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ϕ-fixed Point Results in b-metric Spaces with wt-distance

Ranajit Jyoti*
Binayak S. Choudhury†
Department of Mathematics
Indian Institute of Engineering Science and Technology, Shibpur
Howrah-711103, West Bengal, India
Nikhilesh Metiya‡
Department of Mathematics
Sovarani Memorial College, Jagatballavpur
Howrah-711408, India
Santu Dutta§
Calcutta Institute of Science and Management
Calcutta-700040, West Bengal, India
Sankar P. Mondal¶
Department of Applied Sciences
Maulana Abul Kalam Azad University of Technology, Haringhata
Nadia-741249, West Bengal, India

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Abstract. In this paper, our program is to obtain a ϕ-fixed point result along with some applications. The problem considered here is formulated by combining together several recent trends in metric fixed point theory and its extensions. Two illustrative examples are discussed. It is shown that some results existing in the literature are extended by our main theorem. The application presented is in the area of Volterra and Fredholm integral equations.

Keywords: b-metric space, wt-distance, fixed point, ϕ-fixed point, integral equation.

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Introduction

There have been several emerging aspects in fixed point theory developed in recent times. Our program in this paper is to solve a fixed point problem formulated by combining several of these emerging trends. We describe in brief the areas of the theory which we consider here.

The structure of b-metric spaces is one of many generalizations of the concept of metric space in which several studies originally performed in metric spaces has been successfully extended. The concept was introduced in the work of Czerwick [4] in 1993. Fixed point theory has commendably developed in recent years. [2,3,14,17,18] are some instances of these works. The idea of w-distance was advanced by Kada et al. [9] in an attempt to solve a non-convex minimization problem as well as to generalize certain fixed point theorems. It is essentially an additional distance function

*jyotiranajit97@gmail.com
†binayak12@yahoo.co.in
‡metiya.nikhilesh@gmail.com
§sd83online@gmail.com
¶sankar.mondal02@gmail.com

defined in a metric space. Many metric fixed point results were extended and also some new results were produced by the use of w-distance inequalities. Further, there has also been an extension of w-distance to the metric spaces known as wt-distance [10,16] which has also been utilized in creating new fixed point results on fixed points of functions defined on metric spaces. ϕ-functions were used for the first time by Jleli et al. [8] for investigating fixed points of functions which are also zeroes of ϕ-functions. The motivation for this consideration is that the process of finding the fixed point becomes technically easier once zeroes of the ϕ-function are already known. This category of problems has become to be known as ϕ-fixed point problems which has been considered by several authors in works like [7,11,12].

The present paper combines the above-mentioned trends of fixed point theory in order to formulate a ϕ-fixed point problem on a b-metric space with a wt-distance. A particular function Θ is utilized for defining a wt-distance inequality which is supposed to be satisfied by the mapping under consideration. There are two illustrative examples through one of which it is shown that the main theorem properly generalizes some previously known results. In the last section of the paper we discuss an application of the fixed point theorem obtained in the paper to the problem of integral equations.

1. Mathematical Preliminaries

In this section, we briefly review some basic notations and preliminary results, which we will use in the paper.

Definition 1.1 ([4]). *Let X be a non-empty set. A mapping $\mu : X \times X \to [0,\infty)$ is called a b-metric if there exists a real number $l \geqslant 1$ such that for all $x, y, z \in X$,*

(i) $\mu(x,y) = 0$ if and only if $x = y$;

(ii) $\mu(x,y) = \mu(y,x)$;

(iii) $\mu(x,z) \leqslant l[\mu(x,y) + \mu(y,z)]$.

Then the triplet $(X, \mu, l \geqslant 1)$ is called a b-metric space.

Definition 1.2 ([9,15]). *Let (X,d) be a metric space. A function $p : X \times X \to [0,\infty)$ is called a w-distance on X if p satisfies the following conditions:*

(i) $p(x,z) \leqslant p(x,y) + p(y,z)$, for any $x, y, z \in X$;

(ii) p is lower semi-continuous in the second variable, that is, if $x \in X$ and $y_n \to y$ in X, then

$$p(x,y) \leqslant \lim_{n\to\infty} \inf p(x, y_n);$$

(iii) for any $\epsilon > 0$ there exists $\delta > 0$ such that $p(z,x) \leqslant \delta$ and $p(z,y) \leqslant \delta$ imply $d(x,y) \leqslant \epsilon$.

Definition 1.3 ([6]). *Let $(X, \mu, l \geqslant 1)$ be a b-metric space. A function $P : X \times X \to [0,\infty)$ is called a wt-distance if P satisfies the following conditions:*

(i) $P(x,z) \leqslant l[P(x,y) + P(y,z)]$, for all $x, y, z \in X$;

(ii) P is l-lower semi-continuous in the second variable, that is, if $x \in X$ and $y_n \to y$ in X, then $P(x,y) \leqslant \lim\limits_{n\to\infty} \inf lP(x, y_n)$;

(iii) for any $\epsilon > 0$ there exists $\delta > 0$ such that $P(z,x) \leqslant \delta$ and $P(z,y) \leqslant \delta$ imply $\mu(x,y) \leqslant \epsilon$.

Remark 1.1 ([1,5]). *Clearly, every metric space is a b-metric space with $l = 1$, but the converse is not true in general. Also, every w-distance is a wt-distance but the converse is not true.*

Lemma 1.1 ([13]). *Let $(X, \mu, l \geqslant 1)$ be a b-metric space and P be a wt-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X, $x, y, z \in X$ and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to 0.*

(i) If $\lim_{n\to\infty} P(x_n, x) = \lim_{n\to\infty} P(x_n, y) = 0$*, then* $x = y$*. In particular, if* $P(z, x) = P(z, y) = 0$*, then* $x = y$*.*

(ii) If $P(x_n, y_n) \leqslant \alpha_n$ *and* $P(x_n, y) \leqslant \beta_n$ *for any* $n \in \mathbb{N}$*, then the sequence* $\{y_n\}$ *converges to* y*, that is,* $\mu(y_n,\ y) \to 0$ *as* $n \to \infty$*.*

(iii) The sequence $\{x_n\}$ *is a Cauchy sequence if* $\lim_{m,n\to\infty} P(x_n, x_m) = 0$*, that is, if for each* $\epsilon > 0$ *there exists a natural number* k *such that for* $m > n > k$*,* $P(x_n, x_m) < \epsilon$*.*

(iv) The sequence $\{x_n\}$ *is a Cauchy sequence if* $P(y, x_n) \leqslant \alpha_n$*, for all* $n \in \mathbb{N}$*.*

Let X be a non-empty set, $T : X \to X$ and $\phi : X \to [0, \infty)$ be two mappings. Let F_T denote the set of all fixed points of T, that is, $F_T = \{x \in X : Tx = x\}$. Let Z_ϕ denote the set of all zeroes of ϕ, that is, $Z_\phi = \{x \in X : \phi(x) = 0\}$.

In [8], Jleli et al. introduced the concept of ϕ-fixed point and proved some ϕ-fixed point results using a class of control functions $\mathcal{F}$.

Definition 1.4 ([8]). *Let* X *be a non-empty set,* $T : X \to X$ *and* $\phi : X \to [0, \infty)$ *be two mappings. An element* $x \in X$ *is said to be a* ϕ*-fixed point of* T *if and only if* $x \in F_T \cap Z_\phi$*, that is,* $Tx = x$ *and* $\phi(x) = 0$*.*

Definition 1.5 ([8]). *Let* (X, d) *be a metric space and* $\phi : X \to [0, \infty)$ *be a mapping. A mapping* $T : X \to X$ *is said to be a* ϕ*-Picard operator if and only if there exists* $x_* \in X$ *such that*

(i) $F_T \cap Z_\phi = \{x_*\}$*;*

(ii) $T^n x \to x_*$ *as* $n \to \infty$*, for every* x *in* X*.*

Definition 1.6 ([8]). *Let* (X, d) *be a metric space and* $\phi : X \to [0, \infty)$ *be a mapping. A mapping* $T : X \to X$ *is said to be a weakly* ϕ*-Picard operator if and only if*

(i) $F_T \cap Z_\phi \neq \emptyset$*;*

(ii) the sequence $\{T^n x\}$ *converges to a* ϕ*-fixed point of* T*, for every* $x \in X$*.*

Let $\mathcal{F}$ denote the set of all functions $F : [0, \infty)^3 \to [0, \infty)$ satisfying the following conditions:

(F_1) $\max\{a, b\} \leqslant F(a, b, c)$, for $a, b, c \in [0, \infty)$;

(F_2) $F(0, 0, 0) = 0$;

(F_3) F is continuous.

Theorem 1.1 ([8]). *Let* (X, d) *be a complete metric space,* $\phi : X \to [0, \infty)$ *be a given function and* $F \in \mathcal{F}$*. Let* $T : X \to X$ *be a mapping such that*

$$F(d(Tx, Ty), \phi(Tx), \phi(Ty)) \leqslant kF(d(x, y), \phi(x), \phi(y)), \text{ for } x, y \in X, \tag{1.1}$$

where $k \in (0,\ 1)$ *and* ϕ *is lower semi-continuous. Then*

(i) $F_T \subseteq Z_\phi$*;*

(ii) T *is a* ϕ*-Picard operator.*

Again, Kumrod et al. [11] further generalized the above ϕ-fixed point result of Jleli et al. [8] with the help of a class of control functions Ω. Here, Ω denotes the set of all non-decreasing and continuous functions $\Theta : [0, \infty) \to [0, \infty)$ such that $\sum_{n=0}^{\infty} \Theta^n(t) < \infty$, for every $t > 0$.

Lemma 1.2 ([11]). *If* $\Theta \in \Omega$*, then* $\Theta(t) < t$ *for all* $t > 0$ *and* $\Theta(0) = 0$*.*

Theorem 1.2 ([11]). *Let* (X, d) *be a complete metric space,* $\phi : X \to [0, \infty)$ *be a given function,* $F \in \mathcal{F}$ *and* $\Theta \in \Omega$*. Let* $T : X \to X$ *be a mapping such that*

$$F(d(Tx, Ty), \phi(Tx), \phi(Ty)) \leqslant \Theta(F(d(x, y), \phi(x), \phi(y))), \text{ for } x, y \in X, \tag{1.2}$$

where ϕ *is lower semi-continuous. Then*

(i) $F_T \subseteq Z_\phi$*;*

(ii) T *is a* ϕ*-Picard mapping.*

Recently, Roy et al. [16] established the above result of Kumrod et al. [11] in a metric space with a w-distance. In the following result the used functions $F : [0,\infty)^3 \to [0,\infty)$ is a member of Υ which is a larger class than the class $\mathcal{F}$. Here, Υ denotes the set of all functions $F : [0,\infty)^3 \to [0,\infty)$ satisfying max $\{a, b\} \leqslant F(a, b, c)$, for $a, b, c \in [0,\infty)$.

Theorem 1.3 ([16]). *Let (X, d) be a complete metric space, p be a w-distance on X and $\phi : X \to [0,\infty)$ be a mapping. Let $F \in \Upsilon$ and $\Theta \in \Omega$. Let $T : X \to X$ be a mapping satisfying the following conditions:*

$$F(p(Tx, Ty), \phi(Tx), \phi(Ty)) \leqslant \Theta(F(p(x, y), \phi(x), \phi(y))),\ \textit{for all } x, y \in X \tag{1.3}$$

and

$$\inf\{p(x, y) + p(x, Tx) : x \in X\} > 0, \quad \textit{for every } y \in X \textit{ with } Ty \neq y. \tag{1.4}$$

Then

(i) $F_T \subseteq Z_\phi$;

(ii) T *is a* ϕ*-Picard operator;*

(iii) $p(x_*, x_*) = 0$, *where* $F_T \cap Z_\phi = \{x_*\}$.

In this paper, we take a class of function Δ which is the set of all non-decreasing and continuous functions $\Theta : [0,\infty) \to [0,\infty)$ such that $\sum\limits_{n=0}^{\infty} s^n \Theta^n(t) < \infty$, where $t > 0$ any real number and $s \geqslant 1$ is a constant.

Lemma 1.3. *If $\Theta \in \Delta$, then $\Theta(t) < t$ for every $t > 0$ and $\Theta(0) = 0$.*

Proof. If possible, let $\Theta(t) \geqslant t$ for some $t > 0$. Since Θ is non-decreasing, $\Theta^n(t) \geqslant t$ for all n. As $s \geqslant 1$, $s^n \Theta^n(t) \geqslant t$ for all n. As $\sum\limits_{n=0}^{\infty} s^n \Theta^n(t) < \infty$, where $t > 0$ any real number and $s \geqslant 1$ is a constant, we have $0 = \lim\limits_{n\to\infty} s^n \Theta^n(t) \geqslant t$, that is, $t \leqslant 0$, which is a contradiction. Hence $\Theta(t) < t$, for every $t > 0$. Now, if possible, suppose that $\Theta(0) \neq 0$. Then $\Theta(0) > 0$. Suppose that $\Theta(0) = c$, where $c > 0$ is a constant. As $\Theta(0) > 0$, applying the non-decreasing property of Θ we have $\Theta(c) = \Theta(\Theta(0)) \geqslant \Theta(0) = c$, which is a contradiction to $\Theta(c) < c$. Therefore, $\Theta(0) = 0$. □

2. Main results

Theorem 2.1. *Let $(X, \mu, l \geqslant 1)$ be a complete b-metric space with a wt-distance P on it. Let $F \in \Upsilon$, $\phi : X \to [0,\infty)$ be a mapping and $\Theta : [0,\infty) \to [0,\infty)$ be a non-decreasing and continuous mapping such that $\sum\limits_{n=0}^{\infty} l^n \Theta^n(t) < \infty$, where $t > 0$. Let $T : X \to X$ be a mapping such that*

$$\inf\{P(x, y) + P(x, Tx) : x \in X\} > 0, \quad \textit{for every } y \in X \textit{ with } Ty \neq y \tag{2.5}$$

and

$$F(P(Tx, Ty), \phi(Tx), \phi(Ty)) \leqslant \Theta(N_F^\phi(x, y)),\ \textit{for all } x, y \in X, \tag{2.6}$$

where
$N_F^\phi(x, y) = \max\{F(P(x, y), \phi(x), \phi(y)), F(P(x, Tx), \phi(x), \phi(Tx)), F(P(y, Ty), \phi(y), \phi(Ty))\}$.
Then

(i) $F_T \subseteq Z_\phi$;

(ii) T *has a unique* ϕ*-fixed point;*

(iii) $P(x_*, x_*) = 0$, *where* $F_T \cap Z_\phi = \{x_*\}$.

Proof. Let $x_* \in F_T$. Then $Tx_* = x_*$. Applying (2.6) with $x = y = x_*$, we get

$$F(P(Tx_*, Tx_*), \phi(Tx_*), \phi(Tx_*)) \leqslant \Theta(N_F^{\phi}(x_*, x_*)),$$

that is,

$$F(P(x_*, x_*), \phi(x_*), \phi(x_*)) \leqslant \Theta(N_F^{\phi}(x_*, x_*)). \tag{2.7}$$

Now,

$$\begin{aligned}
&N_F^{\phi}(x_*, x_*) = \\
&= \max\{F(P(x_*, x_*), \phi(x_*), \phi(x_*)), F(P(x_*, Tx_*), \phi(x_*), \phi(Tx_*)), F(P(x_*, Tx_*), \phi(x_*), \phi(Tx_*))\} = \\
&= \max\{F(P(x_*, x_*), \phi(x_*), \phi(x_*)), F(P(x_*, x_*), \phi(x_*), \phi(x_*)), F(P(x_*, x_*), \phi(x_*), \phi(x_*))\} = \\
&= F(P(x_*, x_*), \phi(x_*), \phi(x_*)).
\end{aligned}$$

We have from (2.7) that

$$F(P(x_*, x_*), \phi(x_*), \phi(x_*)) \leqslant \Theta(F(P(x_*, x_*), \phi(x_*), \phi(x_*))),$$

Suppose that $F(P(x_*, x_*), \phi(x_*), \phi(x_*)) > 0$. By Lemma 1.3, we have from the above inequality that

$$F(P(x_*, x_*), \phi(x_*), \phi(x_*)) \leqslant \Theta(F(P(x_*, x_*), \phi(x_*), \phi(x_*))) < F(P(x_*, x_*), \phi(x_*), \phi(x_*)),$$

which is a contradiction. Therefore, we have

$$F(P(x_*, x_*), \phi(x_*), \phi(x_*)) = 0. \tag{2.8}$$

Using the property of F, we have

$$\max\{P(x_*, x_*), \phi(x_*)\} \leqslant F(P(x_*, x_*), \phi(x_*), \phi(x_*)) = 0,$$

which implies that

$$P(x_*, x_*) = \phi(x_*) = 0. \tag{2.9}$$

Thus, $x_* \in F_T$ implies $x_* \in Z_\phi$ and hence $F_T \subseteq Z_\phi$.

Starting with a point $x_0 \in X$, we construct a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, for all $n \in \mathbb{N} \cup \{0\}$. If possible, suppose that $x_{i+1} = x_i$, for some $i \in \mathbb{N} \cup \{0\}$. Then $x_i = x_{i+1} = Tx_i$, that is, x_i is a fixed point of T and consequently $x_i \in F_T \cap Z_\phi$. Hence we shall assume that $x_{n+1} \neq x_n$, for all $n \in \mathbb{N} \cup \{0\}$. Now, $F(P(x_n, x_{n+1}), \phi(x_n), \phi(x_{n+1})) \geqslant 0$, for all $n \in \mathbb{N} \cup \{0\}$. If possible, suppose that $F(P(x_n, x_{n+1}), \phi(x_n), \phi(x_{n+1})) = 0$, for some $K \in \mathbb{N} \cup \{0\}$. Then $F(P(x_K, x_{K+1}), \phi(x_K), \phi(x_{K+1})) = 0$. Using the property of F, we have $P(x_K, x_{K+1}) = \phi(x_K) = 0$. Here, $x_{K+1} \neq x_{K+2} = Tx_{K+1}$. By (2.5), we have

$$\begin{aligned}
0 &< \inf\{P(x, y) + P(x, Tx) : x \in X\}, \text{ for every } y \in X \text{ with } Ty \neq y \\
&\leqslant \inf\{P(x_n, x_{K+1}) + P(x_n, Tx_n) : n \in \mathbb{N} \cup \{0\}\} \\
&= \inf\{P(x_n, x_{K+1}) + P(x_n, x_{n+1}) : n \in \mathbb{N} \cup \{0\}\} \\
&= 0, \ [\text{as } P(x_n, x_{K+1}) + P(x_n, x_{n+1}) = 0, \text{ for } n = K]
\end{aligned}$$

which is a contradiction. Therefore, $F(P(x_n, x_{n+1}), \phi(x_n), \phi(x_{n+1})) > 0$, for all $n \in \mathbb{N} \cup \{0\}$. Now,

$$\begin{aligned}
F(P(x_n, x_{n+1}), \phi(x_n), \phi(x_{n+1})) = F(P(Tx_{n-1}, Tx_n), \phi(Tx_{n-1}), \phi(Tx_n)) \leqslant \\
\leqslant \Theta(N_F^{\phi}(x_{n-1}, x_n)),
\end{aligned} \tag{2.10}$$

where,

$$\begin{aligned}N_F^{\phi}(x_{n-1},x_n)&=\max\{F(P(x_{n-1},x_n),\phi(x_{n-1}),\phi(x_n)),F(P(x_{n-1},Tx_{n-1}),\phi(x_{n-1}),\phi(Tx_{n-1})),\\&\qquad F(P(x_n,Tx_n),\phi(x_n),\phi(Tx_n))\}=\\&=\max\{F(P(x_{n-1},x_n),\phi(x_{n-1}),\phi(x_n)),F(P(x_{n-1},x_n),\phi(x_{n-1}),\phi(x_n)),\\&\qquad F(P(x_n,x_{n+1}),\phi(x_n),\phi(x_{n+1}))\}=\\&=\max\{F(P(x_{n-1},x_n),\phi(x_{n-1}),\phi(x_n)),F(P(x_n,x_{n+1}),\phi(x_n),\phi(x_{n+1}))\}.\end{aligned}$$

If possible, suppose that $N_F^{\phi}(x_{n-1},x_n)=F(P(x_n,x_{n+1}),\phi(x_n),\phi(x_{n+1}))$. By (2.10) and Lemma 1.3, we have

$$\begin{aligned}F(P(x_n,x_{n+1}),\phi(x_n),\phi(x_{n+1}))&\leqslant\Theta(F(P(x_n,x_{n+1}),\phi(x_n),\phi(x_{n+1})))<\\&<F(P(x_n,x_{n+1}),\phi(x_n),\phi(x_{n+1})),\end{aligned}$$

which is a contradiction. Therefore, $N_F^{\phi}(x_{n-1},x_n)=F(P(x_{n-1},x_n),\phi(x_{n-1}),\phi(x_n))$ and hence from (2.10), we get

$$F(P(x_n,x_{n+1}),\phi(x_n),\phi(x_{n+1}))\leqslant\Theta(F(P(x_{n-1},x_n),\phi(x_{n-1}),\phi(x_n))).$$

By repeated application of the above inequality, we have

$$\begin{aligned}F(P(x_n,x_{n+1}),\phi(x_n),\phi(x_{n+1}))&\leqslant\Theta(F(P(x_{n-1},x_n),\phi(x_{n-1}),\phi(x_n)))\leqslant\\&\leqslant\Theta^2(F(P(x_{n-2},x_{n-1}),\phi(x_{n-2}),\phi(x_{n-1})))\leqslant\\&\ \ \vdots\\&\leqslant\Theta^n(F(P(x_0,x_1),\phi(x_0),\phi(x_1))).\end{aligned}$$

Using the property of F, we have

$$P(x_n,x_{n+1})\leqslant\Theta^n(F(P(x_0,x_1),\phi(x_0),\phi(x_1))). \tag{2.11}$$

As $l\geqslant 1$, we have $\Theta^n(F(P(x_0,x_1),\phi(x_0),\phi(x_1)))\leqslant l^n\Theta^n(F(P(x_0,x_1),\phi(x_0),\phi(x_1)))$.
By a property of Θ, we have $\lim\limits_{n\to\infty} l^n\Theta^n(F(P(x_0,x_1),\phi(x_0),\phi(x_1)))=0$.
Hence $\lim\limits_{n\to\infty}\Theta^n(F(P(x_0,x_1),\phi(x_0),\phi(x_1)))=0$. From (2.11), we get

$$\lim_{n\to\infty}P(x_n,x_{n+1})=0. \tag{2.12}$$

Using (2.11), we have for $m>n$, where m and n are natural numbers,

$$\begin{aligned}&P(x_n,x_m)\leqslant\\&\leqslant lP(x_n,x_{n+1})+l^2P(x_{n+1},x_{n+2})+\ldots+l^{m-n-1}[P(x_{m-2},x_{m-1})+P(x_{m-1},x_m)]\leqslant\\&\leqslant lP(x_n,x_{n+1})+l^2P(x_{n+1},x_{n+2})+\ldots+l^{m-n-1}P(x_{m-2},x_{m-1})+l^{m-n}P(x_{m-1},x_m)\leqslant\\&\leqslant l\Theta^n(F(P(x_0,x_1),\phi(x_0),\phi(x_1)))+l^2\Theta^{n+1}(F(P(x_0,x_1),\phi(x_0),\phi(x_1)))+\ldots\\&+l^{m-n-1}\Theta^{m-2}(F(P(x_0,x_1),\phi(x_0),\phi(x_1)))+l^{m-n}\Theta^{m-1}(F(P(x_0,x_1),\phi(x_0),\phi(x_1)))\leqslant\\&\leqslant\frac{1}{l^{n-1}}\sum_{j=n}^{m-1}l^j\Theta^j(F(P(x_0,x_1),\phi(x_0),\phi(x_1)))\leqslant\\&\leqslant\frac{1}{l^{n-1}}\sum_{j=n}^{\infty}l^j\Theta^j(F(P(x_0,x_1),\phi(x_0),\phi(x_1))),\end{aligned} \tag{2.13}$$

which, by a property of Θ, implies that $\lim\limits_{m,n\to\infty} P(x_n, x_m) = 0$, that is, $\{x_n\}$ is a Cauchy sequence in X. As the b-metric space $(X, \mu, l \geqslant 1)$ is complete, there exists $x_* \in X$ such that $x_n \to x_*$ as $n \to \infty$. Since $P(x_n, .)$ is l-lower semi-continuous, we have

$$0 \leqslant P(x_n, x_*) \leqslant \lim_{m\to\infty} \inf lP(x_n, x_m) \leqslant \frac{1}{l^{n-2}} \sum_{j=n}^{\infty} l^j \Theta^j(F(P(x_0, x_1), \phi(x_0), \phi(x_1))),$$

which implies that $\lim\limits_{n\to\infty} P(x_n, x_*) = 0$.

We assume that $Tx_* \neq x_*$. Using (2.5), (2.12) and the result $\lim\limits_{n\to\infty} P(x_n, x_*) = 0$, we have

$$\begin{aligned}
&0 < \inf\{P(x, x_*) + P(x, Tx) : x \in X\} \leqslant \\
&\quad \leqslant \inf\{P(x_n, x_*) + P(x_n, Tx_n) : n \in \mathbb{N}\} = \inf\{P(x_n, x_*) + P(x_n, x_{n+1}) : n \in \mathbb{N}\} = 0,
\end{aligned}$$

which is a contradiction. Therefore, $Tx_* = x_*$ and consequently $x_* \in F_T \cap Z_\phi$.

We now prove that $F_T \cap Z_\phi = \{x_*\}$. If possible, let $x_*, y_* \in F_T \cap Z_\phi$. Then, $Tx_* = x_*$ and $Ty_* = y_*$ and also by (2.9), $P(x_*, x_*) = \phi(x_*) = 0$ and $P(y_*, y_*) = \phi(y_*) = 0$. Now,

$$\begin{aligned}
F(P(x_*, y_*), 0, 0) &= F(P(Tx_*, Ty_*), \phi(Tx_*), \phi(Ty_*)) \\
&\leqslant \Theta(N_F^\phi(x_*, y_*)),
\end{aligned}$$

where,

$$\begin{aligned}
&N_F^\phi(x_*, y_*) = \\
&= \max\{F(P(x_*, y_*), \phi(x_*), \phi(y_*)), F(P(x_*, Tx_*), \phi(x_*), \phi(Tx_*)), F(P(y_*, Ty_*), \phi(y_*), \phi(Ty_*))\} = \\
&= \max\{F(P(x_*, y_*), \phi(x_*), \phi(y_*)), F(P(x_*, x_*), \phi(x_*), \phi(x_*)), F(P(y_*, y_*), \phi(y_*), \phi(y_*))\} = \\
&= \max\{F(P(x_*, y_*), 0, 0), 0, 0\} \quad [\text{using } (2.8)] \\
&= F(P(x_*, y_*), 0, 0).
\end{aligned}$$

Therefore,

$$F(P(x_*, y_*), 0, 0) \leqslant \Theta(F(P(x_*, y_*), 0, 0)),$$

which, by Lemma 1.3, is a contradiction unless $F(P(x_*, y_*), 0, 0) = 0$. Hence $F(P(x_*, y_*), 0, 0) = 0$. Then using the property of F, we have $P(x_*, y_*) = 0$. Also, $P(x_*, x_*) = 0$. By (i) of Lemma 1.1, we have $x_* = y_*$. Therefore, $F_T \cap Z_\phi = \{x_*\}$ and hence T has a unique ϕ-fixed point.

Example 2.1. *Take the complete b-metric space $(X, \mu, l \geqslant 1)$, where $X = [0, 1]$, $\mu(x, y) = (x - y)^2$ and $l = 2$. Take the wt-distance P on X, where $P(x, y) = y^2$, for $x, y \in X$. Let $F \in \Upsilon$ and $\Theta : [0, \infty) \to [0, \infty)$ be defined respectively by $F(a, b, c) = a + b$ and $\Theta(t) = \dfrac{15}{38}t$. Let $T : X \to X$ be defined by*

$$Tx = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \dfrac{1}{2}, \\ \dfrac{1}{2} & \text{if } \dfrac{1}{2} < x \leqslant 1. \end{cases}$$

Consider $\phi : X \to [0, \infty)$, where

$$\phi(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } 0 < x \leqslant \dfrac{1}{2}, \\ 3 & \text{if } \dfrac{1}{2} < x \leqslant 1. \end{cases}$$

Clearly, $\inf\{P(x, y) + P(x, Tx) : x \in X\} > 0$, for every $y \in X$ with $Ty \neq y$. Here, all the conditions of Theorem 2.1 are satisfied and 0 is the only ϕ-fixed point of T.

Theorem 2.2. *Let (X,d) be a complete metric space with a w-distance p on it. Let $T : X \to X$ be a mapping such that*

$$\inf\{p(x,y) + p(x,Tx) : x \in X\} > 0, \quad \textit{for every } y \in X \textit{ with } Ty \neq y \tag{2.14}$$

and

$$F(p(Tx,Ty), \phi(Tx), \phi(Ty)) \leqslant \Theta(N_F^{\phi}(x,y)), \textit{ for } x,y \in X, \tag{2.15}$$

where
$N_F^{\phi}(x,y) = \max\{F(p(x,y),\phi(x),\phi(y)), F(p(x,Tx),\phi(x),\phi(Tx)), F(p(y,Ty),\phi(y),\phi(Ty))\}$, $\Theta \in \Omega$ *and F, ϕ are as defined in Theorem 2.1. Then*

(i) $F_T \subseteq Z_{\phi}$;
(ii) T *is a ϕ-Picard operator;*
(iii) $p(x_*, x_*) = 0$, *where* $F_T \cap Z_{\phi} = \{x_*\}$.

Proof. Here the complete metric space (X,d) is a complete b-metric space (X,μ,l) with $l = 1$ and w-distance p is a wt-distance. Then by an application of Theorem 2.1, we have the required proof. □

Example 2.2. *Take the complete metric space $X = [0,3]$ with the usual metric 'd'. Consider the w-distance p on X, where $p(x,y) = y$, for all $x,y \in X$. Let $F \in \Upsilon$ and $\Theta \in \Omega$ be defined respectively by $F(a,b,c) = a + b$ and $\Theta(t) = \dfrac{49}{100}t$. Let $T : X \to X$ be defined by*

$$Tx = \begin{cases} 0 & \text{if } 0 \leqslant x < \dfrac{5}{2}, \\ 1 & \text{if } \dfrac{5}{2} \leqslant x \leqslant 3. \end{cases}$$

Consider $\phi : X \to [0,\infty)$, where

$$\phi(x) = \begin{cases} 0 & \text{if } x = 0, \\ \dfrac{1}{2} & \text{if } 0 < x < \dfrac{5}{2}, \\ 1 & \text{otherwise.} \end{cases}$$

Here, all the conditions of Theorem 2.2 are satisfied, and 0 is the only ϕ-fixed point of T.

Remark 2.1. *Theorem 2.2 is a generalization of the Theorem 1.3. Taking $x \in [\dfrac{5}{2}, 3]$ and $y = 0$ in Example 2.2, we see $F(p(Tx,Ty),\phi(Tx),\phi(Ty)) = Ty + \phi(Tx) = 0 + \phi(1) = \frac{1}{2}$ and $\Theta(F(p(x,y),\phi(x),\phi(y))) = \dfrac{49}{100}(y + \phi(x)) = \dfrac{49}{100}(0+1) = \dfrac{49}{100}$. The inequality (1.3) of Theorem 1.3 is not satisfied for the case when $x \in \left[\frac{5}{2}, 3\right]$ and $y = 0$. Hence Theorem 1.3 is not applicable to the above example, that is, Example 2.2. This shows that Theorem 2.2 is a proper generalization of the Theorem 1.3.*

3. Applications

In this section we have two applications. Theorem 2.1 is applied to a problem of Volterra integral equation while Theorem 2.2 is applied to obtain a solution of a Fredholm integral equation.
Consider the non-empty set $C[m,n]$ of all real-valued continuous functions defined on $[m,n]$. Take

a complete b-metric μ with a parameter $l = 2$ on $C[m,n]$, where $\mu(x,y) = \sup\limits_{t\in[m,n]} (x(t) - y(t))^2$. Consider the non-homogeneous Volterra integral equation

$$x(t) = g(t) - \int_m^t K(t,s,x(s))ds, \tag{3.16}$$

where $m,n\in\mathbb{R}$ with $m < n$, $m<t<n$, $x\in C[m,n]$, $g\colon [m,n] \to \mathbb{R}$ and $K\colon [m,n]\times[m,n]\times\mathbb{R}\to\mathbb{R}$ are continuous functions.

Take a mapping $T : C[m,n] \to C[m,n]$ defined by

$$(Tx)(t) = g(t) - \int_m^t K(t,s,x(s))ds. \tag{3.17}$$

Theorem 3.1. *The integral equation* (3.16) *has a unique solution in* $C[m,n]$ *if there exists a non-decreasing and continuous function* $\Theta : [0,\infty) \to [0,\infty)$ *such that* $\sum\limits_{n=0}^{\infty} 2^n\Theta^n(t) < \infty$, *where* $t > 0$, *and* K *satisfies the following condition*

$$\begin{aligned}(K(t,s,x(s)))^2 + (K(t,s,y(s)))^2 \leqslant \frac{1}{n-m}\Big[\Theta\Big(\max\Big\{\sup_{t\in[m,n]}(x(t))^2 + \sup_{t\in[m,n]}(y(t))^2,\\ \sup_{t\in[m,n]}(x(t))^2 + \sup_{t\in[m,n]}((Tx)(t))^2, \sup_{t\in[m,n]}(y(t))^2 + \sup_{t\in[m,n]}((Ty)(t))^2\Big\}\Big) - 2(g(t))^2\Big],\end{aligned} \tag{3.18}$$

where $t,s \in [m,n]$ *and* $x,y \in C[m,n]$.

Proof. Let P be a wt-distance on $C[m,n]$ defined as $P(x,y) = \sup\limits_{t\in[m,n]}(x(t))^2 + \sup\limits_{t\in[m,n]}(y(t))^2$. Take a function $\phi : C[m,n] \to [0,\infty)$, where $\phi(x) = 0$ for all $x \in C[m,n]$ and $F \in \Upsilon$, where $F(x,y,z) = x + y + z$.
For $x,y \in C[m,n]$, we get

$$\begin{aligned}((Tx)(t))^2 + ((Ty)(t))^2 &= \left(g(t) - \int_m^t K(t,s,x(s))ds\right)^2 + \left(g(t) - \int_m^t K(t,s,y(s))ds\right)^2 \leqslant\\ &\leqslant 2\,(g(t))^2 + \left(\int_m^t K(t,s,x(s))ds\right)^2 + \left(\int_m^t K(t,s,y(s))ds\right)^2 \leqslant\\ &\leqslant 2\,(g(t))^2 + \int_m^t (K(t,s,x(s)))^2\,ds + \int_m^t (K(t,s,y(s)))^2\,ds \leqslant\\ &\leqslant 2\,(g(t))^2 + \int_m^t \left[(K(t,s,x(s)))^2 + (K(t,s,y(s)))^2\right]ds \leqslant\\ &\leqslant 2\,(g(t))^2 + \int_m^t \frac{1}{n-m}\left[\Theta(N_F^\phi(x,y)) - 2\,(g(t))^2\right]ds =\\ &= 2\,(g(t))^2 + \frac{1}{n-m}\left[\Theta(N_F^\phi(x,y)) - 2\,(g(t))^2\right]\int_m^t ds \leqslant\\ &\leqslant \Theta(N_F^\phi(x,y)).\end{aligned}$$

Therefore,

$$\sup_{t\in[m,n]}((Tx)(t))^2 + \sup_{t\in[m,n]}((Ty)(t))^2 \leqslant \Theta(N_F^\phi(x,y)).$$

Thus, $F(P(Tx,Ty),\phi(Tx),\phi(Ty)) \leqslant \Theta(N_F^{\phi}(x,y))$, for all $x,y \in C[m,n]$. All the conditions of Theorem 2.1 are satisfied and the equation (3.16) has a unique solution.

Consider the complete metric space $C[m,n]$ of all real-valued continuous functions defined on $[m,n]$ with the metric 'd', where $d(x,y) = \sup\limits_{t\in[m,n]} |x(t)-y(t)|$. Consider the non-homogeneous Fredholm integral equation

$$x(t) = f(t) + \int_m^n G(t,s,x(s))ds, \tag{3.19}$$

where $m,n \in \mathbb{R}$ with $m<n$, $x \in C[m,n]$, $f:[m,n]\to\mathbb{R}$ and $G:[m,n]\times[m,n]\times\mathbb{R}\to\mathbb{R}$ are continuous functions.

Consider a mapping $T\colon C[m,n]\to C[m,n]$ defined by

$$(Tx)(t) = f(t) + \int_m^n G(t,s,x(s))ds. \tag{3.20}$$

Theorem 3.2. *The integral equation* (3.19) *has a unique solution in* $C[m,n]$ *if there exists* $\Theta\in\Omega$ *such that* G *satisfies the following condition*

$$\begin{aligned}|G(t,s,x(s))| + |G(t,s,y(s))| \leqslant \frac{1}{n-m}\Bigg[\Theta\Bigg(&\max\Bigg\{\sup_{t\in[m,n]}|x(t)| + \sup_{t\in[m,n]}|y(t)|,\\ &\sup_{t\in[m,n]}|x(t)| + \sup_{t\in[m,n]}|(Tx)(t)|, \sup_{t\in[m,n]}|y(t)| + \sup_{t\in[m,n]}|(Ty)(t)|\Bigg\}\Bigg) - 2|f(t)|\Bigg],\end{aligned} \tag{3.21}$$

where $t,s\in[m,n]$ *and* $x,y\in C[m,n]$.

Proof.

Let p be a w-distance on $C[m,n]$ defined by $p(x,y) = \sup\limits_{t\in[m,n]}|x(t)| + \sup\limits_{t\in[m,n]}|y(t)|$. We take a function $\phi : C[m,n]\to[0,\infty)$, where $\phi(x)=0$ for all $x\in C[m,n]$ and $F\in\Upsilon$, where $F(x,y,z)=x+y+z$.

For $x,y\in C[m,n]$, we get

$$\begin{aligned}|(Tx)(t)| + |(Ty)(t)| &= |f(t) + \int_m^n G(t,s,x(s))ds| + |f(t) + \int_m^n G(t,s,y(s))ds| \leqslant\\ &\leqslant 2|f(t)| + \Big|\int_m^n G(t,s,x(s))ds| + |\int_m^n G(t,s,y(s))ds\Big| \leqslant\\ &\leqslant 2|f(t)| + \int_m^n |G(t,s,x(s))|ds + \int_m^n |G(t,s,y(s))|ds \leqslant\\ &\leqslant 2|f(t)| + \int_m^n \left(|G(t,s,x(s))| + |G(t,s,y(s))|\right)ds \leqslant\\ &\leqslant 2|f(t)| + \int_m^n \frac{1}{n-m}\Big[\Theta(N_F^{\phi}(x,y)) - 2|f(t)|\Big]ds =\\ &= 2|f(t)| + \frac{1}{n-m}\Big[\Theta(N_F^{\phi}(x,y)) - 2|f(t)|\Big]\int_m^n ds =\\ &= \Theta(N_F^{\phi}(x,y)).\end{aligned}$$

Therefore,

$$\sup_{t\in[m,n]}|(Tx)(t)| + \sup_{t\in[m,n]}|(Ty)(t)| \leqslant \Theta(N_F^{\phi}(x,y)).$$

Hence, $F(p(Tx,Ty),\phi(Tx),\phi(Ty)) \leqslant \Theta(N_F^{\phi}(x,y))$, for all $x,y\in C[m,n]$. All the conditions of Theorem 2.2 are satisfied and the equation (3.19) has a unique solution.

Conclusion

In fixed point theory w-distances have proved to be useful in obtaining several new results and also in extending results already existing in the literature. The present paper is another demonstration of the above fact obtained by establishing a new ϕ-fixed point result in the framework of b-metric spaces.

We hope that the present approach can be utilized in other contexts gainfully for deducing new results. This is supposed to form the subject of our future works.

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ϕ-неподвижная точка приводит к b-метрическим пространствам с wt-расстоянием

Ранаджит Джоти
Бинаяк С. Чоудхури
Индийский институт инженерных наук и технологий, Шибпур
Ховрах-711103, Западная Бенгалия, Индия
Никхилеш Метия
Соварани мемориал колледж, Джагатбаллавпур
Овра-711408, Индия
Санту Дутта
Калькуттский институт науки и управления
Калькутта-700040, Западная Бенгалия, Индия
Санкар П. Мондал
Маулана Абул Калам Асад университет технологий, Аригата
Надия-741249, Западная Бенгалия, Индия

Аннотация. В этой статье наша программа заключается в получении результата ϕ-неподвижной точки вместе с некоторыми приложениями. Рассматриваемая здесь проблема сформулирована путем объединения нескольких последних тенденций в метрической теории неподвижной точки и ее расширений. Обсуждаются два иллюстративных примера. Показано, что некоторые результаты, существующие в литературе, расширяются нашей основной теоремой. Представленное приложение находится в области интегральных уравнений Вольтерра и Фредгольма.

Ключевые слова: b-метрическое пространство, wt-расстояние, неподвижная точка, ϕ-неподвижная точка, интегральное уравнение.