by some tuple from AGS($\mathcal{N}$). Thus, there is a bijection between the set of all subnets of the network $\mathcal{N}$ and the set AGS($\mathcal{N}$) $\setminus \{\overline{\varnothing}\}$.

Remark 1.2. The changes made to Definitions 3 and 4 from [1] does not change the contents of the set AGS($\mathcal{N}$). Additionally, these changes do not affect definitions of operation: $(+)$ and $(*)$. These changes allow more elements in AGS($\mathcal{N}$) to be associated with subnets. Continuous tuples with only one layer different from the empty set did not induce any subnetworks due to Definition 4 of [1]. Because the subnets from that definition had at least two layers. If $\mathcal{G}_1(\mathcal{N})$ is the set of all subnets by Definition 4 of [1] and $\mathcal{G}_2(\mathcal{N})$ is the set of all subnets by Definition 1.3, then the inclusion $\mathcal{G}_1(\mathcal{N}) \subset \mathcal{G}_2(\mathcal{N})$.

2. Generalized subnets

The concept of a neural network subnet, introduced by the Definition 1.3, describes objects obtained from the original network by switching off a certain set of neurons and the deleting of synaptic connections associated with disconnected neurons. Let's build a model of a generalized subnetwork that describes objects that can be obtained by turning off a certain set of neurons and resetting the weights of a given set of synaptic connections to zero.

The set of all subsets of the set X, as usual, will be denoted by 2^X. Let $\mathcal{N} = (M_1, \ldots, M_n, f, g, l)$. Then we introduce the set

$$\widehat{\mathrm{AGS}}(\mathcal{N}) := \mathrm{AGS}(\mathcal{N}) \times 2^{Syn(\mathcal{N})}.$$

Elements from $\widehat{\mathrm{AGS}}(\mathcal{N})$ will be denoted by capital Latin letters with a cap.

Definition 2.1. Let $\mathcal{N}' = (Y_1, \ldots, Y_m, f', g', l')$ is subnet of network $\mathcal{N}$, which is induced by the tuple $\overline{X}$ from $\mathrm{AGS}(\mathcal{N})$. We assume that $S' := (Y_1 \times Y_2) \cup (Y_2 \times Y_3) \cup \cdots \cup (Y_{m-1} \times Y_m)$ and Q is a certain subset of set $Syn(\mathcal{N})$. Let us introduce the mapping

$$f''(s) := \begin{cases} f'(s), & s \notin Q, \\ 0, & s \in Q \end{cases} \qquad (s \in S').$$

Then the object $\mathcal{N}' := (Y_1, \ldots, Y_m, f'', g', l')$ will be called a *generalized subnetwork* of the network $\mathcal{N}$. We will say that the tuple $\widehat{U} = (\overline{X}, Q)$ *induces* a generalized subnet $\mathcal{N}'$. Cortege $(Y_1, \ldots, Y_m)$ is the main tuple of neurons of the generalized subnet $\mathcal{N}'$.

Remark 2.1. A generalized subnet $\mathcal{N}'$ is an object that satisfies the definition of 1.1. Various tuples from $\widehat{\mathrm{AGS}}(\mathcal{N})$ can induce one generalized subnet $\mathcal{N}'$ of the network $\mathcal{N}$ (an important difference with the case of simple subnets, see remark 1.1). A tuple $\widehat{U}$ induces a subnet of the network $\mathcal{N}$ if and only if it contains in the set

$$\widehat{\mathrm{AGS}}(\mathcal{N}) \setminus \{(\overline{\varnothing}, W) \mid W \subseteq Syn(\mathcal{N})\}.$$

Definition 2.2. We assume that the neural network $\mathcal{N} = (M_1, \ldots, M_n, f, g, l)$ is defined. Next, $\widehat{U}_1 = (\overline{X}_1, Q_1)$, $\widehat{U}_2 = (\overline{X}_2, Q_2)$ — these are two arbitrary elements from $\widehat{\mathrm{AGS}}(\mathcal{N})$. Let us define binary algebraic operations $(+)$ and $(*)$ on the set $\widehat{\mathrm{AGS}}(\mathcal{N})$:

$$\widehat{U}_1 + \widehat{U}_2 := \begin{cases} (\overline{X}_1 \cup \overline{X}_2, Q_1 \cup Q_2), & \text{if } \overline{X}_1 \cup \overline{X}_2 \in \mathrm{AGS}(\mathcal{N}), \\ (\overline{\varnothing}, \varnothing), & \text{if } \overline{X}_1 \cup \overline{X}_2 \notin \mathrm{AGS}(\mathcal{N}); \end{cases} \tag{1}$$

$$\widehat{U}_1 * \widehat{U}_2 := \begin{cases} (\overline{X}_1 \cap \overline{X}_2, Q_1 \cap Q_2), & \text{if } \overline{X}_1 \cap \overline{X}_2 \in \mathrm{AGS}(\mathcal{N}), \\ (\overline{\varnothing}, \varnothing), & \text{if } \overline{X}_1 \cap \overline{X}_2 \notin \mathrm{AGS}(\mathcal{N}). \end{cases} \tag{2}$$

Then the groupoid $\widehat{\mathrm{AGS}}(\mathcal{N}) := (\widehat{\mathrm{AGS}}(\mathcal{N}), +)$ will be called *the additive groupoid of generalized neural network subnets* $\mathcal{N}$ and groupoid $\widehat{\mathrm{MGS}}(\mathcal{N}) := (\widehat{\mathrm{AGS}}(\mathcal{N}), *)$ we will call *multiplicative groupoid of generalized neural network subnets* $\mathcal{N}$.

Remark 2.2. Operations in groupoids $\widehat{\mathrm{AGS}}(\mathcal{N})$ and $\widehat{\mathrm{MGS}}(\mathcal{N})$ are also denoted as in the groupoids $\mathrm{AGS}(\mathcal{N})$ and $\mathrm{MGS}(\mathcal{N})$, respectively. In practice this does not lead to confusion. It is always clear from the context what operation is meant. Further, for tuples from $\widehat{\mathrm{AGS}}(\mathcal{N})$ it will be convenient to use the operation of componentwise union and intersection. If $\widehat{U}_1 = (\overline{X}_1, Q_1)$ and $\widehat{U}_2 = (\overline{X}_2, Q_2)$ then $\widehat{U}_1 \cup \widehat{U}_2 := (\overline{X}_1 \cup \overline{X}_2, Q_1 \cup Q_2)$ and $\widehat{U}_1 \cap \widehat{U}_2 := (\overline{X}_1 \cap \overline{X}_2, Q_1 \cap Q_2)$.

Remark 2.3. The additive generalized subnet groupoid models the merging of two subnets into one when possible and returns the tuple $(\overline{\varnothing}, \varnothing)$ when this is not possible. The multiplicative groupoid of generalized subnets models the intersection of two subnets (i.e., returns a subnet that is contained in both networks) when possible and returns the tuple $(\overline{\varnothing}, \varnothing)$ when this is not possible.

3. Basic algebraic properties

The main result of this section is expressed in the form of Theorems 3.1 and 3.2. First, we formulate and prove some algebraic properties of additive and multiplicative groupoids of generalized subnets (see Properties 3.1, 3.2 and 3.3).

Property 3.1. *For any neural network $\mathcal{N}$ the following statements are satisfied:*
1) groupoids $\widehat{\mathrm{AGS}}(\mathcal{N})$ and $\widehat{\mathrm{MGS}}(\mathcal{N})$ are commutative and idempotent;
2) the tuple $(\overline{\varnothing}, \varnothing)$ is a neutral element of the groupoid $\widehat{\mathrm{AGS}}(\mathcal{N})$;
3) the tuple $(\overline{\varnothing}, \varnothing)$ has the multiplicative zero property in the groupoid $\widehat{\mathrm{MGS}}(\mathcal{N})$;
4) the tuple $((M_1, \ldots, M_n), Syn(\mathcal{N}))$ is a neutral element in the groupoid $\widehat{\mathrm{MGS}}(\mathcal{N})$, where $(M_1, \ldots, M_n)$ is the main tuple of neurons of the network $\mathcal{N}$;
5) the tuple $((M_1, \ldots, M_n), Syn(\mathcal{N}))$ has the multiplicative zero property in the groupoid $\widehat{\mathrm{AGS}}(\mathcal{N})$.

Proof. Commutativity and idempotency of the groupoids $\widehat{\mathrm{AGS}}(\mathcal{N})$ and $\widehat{\mathrm{MGS}}(\mathcal{N})$ is trivial follows from (1) and (2).

Statements 2) – 5) follow from the definitions of the operations $(+)$ and $(*)$. Indeed, let $\widehat{U} = (\overline{X}, Q)$ be an arbitrary element of the set $\widehat{\mathrm{AGS}}(\mathcal{N})$ (hence, it is an element of groupoids $\widehat{\mathrm{AGS}}(\mathcal{N})$ and $\widehat{\mathrm{MGS}}(\mathcal{N})$). We assume that $(M_1, \ldots, M_n)$ is the main tuple of neurons in the network $\mathcal{N}$. Then the equalities

$$\widehat{U} + (\overline{\varnothing}, \varnothing) = (\overline{X} \cup \overline{\varnothing}, Q \cup \varnothing) = \widehat{U}, \quad \widehat{U} * (\overline{\varnothing}, \varnothing) = (\overline{X} \cap \overline{\varnothing}, Q \cap \varnothing) = (\overline{\varnothing}, \varnothing),$$

$$\widehat{U} * ((M_1, \ldots, M_n), Syn(\mathcal{N})) = (\overline{X} \cap (M_1, \ldots, M_n), Q \cap Syn(\mathcal{N})) = (\overline{X}, Q),$$

$$\widehat{U} + ((M_1, \ldots, M_n), Syn(\mathcal{N})) = (\overline{X} \cup (M_1, \ldots, M_n), Q \cup Syn(\mathcal{N})) = ((M_1, \ldots, M_n), Syn(\mathcal{N}))$$

show the validity of statements 2)–5). □

Property 3.2. *If $\overline{X}_1 \cup \overline{X}_2, \overline{Y}_1 \cap \overline{Y}_2 \in \mathrm{AGS}(\mathcal{N})$, then for elements $\widehat{U}_1 = (\overline{X}_1, Q_1)$, $\widehat{U}_2 = (\overline{X}_2, Q_2)$ of the groupoid $\widehat{\mathrm{AGS}}(\mathcal{N})$ and elements $\widehat{U}_3 = (\overline{Y}_1, W_1)$, $\widehat{U}_4 = (\overline{Y}_2, W_2)$ of the groupoid $\widehat{\mathrm{MGS}}(\mathcal{N})$ the equalities hold*

$$\widehat{U}_1 + \widehat{U}_2 = (\overline{X}_1 + \overline{X}_2, Q_1 \cup Q_2), \quad \widehat{U}_3 * \widehat{U}_4 = (\overline{Y}_1 * \overline{Y}_2, W_1 \cap W_2). \tag{3}$$

Proof. Since on the left side of the equalities (3) the operations $(+)$ and $(*)$ are operations of groupoids $\widehat{\mathrm{AGS}}(\mathcal{N})$ and $\widehat{\mathrm{MGS}}(\mathcal{N})$, and on the right side these are groupoid operations $\mathrm{AGS}(\mathcal{N})$ and $\mathrm{MGS}(\mathcal{N})$, then by equalities (1) and (2) the equalities are satisfied

$$(\overline{X}_1 + \overline{X}_2, Q_1 \cup Q_2) = \begin{cases} (\overline{X}_1 \cup \overline{X}_2, Q_1 \cup Q_2), & \text{if } \overline{X}_1 \cup \overline{X}_2 \in \mathrm{AGS}(\mathcal{N}) \\ (\overline{\varnothing}, \varnothing), & \text{if } \overline{X}_1 \cup \overline{X}_2 \notin \mathrm{AGS}(\mathcal{N}) \end{cases} = \widehat{U}_1 + \widehat{U}_2,$$

$$(\overline{X}_1 * \overline{X}_2, Q_1 \cap Q_2) = \begin{cases} (\overline{X}_1 \cap \overline{X}_2, Q_1 \cap Q_2), & \text{if } \overline{X}_1 \cap \overline{X}_2 \in \mathrm{AGS}(\mathcal{N}) \\ (\overline{\varnothing}, \varnothing), & \text{if } \overline{X}_1 \cap \overline{X}_2 \notin \mathrm{AGS}(\mathcal{N}) \end{cases} = \widehat{U}_1 * \widehat{U}_2,$$

which give equalities (3). □

We define mapping $\Psi : \widehat{\mathrm{AGS}}(\mathcal{N}) \to \mathrm{AGS}(\mathcal{N})$ as follows $\Psi((\overline{X}, Q)) := \overline{X}$, where $(\overline{X}, Q) \in \widehat{\mathrm{AGS}}(\mathcal{N})$ and $\overline{X} \in \mathrm{AGS}(\mathcal{N})$.

Property 3.3. *The following statements are true:*
1) the mapping Ψ *is a homomorphism of the groupoid* $\widehat{\mathrm{AGS}}(\mathcal{N})$ *into the groupoid* $\mathrm{AGS}(\mathcal{N})$*;*
2) the mapping Ψ *is a homomorphism of the groupoid* $\widehat{\mathrm{MGS}}(\mathcal{N})$ *into the groupoid* $\mathrm{MGS}(\mathcal{N})$*;*
3) the sets $\Phi(\widehat{\mathrm{AGS}}(\mathcal{N})) = \mathrm{AGS}(\mathcal{N})$ *are equal.*

Proof. Let

$$\widehat{U}_1 = (\overline{X}_1, Q_1),\ \widehat{U}_2 = (\overline{X}_2, Q_2),\ \widehat{U}_3 = (\overline{Y}_1, W_1),\ \widehat{U}_4 = (\overline{Y}_2, W_2)$$

these are arbitrary elements of the groupoids $\widehat{\mathrm{AGS}}(\mathcal{N})$ and $\widehat{\mathrm{MGS}}(\mathcal{N})$. We assume that $\overline{X}_1 \cup \overline{X}_2$ and $\overline{Y}_1 \cap \overline{Y}_2$ are continuous tuples (i.e. tuples from $\mathrm{AGS}(\mathcal{N})$). Then, by virtue of the equalities (3), the equalities

$$\Psi(\widehat{U}_1+\widehat{U}_2) = \Psi((\overline{X}_1+\overline{X}_2, Q_1\cup Q_2)) = \overline{X}_1+\overline{X}_2 = \Psi((\overline{X}_1, Q_1))+\Psi((\overline{X}_2, Q_2)) = \Psi(\widehat{U}_1)+\Psi(\widehat{U}_2),$$

$$\Psi(\widehat{U}_3 * \widehat{U}_4) = \Psi((\overline{Y}_1 * \overline{Y}_2, W_1 \cap W_2)) = \overline{Y}_1 * \overline{Y}_2 = \Psi((\overline{Y}_1, W_1)) * \Psi((\overline{Y}_2, W_2)) = \Psi(\widehat{U}_3) * \Psi(\widehat{U}_4).$$

Let now the tuples $\overline{X}_1 \cup \overline{X}_2$ and $\overline{Y}_1 \cap \overline{Y}_2$ not belong to $\mathrm{AGS}(\mathcal{N})$. Then we have equalities

$$\Psi(\widehat{U}_1 + \widehat{U}_2) = \Psi((\overline{\varnothing}, \varnothing)) = \overline{\varnothing} = \overline{X}_1 + \overline{X}_2 = \Psi((\overline{X}_1, Q_1)) + \Psi((\overline{X}_2, Q_2)) = \Psi(\widehat{U}_1) + \Psi(\widehat{U}_2),$$

$$\Psi(\widehat{U}_3 * \widehat{U}_4) = \Psi((\overline{\varnothing}, \varnothing)) = \overline{\varnothing} = \overline{Y}_1 * \overline{Y}_2 = \Psi((\overline{Y}_1, Q_1)) * \Psi((\overline{Y}_2, Q_2)) = \Psi(\widehat{U}_3) * \Psi(\widehat{U}_4).$$

These equalities show that Ψ is a homomorphism from statements 1) and 2). Statements 1) and 2) have been proven. Statement 3) follows from the definition of the set $\widehat{\mathrm{AGS}}(\mathcal{N})$. The property is proved. □

Theorem 3.1. *For any neural network* $\mathcal{N}$ *the following statements are equivalent.*
1) The condition $n(\mathcal{N}) \in \{1, 2\}$ *is satisfied.*
2) The groupoid $\mathrm{AGS}(\mathcal{N})$ *is associative.*
3) The groupoid $\widehat{\mathrm{AGS}}(\mathcal{N})$ *is associative.*

Proof. Let us show that statement 1) is equivalent to statement 2). Let statement 1) be true. Then for any $\overline{X}_1, \overline{X}_2 \in \mathrm{AGS}(\mathcal{N})$ tuple $\overline{X}_1 + \overline{X}_2$ will be a continuous tuple (since there is no way to get a discontinuous tuple). Therefore, for any $\overline{Y}_1, \overline{Y}_2, \overline{Y}_3 \in \mathrm{AGS}(\mathcal{N})$ the relations

$$(\overline{Y}_1+\overline{Y}_2)+\overline{Y}_3 = (\overline{Y}_1\cup\overline{Y}_2)\cup\overline{Y}_3 = \overline{Y}_1\cup\overline{Y}_2\cup\overline{Y}_3, \quad \overline{Y}_1+(\overline{Y}_2+\overline{Y}_3) = \overline{Y}_1\cup(\overline{Y}_2\cup\overline{Y}_3) = \overline{Y}_1\cup\overline{Y}_2\cup\overline{Y}_3.$$

These relations show that the groupoid $\mathrm{AGS}(\mathcal{N})$ is associative. Therefore, the groupoid $\widehat{\mathrm{AGS}}(\mathcal{N})$ is associative. Thus, from 1) it follows 2).

On the other hand, suppose that statement 2) holds and statement 1) does not hold. Then $n(\mathcal{N}) > 2$. In this case, for any network $\mathcal{N}$ it is always possible to specify three tuples $\overline{X}_1$, $\overline{X}_2$, $\overline{X}_3$ for which the condition is satisfied $\overline{X}_1 + (\overline{X}_2 + \overline{X}_3) \neq (\overline{X}_1 + \overline{X}_2) + \overline{X}_3$. For example, you can take tuples:

$$\overline{X}_1 = (\{a\}, \varnothing, \varnothing, \ldots, \varnothing),\ \overline{X}_2 = (\varnothing, \{b\}, \varnothing, \ldots, \varnothing),\ \overline{X}_3 = (\varnothing, \varnothing, \{c\}, \ldots, \varnothing).$$

A contradiction has been obtained. It shows that from 2) follows (1). This means that statements 1) and 2) are equivalent.

Let us show that statements 2) and 3) are equivalent. Let statement 2) be true. The groupoid $\mathrm{AGS}(\mathcal{N})$ is associative if and only if $n(\mathcal{N}) \in \{1, 2\}$. Then, as noted above, for any $\overline{X}_1, \overline{X}_2 \in \mathrm{AGS}(\mathcal{N})$ tuple $\overline{X}_1 \cup \overline{X}_2$ is a continuous tuple. Therefore, by virtue of the equality (3) for any elements $\widehat{U}_1 = (\overline{Y}_1, Q_1)$, $\widehat{U}_2 = (\overline{Y}_2, Q_2)$, $\widehat{U}_3 = (\overline{Y}_3, Q_3)$ groupoid $\widehat{\mathrm{AGS}}(\mathcal{N})$ the following relations hold:

$$(\widehat{U}_1 + \widehat{U}_2) + \widehat{U}_3 = (\overline{Y}_1 + \overline{Y}_2, Q_1 \cup Q_2) + (\overline{Y}_3, Q_3) = ((\overline{Y}_1 + \overline{Y}_2) + \overline{Y}_3, (Q_1 \cup Q_2) \cup Q_3) =$$

$$= (\overline{Y}_1 \cup \overline{Y}_2 \cup \overline{Y}_3, Q_1 \cup Q_2 \cup Q_3) = (\overline{Y}_1 + (\overline{Y}_2 + \overline{Y}_3), Q_1 \cup (Q_2 \cup Q_3)) = \widehat{U}_1 + (\widehat{U}_2 + \widehat{U}_3).$$

From this we obtain the associativity of the groupoid $\widehat{\mathrm{AGS}}(\mathcal{N})$. Thus, from 2) follows 3).

Statement 3) implies statement 2). Indeed, since Ψ is a homomorphism of $\widehat{\mathrm{AGS}}(\mathcal{N})$ into $\mathrm{AGS}(\mathcal{N})$ and $\Phi(\widehat{\mathrm{AGS}}(\mathcal{N})) = \mathrm{AGS}(\mathcal{N})$ (see property 3.3), then from the associativity of the groupoid $\widehat{\mathrm{AGS}}(\mathcal{N})$ implies the associativity of the groupoid $\mathrm{AGS}(\mathcal{N})$.

Statements 2) and 3) are equivalent. Since 2) is equivalent to 1), then 3) is equivalent to 1). The theorem is proved. □

Remark 3.1. Statement 2 of [1] states that the groupoid $\mathrm{AGS}(\mathcal{N})$ is associative if and only if $\mathcal{N}$ is a two-layer neural network. The discrepancy with the results of Theorem 3.1 is due to the fact that in the work [1] single-layer neural networks were not considered. Taking this fact into account, it can be argued that the results of Statement 2 of [1] and Theorem 3.1 are consistent.

Theorem 3.2. *For any neural network $\mathcal{N}$ the following statements are equivalent.*
1) In a neural network $\mathcal{N} = (M_1, \ldots, M_n, f, g, l)$, only the input layer M_1 and the output layer M_n can contain more than one neuron.
2) The groupoid $\mathrm{MGS}(\mathcal{N})$ is associative.
3) The groupoid $\widehat{\mathrm{MGS}}(\mathcal{N})$ is associative.

Proof. Let us show that statements 1) and 2) are equivalent. Let statement 1) be true. If $n(\mathcal{N}) \in \{1, 2\}$, then for any $\overline{X} = (X_1, \ldots, X_n)$ and $\overline{Y} = (Y_1, \ldots, Y_n)$ from $\mathrm{MGS}(\mathcal{N})$ the tuple $\overline{X} \cap \overline{Y}$ is continuous. In this case, $\overline{X} * \overline{Y} = \overline{X} \cap \overline{Y}$. Due to the associativity of the operation $(\cap)$ on sets, we have the associativity of the operation $(\cap)$ on tuples from $\mathrm{MGS}(\mathcal{N})$. Therefore the groupoid $\mathrm{MGS}(\mathcal{N})$ is associative.

We assume that $n(\mathcal{N}) > 2$ and $\overline{X} = (X_1, \ldots, X_n)$, $\overline{Y} = (Y_1, \ldots, Y_n)$ are two elements of the groupoid $\mathrm{MGS}(\mathcal{N})$ such that the tuple $\overline{X} \cap \overline{Y}$ is not continuous. This means that the following conditions are met:
(c.1) there is an index $i \in \{1, \ldots, n\}$ such that $X_i \cap Y_i = \varnothing$;
(c.2) there are indices $u, v \in \{1, \ldots, n\}$ such that the conditions are satisfied

$$u < i < v,\ X_u \cap Y_u \neq \varnothing,\ X_v \cap Y_v \neq \varnothing.$$

From (c.2) the conditions follow: $i \neq 1$ and $i \neq n$. Condition (c.1) cannot be satisfied. Indeed, since Statement 3) holds, then $X_i = Y_i = \{a\}$, where a is an element of layer M_i. Thus, we have shown that for any $\overline{X}$ and $\overline{Y}$ from The $\mathrm{MGS}(\mathcal{N})$ tuple $\overline{X} \cap \overline{Y}$ is continuous. Consequently, the identity $\overline{X} * \overline{Y} = \overline{X} \cap \overline{Y}$ holds. Therefore the groupoid $\mathrm{MGS}(\mathcal{N})$ is associative. Statement 1) gives statement 2).

Let us show that statement 2) implies statement 1). Let the groupoid $\mathrm{MGS}(\mathcal{N})$ is associative and statement 1) does not hold. Since statement 1) does not hold, then $n(\mathcal{N}) > 2$ and there is an index $i \notin \{1, n\}$ such that layer M_i contains more than one neuron. For any tuple $\overline{Y}$ from $\mathrm{MGS}(\mathcal{N})$ we denote by $K_s(\overline{Y})$ the s-th component of the tuple $\overline{Y}$. For any network $\mathcal{N}$ with the specified conditions, we can define tuples $\overline{Y}_1$, $\overline{Y}_2$ and $\overline{Y}_3$ from $\mathrm{MGS}(\mathcal{N})$ so that the following conditions are satisfied:

$$K_{i-1}(\overline{Y}_1) = \{a\},\ K_i(\overline{Y}_1) = \{b\},\ K_{i+1}(\overline{Y}_1) = \{c\},\ K_s(\overline{Y}_1) = \varnothing \quad (s \notin \{i-1, i, i+1\});$$

$$K_{i-1}(\overline{Y}_2) = \{a\},\ K_i(\overline{Y}_2) = \{m\},\ K_{i+1}(\overline{Y}_2) = \{c\},\ K_s(\overline{Y}_2) = \varnothing\ \ (s \notin \{i-1, i, i+1\});$$

$$K_{i-1}(\overline{Y}_3) = \varnothing,\ K_i(\overline{Y}_3) = \varnothing,\ K_{i+1}(\overline{Y}_3) = \{c\},\ K_s(\overline{Y}_3) = \varnothing\ \ (s \notin \{i-1, i, i+1\})$$

$$(a \in M_{i-1},\ b, m \in M_i,\ c \in M_{i+1}).$$

Then the equalities hold $(\overline{Y}_1 * \overline{Y}_2) * \overline{Y}_3 = \overline{\varnothing}$, $\overline{Y}_1 * (\overline{Y}_2 * \overline{Y}_3) = \overline{Y}_3$. The equality data shows the lack of associativity in the groupoid $\mathrm{MGS}(\mathcal{N})$ if $|M_i| > 2$ and $i \neq 1, n$. This contradiction shows that statement 1) must be true if statement 2) is true. Statements 1) and 2) are equivalent.

Let us show that from statement 2) and 3) are equivalent. Let 2 be fulfilled The groupoid $\mathrm{MGS}(\mathcal{N})$ is associative if and only if Statement 1) holds. Therefore, for any $\overline{X}_1, \overline{X}_2 \in \mathrm{MGS}(\mathcal{N})$ tuple $\overline{X}_1 \cap \overline{X}_2$ is a continuous tuple. Therefore, by virtue of equalities (3) for any elements $\widehat{U}_1 = (\overline{Y}_1, Q_1)$, $\widehat{U}_2 = (\overline{Y}_2, Q_2)$, $\widehat{U}_3 = (\overline{Y}_3, Q_3)$ groupoid $\widehat{\mathrm{MGS}}(\mathcal{N})$ the following relations hold:

$$(\widehat{U}_1 * \widehat{U}_2) * \widehat{U}_3 = (\overline{Y}_1 * \overline{Y}_2, Q_1 \cap Q_2) * (\overline{Y}_3, Q_3) = ((\overline{Y}_1 * \overline{Y}_2) * \overline{Y}_3, (Q_1 \cap Q_2) \cap Q_3) =$$

$$= (\overline{Y}_1 \cap \overline{Y}_2 \cap \overline{Y}_3, Q_1 \cap Q_2 \cap Q_3) = (\overline{Y}_1 * (\overline{Y}_2 * \overline{Y}_3), Q_1 \cap (Q_2 \cap Q_3)) = \widehat{U}_1 * (\widehat{U}_2 * \widehat{U}_3).$$

From this we obtain the associativity of the groupoid $\widehat{\mathrm{MGS}}(\mathcal{N})$. Thus, from 2) follows 3).

Statement 3) implies statement 2). Indeed, since the groupoid $\mathrm{MGS}(\mathcal{N})$ is a homomorphic image of the groupoid $\widehat{\mathrm{MGS}}(\mathcal{N})$ (see property 3.3), then the associativity of $\widehat{\mathrm{MGS}}(\mathcal{N})$ implies the associativity of $\mathrm{MGS}(\mathcal{N})$. Statements 2) and 3) are equivalent. Since 2) is equivalent to 1), then 3) is equivalent to 1). The theorem is proved. □

4. Generalized subnets and subgroupoids

Theorem 4.1. *Let $\mathcal{N}'$ be a generalized subnet of the neural network $\mathcal{N}$. Then the set $\widehat{\mathrm{AGS}}(\mathcal{N})$ has a subset $T(\mathcal{N}')$ such that this subset is a subgroupoid in the groupoid $\widehat{\mathrm{AGS}}(\mathcal{N})$ and a subgroupoid in the groupoid $\widehat{\mathrm{MGS}}(\mathcal{N})$. In this case, the isomorphisms hold*

$$(T(\mathcal{N}'), +) \cong \widehat{\mathrm{AGS}}(\mathcal{N}'), \quad (T(\mathcal{N}'), *) \cong \widehat{\mathrm{MGS}}(\mathcal{N}').$$

Proof. Let the tuple $\widehat{U} = (\overline{X}, Q)$ induce a generalized subnet $\mathcal{N}'$. Let's build a set

$$T(\mathcal{N}') := \{(\overline{V}, W) \in \widehat{\mathrm{AGS}}(\mathcal{N}) \mid \overline{V} \subseteq \overline{X},\ W \subseteq Syn(\mathcal{N}')\}.$$

From the construction it is clear that $T(\mathcal{N}') \subseteq \widehat{\mathrm{AGS}}(\mathcal{N})$ does not depend on the set Q. The set $T(\mathcal{N}')$ contains the tuple $(\overline{\varnothing}, \varnothing)$. Moreover, $T(\mathcal{N}')$ is closed under the operation $(+)$ in the groupoid $\widehat{\mathrm{AGS}}(\mathcal{N})$. Indeed, if $\widehat{T_1}, \widehat{T_2}$ are two arbitrary elements from $T(\mathcal{N}')$, then at least one of the conditions is satisfied: $\widehat{T_1} + \widehat{T_2} = (\overline{\varnothing}, \varnothing)$, $\widehat{T_1} + \widehat{T_2} = \widehat{T_1} \cup \widehat{T_2}$. In both cases we have $\widehat{T_1} + \widehat{T_2} \in T(\mathcal{N}')$. Thus, $T(\mathcal{N}')$ is a subgroupoid of the groupoid $\widehat{\mathrm{AGS}}(\mathcal{N})$. Similarly, we obtain that $T(\mathcal{N}')$ is a subgroupoid of the groupoid $\widehat{\mathrm{MGS}}(\mathcal{N})$.

Let us show that $(T(\mathcal{N}'), +)$ is isomorphic to $\widehat{\mathrm{AGS}}(\mathcal{N}')$. Since $\widehat{U} = (\overline{X}, Q)$ induces a generalized subnet $\mathcal{N}'$, then $\overline{X}$ is a continuous tuple. We assume that the first non-empty component of the tuple $\overline{X}$ has number u, and the last non-empty component has number v. Since $\overline{X}$ is a continuous tuple, the neural network $\mathcal{N}'$ has exactly $v - u + 1$ layers (follows from the definition). As before, let $K_s(\overline{Y})$ denote the s-th component of the tuple $\overline{Y}$ from $\mathrm{AGS}(\mathcal{N})$. Let us define a mapping $\alpha : T(\mathcal{N}') \to \widehat{\mathrm{AGS}}(\mathcal{N}')$ so that for an arbitrary element $(\overline{Y}, W) \in T(\mathcal{N}')$ and arbitrary $s \in \{1, \ldots, v - u + 1\}$ the equalities hold

$$\alpha((\overline{Y}, W)) := (\alpha(\overline{Y}), W), \quad K_s(\alpha(\overline{Y})) := K_{u+s-1}(\overline{Y}), \tag{4}$$

where $\alpha(\overline{Y})$ is the first component of the tuple $\alpha((\overline{Y}, W))$ by definition. Since the tuple $\overline{X}$ is continuous and by virtue of the construction of the set $T(\overline{Y})$, then for any $\overline{Y} \in T(\mathcal{N}')$ and an arbitrary index $d \notin \{u, u+1, \ldots, v\}$ we have $K_d(\overline{Y}) = \varnothing$. Therefore the α-images of two distinct elements from $T(\mathcal{N}')$ are different (α is injective). The surjectivity of α follows easily from the definitions of the set $T(\mathcal{N}')$ and $\widehat{\mathrm{AGS}}(\mathcal{N}')$. Thus, α is a bijection of the set $T(\mathcal{N}')$ onto the set $\widehat{\mathrm{AGS}}(\mathcal{N}')$.

In what follows, operations in the groupoids $\widehat{\mathrm{AGS}}(\mathcal{N}')$ and $\mathrm{AGS}(\mathcal{N}')$ will be denoted by $(+')$. Let $\widehat{U}_1 = (\overline{Y}_1, W_1)$ and $\widehat{U}_2 = (\overline{Y}_2, W_2)$ be two arbitrary elements of $T(\mathcal{N}')$. There are possible cases: either $\overline{Y}_1 \cup \overline{Y}_2$ is a continuous tuple, or $\overline{Y}_1 \cup \overline{Y}_2$ is not a continuous tuple. Let the first case be true. Then the identity $\overline{Y}_1 + \overline{Y}_2 = \overline{Y}_1 \cup \overline{Y}_2$ is true, due to (4) tuple $\alpha(\overline{Y}_1 \cup \overline{Y}_2)$ is a continuous tuple. In addition, for any index $s \in \{1, \ldots, v-u+1\}$ the following equalities hold:

$$\alpha(\widehat{U}_1 + \widehat{U}_2) = \alpha((\overline{Y}_1, W_1) + (\overline{Y}_2, W_2)) = \alpha(\overline{Y}_1 + \overline{Y}_2, W_1 \cup W_2) = (\alpha(\overline{Y}_1 \cup \overline{Y}_2), W_1 \cup W_2),$$

$$K_s(\alpha(\overline{Y}_1 \cup \overline{Y}_2)) = K_{u+s-1}(\overline{Y}_1 \cup \overline{Y}_2) = K_{u+s-1}(\overline{Y}_1) \cup K_{u+s-1}(\overline{Y}_2) = K_s(\alpha(\overline{Y}_1)) \cup K_s(\alpha(\overline{Y}_2)).$$

From the last chain of equalities we obtain the condition $\alpha(\overline{Y}_1) \cup \alpha(\overline{Y}_2) \in \mathrm{AGS}(\mathcal{N}')$ and the relations

$$\alpha(\overline{Y}_1 \cup \overline{Y}_2) = \alpha(\overline{Y}_1) \cup \alpha(\overline{Y}_2) = \alpha(\overline{Y}_1) +' \alpha(\overline{Y}_2).$$

Therefore we have the relations

$$\alpha(\widehat{U}_1 + \widehat{U}_2) = (\alpha(\overline{Y}_1 \cup \overline{Y}_2), W_1 \cup W_2) = (\alpha(\overline{Y}_1) \cup \alpha(\overline{Y}_2), W_1 \cup W_2) = (\alpha(\overline{Y}_1) +' \alpha(\overline{Y}_2), W_1 \cup W_2) =$$

$$= (\alpha(\overline{Y}_1), W_1) +' (\alpha(\overline{Y}_2), W_2) = \alpha(\widehat{U}_1) +' \alpha(\widehat{U}_2).$$

Now let $\overline{Y}_1 \cup \overline{Y}_2$ be not a continuous tuple. Then, by virtue of (4), we have the relations $(\alpha(\overline{\varnothing}), \varnothing) = (\overline{\varnothing}, \varnothing)'$, where $(\overline{\varnothing}, \varnothing)'$ is a tuple in the groupoid $\widehat{\mathrm{AGS}}(\mathcal{N}')$.

Let there exist parameters $d, m, k \in \{u, u+1 \ldots, v\}$ such that the conditions

$$d < m < k, \quad K_d(\overline{Y}_1 \cup \overline{Y}_2) = K_d(\overline{Y}_1) \cup K_d(\overline{Y}_2) \neq \varnothing, \quad K_k(\overline{Y}_1 \cup \overline{Y}_2) = K_k(\overline{Y}_1) \cup K_k(\overline{Y}_2) \neq \varnothing,$$

$$K_m(\overline{Y}_1 \cup \overline{Y}_2) = K_m(\overline{Y}_1) \cup K_m(\overline{Y}_2) = \varnothing.$$

The last statement is a necessary and sufficient condition for the fact that $\overline{Y}_1 \cup \overline{Y}_2 \notin \mathrm{AGS}(\mathcal{N})$. From the given equalities we derive the conditions

$$K_{d-u+1}(\alpha(\overline{Y}_1)) \cup K_{d-u+1}(\alpha(\overline{Y}_2)) = K_d(\overline{Y}_1) \cup K_d(\overline{Y}_2) \neq \varnothing,$$

$$K_{k-u+1}(\alpha(\overline{Y}_1)) \cup K_{k-u+1}(\alpha(\overline{Y}_2)) = K_k(\overline{Y}_1) \cup K_k(\overline{Y}_2) \neq \varnothing,$$

$$K_{m-u+1}(\alpha(\overline{Y}_1)) \cup K_{m-u+1}(\alpha(\overline{Y}_2)) = K_m(\overline{Y}_1) \cup K_m(\overline{Y}_2) = \varnothing, \ \ d-u+1 < m-u+1 < k-u+1.$$

Therefore, the equality $\alpha(\widehat{U}_1) +' \alpha(\widehat{U}_2) = (\overline{\varnothing}, \varnothing)'$ holds. From here we get

$$\alpha(\widehat{U}_1 + \widehat{U}_2) = \alpha((\overline{Y}_1, W_1) + (\overline{Y}_2, W_2)) = \alpha((\overline{\varnothing}, \varnothing)) = (\alpha(\overline{\varnothing}), \varnothing) = (\overline{\varnothing}, \varnothing)' = \alpha(\widehat{U}_1) +' \alpha(\widehat{U}_2).$$

This means that α is an isomorphism of the groupoid $T(\mathcal{N}')$ and $\widehat{\mathrm{AGS}}(\mathcal{N}')$.

Similarly, it is proved that α is an isomorphism between the groupoid $(T(\mathcal{N}'), *)$ and $\widehat{\mathrm{MGS}}(\mathcal{N}')$. Indeed, in the above reasoning, the operation $(\cup)$ must be replaced by the operation $(\cap)$, and the operation $(+')$ by the operation $(*')$, which is an operation in the groupoid $\widehat{\mathrm{MGS}}(\mathcal{N}')$. The theorem is proved. □

The problems below are of interest.

Problems 4.1. Describe all subgroupoids H of the groupoid $X(\mathcal{N})$ such that $H \cong Y(\mathcal{N}')$ for a suitable generalized subnet $\mathcal{N}'$ networks $\mathcal{N}$, where:

$$(a)\ X(\mathcal{N}) := \widehat{\mathrm{AGS}}(\mathcal{N}),\ Y(\mathcal{N}') := \widehat{\mathrm{AGS}}(\mathcal{N}'); \quad (b)\ X(\mathcal{N}) := \widehat{\mathrm{MGS}}(\mathcal{N}),\ Y(\mathcal{N}') := \widehat{\mathrm{MGS}}(\mathcal{N}').$$

Problems 4.2. Describe all subgroupoids H of the groupoid $X(\mathcal{N})$ such that H is not isomorphic $Y(\mathcal{N}')$ for any generalized subnet $\mathcal{N}'$ of the $\mathcal{N}$ network, where:

$$(a)\ X(\mathcal{N}) := \widehat{\mathrm{AGS}}(\mathcal{N}),\ Y(\mathcal{N}') := \widehat{\mathrm{AGS}}(\mathcal{N}'); \quad (b)\ X(\mathcal{N}) := \widehat{\mathrm{MGS}}(\mathcal{N}),\ Y(\mathcal{N}') := \widehat{\mathrm{MGS}}(\mathcal{N}').$$

Solutions to problems 4.1 (a) and 4.2 (a) will give a description of all subgroupoids of the groupoid $\widehat{\mathrm{AGS}}(\mathcal{N})$ (similar, problems 4.1 (b) and 4.2 (b) give a description of all subgroupoids of the groupoid $\widehat{\mathrm{MGS}}(\mathcal{N})$).

Problems 4.3. Give a description of all subgroupoids of the H groupoid $\widehat{\mathrm{AGS}}(\mathcal{N})$ such that H is isomorphic $\widehat{\mathrm{AGS}}(\mathcal{N}')$, where $\mathcal{N}'$ is the appropriate neural network. A similar question for the groupoid $\widehat{\mathrm{MGS}}(\mathcal{N})$.

In the above problems, a description of subgroupoids is understood as a description that provides information about what elements a subgroupoid with the desired property contains.

Problems 4.4. Describe all pairs of neural networks $(\mathcal{N}, \mathcal{K})$ for which isomorphism holds: a) $\widehat{\mathrm{AGS}}(\mathcal{N}) \cong \widehat{\mathrm{MGS}}(\mathcal{K})$; b) $\widehat{\mathrm{AGS}}(\mathcal{N}) \cong \widehat{\mathrm{AGS}}(\mathcal{K})$; c) $\widehat{\mathrm{MGS}}(\mathcal{N}) \cong \widehat{\mathrm{MGS}}(\mathcal{K})$.

Problems 4.5. Give an element-by-element description of the monoids of all endomorphisms of the groupoids $\widehat{\mathrm{AGS}}(\mathcal{N})$ and $\widehat{\mathrm{MGS}}(\mathcal{N})$.

Problems 4.6. Give an element-by-element description of the sets of all congruences of groupoids $\widehat{\mathrm{AGS}}(\mathcal{N})$ and $\widehat{\mathrm{MGS}}(\mathcal{N})$.

Problems 4.5 and 4.6 are closely related (the connection between endomorphisms and congruences of universal algebras is well known; see, for example, the homomorphism theorem).

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О некоторых коммутативных и идемпотентных конечных группоидах, связанных с подсетями многослойных нейронных сетей прямого распространения сигнала

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Аннотация. В работе изучаются коммутативные и идемпотентные конечные группоиды, которые связанны с подсетями многослойных нейронных сетей прямого распространения сигнала (далее, просто нейронные сети). Ранее вводилось понятие подсети нейронной сети. В данной работе вводится понятие обобщенной подсети нейронной сети. Это понятие обобщает ранее введенное понятие. Полученные группоиды получают название обобщенных подсетей заданной нейронной сети. Данные группоиды моделируют объединение и пересечение обобщенных подсетей некоторой нейронной сети. Выявлены условия, которым должна удовлетворять архитектура нейронной сети, чтобы аддитивный группоид обобщенных подсетей был ассоциативен. Получены условия, которым должна удовлетворять архитектура нейронной сети, чтобы мультипликативный группоид обобщенных подсетей был ассоциативен. Изучаются подгруппоиды построенных группоидов.

Ключевые слова: группоид, многослойная нейронная сеть прямого распространения сигнала, подсеть многослойной нейронной сети прямого распространения сигнала, аддитивный группоид обобщенных подсетей, мультипликативный группоид обобщенных подсетей, обобщенная подсеть.