EDN: BAENPA
УДК 512.5

Generation of the Group $SL_6(\mathbb{Z}+i\mathbb{Z})$ by Three Involutions

Rodion I. Gvozdev*
Siberian Federal University
Krasnoyarsk, Russian Federation

Received 10.06.2024, received in revised form 15.07.2024, accepted 20.08.2024

Abstract. It is proved that the group $SL_6(\mathbb{Z}+i\mathbb{Z})$ is generated by three involutions. Previously, the solution of the problem on the existence of generating triples of involutions two of which commute was completed for the groups $SL_n(\mathbb{Z}+i\mathbb{Z})$ and $PSL_n(\mathbb{Z}+i\mathbb{Z})$ (Math. notes, 115 (2024), no. 3). The question of generating these groups by three involutions remained unresolved only for $SL_6(\mathbb{Z}+i\mathbb{Z})$.

Keywords: special linear group, the ring of Gaussian integers, generating triples of involutions.

Citation: R.I. Gvozdev, Generation of the Group $SL_6(\mathbb{Z}+i\mathbb{Z})$ by Three Involutions, J. Sib. Fed. Univ. Math. Phys., 2024, 17(6), 693–697. EDN: BAENPA.

Introduction

In [1], there was completed the solution of the problem on the existence of generating triples of involutions two of which commute for the groups $SL_n(\mathbb{Z}+i\mathbb{Z})$ and $PSL_n(\mathbb{Z}+i\mathbb{Z})$. Exactly, the remaining cases of the groups SL_5, SL_{10} and PSL_6 were considered. Taking into account the work of [2–6], the following final result was obtained in the article [1]

Theorem 1. *a) The group $SL_n(\mathbb{Z}+i\mathbb{Z})$ is generated by three involutions two of which commute if and only if $n \geqslant 5$ and $n \neq 6$.*
b) The group $PSL_n(\mathbb{Z}+i\mathbb{Z})$ is generated by three involutions two of which commute if and only if $n \geqslant 5$.

For exceptional groups of small dimension from Theorem 1, the following is known. In article [5] were found generating triples of involutions, without the permutation condition of two of them, for groups $PSL_2(\mathbb{Z}+i\mathbb{Z})$, $SL_3(\mathbb{Z}+i\mathbb{Z})$ and $SL_4(\mathbb{Z}+i\mathbb{Z})$. Group $SL_2(\mathbb{Z}+i\mathbb{Z})$ is not generated by any set of involutions, since the involution in this group is unique. In [6] it is proved that, for any integral domain D of characteristic other than 2, the group $SL_6(D)$ is not generated by three involutions, two of which commute. In particular, this implies a similar statement for the group $SL_6(\mathbb{Z}+i\mathbb{Z})$. In this paper we consider the only remained unresolved case of the question of generation of the groups $SL_n(\mathbb{Z}+i\mathbb{Z})$ and $PSL_n(\mathbb{Z}+i\mathbb{Z})$ by three involutions, exactly, proved

Theorem 2. *The group $SL_6(\mathbb{Z}+i\mathbb{Z})$ is generated by three involutions.*

*gvozdev.rodion@bk.ru

1. Preliminaries

Let R be an arbitrary Euclidean ring. We write e_{rs} for the matrix with (r,s)-entry 1 and all other entries equal to 0. Elementary transvections are matrices of the form $t_{rs}(k) = E_n + ke_{rs}$, where E_n is a unit matrix. Let I be some ideal in the ring R, then we denote

$$t_{rs}(I) = \langle t_{rs}(x) \mid x \in I\rangle,\ r,s = 1,2,\ldots,n,\ r \neq s.$$

Here and below, for any nonempty subset M of some group, we denote by $\langle M\rangle$ the subgroup generated by M. The following lemma is well known (e.g., see [7, p. 107]).

Lemma 1. *The group $SL_n(R)$ over an Euclidean ring R is generated by the subgroups $t_{rs}(R)$, $r,s = 1,\ldots,n,\ r \neq s$.*

The ring $\mathbb{Z}$ of integers and the ring $\mathbb{Z}+i\mathbb{Z}$, where $i^2 = -1$, of Gaussian integers are Euclidean (e.g., see [8, p. 439]), and since $t_{rs}(\mathbb{Z}) = \langle t_{rs}(1)\rangle$ and $t_{rs}(\mathbb{Z}+i\mathbb{Z}) = \langle t_{rs}(1), t_{rs}(i)\rangle$, we see that the following corollary of Lemma 1 holds.

Lemma 2. *a) The group $SL_n(\mathbb{Z})$ is generated by the transvections $t_{rs}(1)$, $r,s = 1,\ldots,n$, $r \neq s$.*
б) The group $SL_n(\mathbb{Z}+i\mathbb{Z})$ is generated by the transvections $t_{rs}(1)$, $t_{rs}(i)$, $r,s = 1,\ldots,n$, $r \neq s$.

It is known that the symmetric group S_n is isomorphic to the quotient group N/D of the monomial group N in $SL_n(R)$ by the subgroup of its diagonal matrices D. Each monomial matrix ϵ, with a nonzero element in position (s_k, r_k) and zeros in the remaining positions of this column corresponds to the substitution $\tilde{\epsilon}$, which transfers the element r_k into the element s_k, respectively, $k = 1,2,\ldots,n$. Conversely, each substitution on n symbols corresponds to monomial matrices from $SL_n(R)$. It should be noted the following elementary but useful statement in the form of a lemma.

Lemma 3. *Let η be a monomial (0,1)-matrix (permutation matrix), which corresponds to the element $\tilde{\eta}$ of the symmetric group, sending a pair of symbols (l,k) into a pair of (r,s), respectively, $l \neq k, r \neq s$, $l,k,r,s \in \{1,2,\ldots,n\}$. Then the equality $t_{lk}(u)^{\eta} = t_{rs}(u)$ is valid for any element $u \in R$.*

We regard $SL_n(\mathbb{Z}+i\mathbb{Z})$ as acting on the n-dimensional space of the column vectors V with components from the field of complex numbers. Let $v^t = (v_1,\ldots,v_n)^t$, $u^t = (u_1,\ldots,u_n)^t$ be some nonzero elements from V. If $g = E_n + v^t \times u$, then the rank of the matrix $g - E_n$ is equal to one. The matrix $g = E_n + v^t \times u$ is called a *transvection* if the vectors v^t and u^t are orthogonal, i.e., $uv^t = 0$. As is known, every transvection in $SL_n(F)$ over the field F is conjugate to an elementary transvection (e.g., see [9, p. 65]). In proving Theorem 2, the method of searching for elementary transvections described in [1] will also be used. Denote by M the subgroup generated by the given involutions. This method consists in finding in the group M matrices of the form $g_l = E_n + v^t \times u$, which are transvections or products of commuting transvections. It is further proved that M contains such matrices h that either $hv^t = \pm v^t$ or $uh^{-1} = \pm u$. Then ${g_l}^h = E_n + c^t \times a$, where $c^t \neq \pm v^t$ and $a = u$, or $c^t = v^t$ and $a \neq \pm u$. After obtaining a sufficient number of g_l matrices, it turns out that some elementary transvections $t_{rs}(x)$ lie in a subgroup of the group M generated by g_l matrices.

2. Proof of the Theorem 2

The proof is constructive and consists in the fact that all generators are indicated explicitly. Let

$$\alpha = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & i & -i \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\beta = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrices α, β, γ are contained in the group $SL_6(\mathbb{Z}+i\mathbb{Z})$ and are involutions. Denote $M = \langle \alpha, \beta, \gamma \rangle$ and show that $M = SL_6(\mathbb{Z}+i\mathbb{Z})$.

Lemma 4. *Any transvection $t_{rs}(x)$, $r \neq s$, $1 \leqslant r \leqslant 5$, $1 \leqslant s \leqslant 6$, $x \in \mathbb{Z}+i\mathbb{Z}$, by conjugation of elements from M can be transferred to the transvection $t_{12}(x)$.*

Proof. For the element g of the group G and its subgroup H, we denote $g^H = \{hgh^{-1}, h \in H\}$. We show directly that $t_{12}(x)^M$ contains all transvections $t_{rs}(x)$, $r \neq s$, $1 \leqslant r \leqslant 5$, $1 \leqslant s \leqslant 6$. Since the dihedral group $\langle \beta, \gamma \rangle$ consists of permutation matrices, for the proof it will be convenient to consider the action of its image in the symmetric group S_6 on the set of pairs of indices (r, s) corresponding to the transvections $t_{rs}(x)$ of $t_{12}(x)^M$, and use lemma 3. We will denote by O_{rs} the orbit $(r,s)^{\langle \tilde\beta, \tilde\gamma \rangle}$. We have

$$\tilde\beta = (12)(34), \;\; \tilde\gamma = (23)(45),$$

$$O_{12} = \{(1,2),(1,3),(2,1),(2,4),(3,1),(3,5),(4,2),(4,5),(5,3),(5,4)\},$$

$$t_{35}(x)^\alpha = t_{36}(x),$$

$$O_{36} = \{(1,6),(2,6),(3,6),(4,6),(5,6)\},$$

$$t_{26}(x)^\alpha = t_{25}(x),$$

$$O_{25} = \{(1,4),(1,5),(2,3),(2,5),(3,2),(3,4),(4,1),(4,3),(5,1),(5,2)\}.$$

It is easy to see that the orbits O_{12}, O_{36}, O_{25} contain all pairs of indices (r,s), $r \neq s$, $1 \leqslant r \leqslant 5$, $1 \leqslant s \leqslant 6$. According to Lemma 3, by conjugating the transvections $t_{12}(x)$, $t_{36}(x)$, $t_{25}(x)$ with elements from $\langle \beta, \gamma \rangle$, we can obtain any transvection $t_{rs}(x)$, $r \neq s$, $1 \leqslant r \leqslant 5$, $1 \leqslant s \leqslant 6$. The lemma is proved. $\square$

Lemma 5. *If for some $r_1 \neq s_1,\ 1 \leqslant r_1 \leqslant 5,\ 1 \leqslant s_1 \leqslant 6$ and for some $x \in \mathbb{Z}+i\mathbb{Z}$ the transvection $t_{r_1 s_1}(x)$ lies in M, then M contains all the transvections $t_{rs}(x)$, $t_{rs}(ix)$, $1 \leqslant r, s \leqslant 6$.*

Proof. According to Lemma 4, the transvections $t_{12}(x)$, $t_{42}(x)$, $t_{52}(x)$ lie in M. The further part of the proof of the theorem was obtained with significant use of computer calculations. We have

$$t_{12}(x)^{(\gamma\beta)^2\alpha\gamma\beta\alpha\gamma\beta\gamma}t_{42}(-x) = t_{32}(ix)t_{62}(x),$$

$$t_{12}(x)^{(\gamma\beta\alpha)^3\gamma\beta\gamma}t_{42}(-x) = t_{12}(ix)t_{32}(-ix)t_{62}(x),$$

$$t_{12}(x)^{\gamma\beta\alpha\beta\gamma\beta\alpha\gamma\beta}t_{52}(x) = t_{12}(-ix)t_{62}(-x).$$

From the last three equalities it follows that $t_{32}(ix), t_{62}(x) \in M$. Again by Lemma 4, $t_{r_1 s_1}(ix)$ lies in M for all $r_1 \neq s_1,\ 1 \leqslant r_1 \leqslant 5,\ 1 \leqslant s_1 \leqslant 6$. From here we get $t_{62}(ix) \in M$. Finally

$$t_{62}(x)^{\beta} = t_{61}(x),\ t_{62}(x)^{\gamma} = t_{63}(x),$$

$$t_{63}(x)^{\beta} = t_{64}(x),\ t_{64}(x)^{\beta} = t_{65}(x).$$

The lemma is proved. □

Lemma 6. *The group M contains the subgroup $t_{rs}(I)$ for any $r \neq s$, $1 \leqslant r, s \leqslant 6$, where I is the ideal in the ring $\mathbb{Z}+i\mathbb{Z}$ generated by element 3.*

Proof. Let

$$g_1 = (\alpha\beta)^2,\ g_2 = g_1 g_1^{\gamma\beta\alpha\gamma(\beta\gamma)^3},$$

$$g_3 = g_2^{-1} g_1 g_1^{\gamma\beta\alpha\gamma(\beta\gamma)^2\alpha\beta\gamma},$$

$$g_4 = [g_1^{-1}, (\gamma\beta)^3\gamma\alpha\beta\gamma\beta],\ g_5 = g_2^{(\gamma\beta)^3\gamma\alpha\beta\gamma\beta},$$

$$g_6 = (g_5 g_4^{-1})^{(\gamma\beta\alpha\beta\gamma\alpha)^2}(g_4 g_5^{-1})^{\gamma\beta\alpha\beta\gamma},$$

$$g_7 = g_4 g_5^{-1} g_2^{-1} g_6^{\alpha^{\alpha\gamma\beta}(\gamma\beta)^4\gamma\alpha\beta},\ g_8 = g_7^{-1} g_7^{(\gamma\beta)^3\gamma\alpha\gamma\beta}.$$

Each of the matrices $g_1, \ldots, g_8$ is the product of two commuting transvections and does not differ from the identity matrix in the lower left block 4×4. Next, we get the following transvections:

$$g_9 = g_8^{(\gamma\beta)^4\gamma} g_8^{(\gamma\beta)^3\gamma\alpha\beta\gamma} = t_{13}(3i-3)t_{16}(3i-3),\ (\alpha\gamma)^6 = t_{21}(-3)t_{31}(3),$$

$$(t_{21}(-3)t_{31}(3))^{\gamma\alpha\beta\gamma\beta\alpha\beta\gamma(\alpha\beta)^2}(t_{21}(-6)t_{31}(6))^{\gamma(\alpha\beta)^3\gamma\alpha\gamma} = t_{51}(9),$$

$$g_9^{(\gamma\beta)^4\gamma\alpha\beta\gamma}(g_9^{-1})^{(\gamma\beta)^4\gamma\alpha\beta} g_9^{(\gamma\beta)^3\gamma\beta\alpha\gamma\beta\gamma} = t_{12}(6-6i).$$

Inclutions $t_{rs}(6-6i)$, $t_{rs}(6+6i) \in M$ and $t_{rs}(9)$, $t_{rs}(9i) \in M$ follow from the last two equalities and Lemma 5, for all $1 \leqslant r, s \leqslant 6$. Therefore, $t_{rs}(12)$, $t_{rs}(12i) \in M$, and finally, $t_{rs}(3)$, $t_{rs}(3i) \in M$. The lemma is proved. □

Now, using Lemmas 5 and 6, we complete the proof of the theorem. The group M contains the elements $g_7 = t_{13}(5i-2)t_{23}(1-i)t_{14}(1+2i)t_{24}(1-i)t_{15}(3i-3)t_{16}(-1-2i)t_{26}(2-2i)$,

$$g_{10} = g_8^{[\gamma,(\alpha\beta)^2]\gamma\beta} = t_{13}(2+4i)t_{23}(2-2i)t_{14}(-2-4i)t_{24}(1-i)t_{15}(2+4i)t_{25}(i-1).$$

Direct calculations give that

$$g_7 g_{10}^{\alpha^{\alpha\beta}} = t_{13}(7i-4)t_{23}(-1-5i)t_{14}(3i)t_{24}(3+3i)t_{15}(6i-6)t_{25}(-6-12i)t_{16}(3-6i)t_{26}(6+6i) \in M.$$

Since, according to Lemma 6, the multipliers starting from $t_{14}(3i)$ and further lie in M, we have the inclusion $g_{11} = t_{13}(i-1)t_{23}(i-1) \in M$. Finally $g_{11}g_{11}^{\alpha\beta\gamma\beta\alpha(\beta\gamma)^2\beta}(g_{12}^{-1})^{\alpha\beta\gamma\beta\alpha\gamma\alpha\beta\alpha\gamma} = t_{23}(3i-3)t_{43}(1-i)$. Therefore, $t_{43}(1-i) \in M$. Hence $t_{31}(1-i) \in M$ by Lemma 5 and therefore $[t_{31}(1-i), t_{43}(1-i)] = t_{41}(2i)$. Again, according to Lemma 5, we have inclusions $t_{rs}(2), t_{rs}(2i) \in M$. Applying Lemma 6, we get $t_{rs}(1)$, $t_{rs}(i) \in M$, for all $1 \leqslant r, s \leqslant 6$. Thus, $M = SL_6(\mathbb{Z}+i\mathbb{Z})$ by Lemma 2. The theorem is proved.

This work is supported by the Krasnoyarsk Mathematical Center and financed by the Ministry of Science and Higher Education of the Russian Federation (Agreement no. 075-02-2024-1429).

References

[1] M.A.Vsemirnov, R.I.Gvozdev, Ya.N.Nuzhin, T.B.Shaipova, On generation of the groups $SL_n(\mathbb{Z}+i\mathbb{Z})$ and $PSL_n(\mathbb{Z}+i\mathbb{Z})$ by three involutions, two of which commute II, *Math. notes*, **115**(2024), no. 3, 289–300. DOI: 10.1134/S0001434624030015

[2] M.C.Tamburini, P.Zucca, Generation of Certain Matrix Groups by Three Involutions, *J. of Algebra*, **195**(1997), no. 2, 650–661.

[3] D.V.Levchuk, Ya.N.Nuzhin, On generation of the group $PSL_n(\mathbb{Z}+i\mathbb{Z})$ by three involutions, two of which commute, *Journal of Siberian Federal University. Math. and Phys.*, **1**(2008), no. 2, 133–139.

[4] D.V.Levchuk, On generation of the group $SL_7(\mathbb{Z}+i\mathbb{Z})$ by three involutions, two of which commute, *Vestn. Novosib. Gos. Univ., Ser. Mat. Mekh. Inform.*, **9**(2009), no. 1, 35–38 (in Russian).

[5] R.I.Gvozdev, Ya.N.Nuzhin, T.B.Shaipova, On generation of the groups $SL_n(\mathbb{Z}+i\mathbb{Z})$ and $PSL_n(\mathbb{Z}+i\mathbb{Z})$ by three involutions, two of which commute, *The Bulletin of Irkutsk State University*, **40**(2022), 49–62 (in Russian).

[6] Ya.N.Nuzhin, Tensor representations and generating sets of involutions of some matrix groups, *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, **26**(2020), no. 3, 133–141 (in Russian). DOI: 10.21538/0134-4889-2020-26-3-133-141

[7] R.Steinberg, Lectures on Chevalley groups, Mir, Moscow, 1975 (in Russian).

[8] A.I.Kostrikin, Introduction to algebra, Nauka, Moscow, 1977 (in Russian).

[9] D.A.Suprunenko, Matrix groups, Nauka, Moscow, 1977 (in Russian).

Порождение группы $SL_6(\mathbb{Z}+i\mathbb{Z})$ тремя инволюциями

Родион И. Гвоздев
Сибирский федеральный университет
Красноярск, Российская Федерация

Аннотация. Доказано, что группа $SL_6(\mathbb{Z}+i\mathbb{Z})$ порождается тремя инволюциями. Ранее было завершено решение задачи о существовании порождающих троек инволюций, две из которых перестановочны, для групп $SL_n(\mathbb{Z}+i\mathbb{Z})$ и $PSL_n(\mathbb{Z}+i\mathbb{Z})$ (Матем. заметки, 115 (2024), №3). Вопрос о порождении тремя инволюциями данных групп оставался нерешенным только для $SL_6(\mathbb{Z}+i\mathbb{Z})$.

Ключевые слова: специальная линейная группа, кольцо целых гауссовых чисел, порождающие тройки инволюций.