EDN: VBJGHW
УДК 532.5

Axisymmetric Ideal Fluid Flows Effectively not Being Tied to Vortex Zones

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Received 10.06.2024, received in revised form 05.07.2024, accepted 26.08.2024

Abstract. The paper formulates a model of axisymmetric flow of an ideal fluid with n effectively inviscid vortex zones, generalizing the well-known model of M. A. Lavrentiev on the gluing of vortex and potential flows in a plane case. The possibility is shown within the framework of such a model of the existence in space of a liquid sphere streamlined around by a potential axisymmetric flow, consisting of n spherical layers of axisymmetric vortex flows. This model example generalizes the spherical Hill vortex with one vortex zone, known in hydrodynamics. Such a vortex flow with n spherical layers is also possible in a sphere, and, unlike a flow in space, such a flow is not unique. The problem of an axisymmetric vortex flow in a limited region is considered; its formulation generalizes the plane flow of an ideal fluid in a field of Coriolis forces.

Keywords: ideal fluid, vortex flows, spherical Hill vortex

Citation: I.I. Vainshtein, Axisymmetric Ideal Fluid Flows Effectively not Being Tied to Vortex Zones, J. Sib. Fed. Univ. Math. Phys., 2024, 17(5), 665–678. EDN: VBJGHW.

Introduction. Setting of the problem

A large number of works and monographs are devoted to the study of vortex flows. The topic of vortex flows is presented in every hydrodynamics course. The monographs by M. A. Gol'dshtik "Vortex Flows" [1], M. A. Lavrentiev, B. V. Shabat "Problems of Hydrodynamics and Their Mathematical Models" [2] can be considered fundamental in this research area. The monographs indicate various examples of vortex flows in nature and technology, present a study of problems of signifit scientific and practical interest, and formulate various mathematical problems for research.

The paper examines one of them, related to the existence and non-uniqueness of axisymmetric flows according to the scheme of M. A. Lavrentiev [1, 2] with n effectively inviscid vortex zones in an unbounded and limited region.

The stationary vortex flow of an ideal incompressible fluid in the plane case is described by the equation

$$\Delta\Psi = \frac{\partial\Psi(x,y)}{\partial x^2} + \frac{\partial\Psi(x,y)}{\partial y^2} = F(\Psi),\ v_x = \frac{\partial\Psi}{\partial y},\ v_y = -\frac{\partial\Psi}{\partial x}, \tag{1}$$

$$L\Psi(z,r) = \frac{\partial^2\Psi(z,r)}{\partial z^2} + \frac{\partial^2\Psi(z,r)}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi(z,r)}{\partial r} = H'(\Psi)r^2 - \Gamma'(\Psi)\Gamma(\Psi), \tag{2}$$

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$v_r = -\dfrac{1}{r}\dfrac{\partial \Psi}{\partial z}$, $v_z = \dfrac{1}{r}\dfrac{\partial \Psi}{\partial r}$ in axisymmetric. Functions $F(\Psi),\ H'(\Psi),\ \Gamma(\Psi)$ are arbitrary functions of the flow function Ψ[1]. Various approaches to defining the functions $F(\Psi),\ H(\Psi),\ \Gamma(\Psi)$ when solving specific problems are also available in [1].

The right-hand sides of equations (1), (2) determine the value of vorticity $\omega(x,y),\ \omega(z,r)$. When the vorticity is zero, the flow is potential.

Thus, equations (1), (2) of the motion of an ideal fluid in terms of the flow function make it possible to study the motion of an ideal fluid with potential and vortex zones. With the natural requirement of continuity of the velocity field, one should require the continuity of the first partial derivatives of the flow function when passing through the common boundary of these zones.

It is important to note that equation (2) in the appropriate notation is called the Grad-Shafranov equation [3] in plasma theory, on the basis of which tokamaks are calculated and built.

The paper considers flows with effectively inviscid vortex zones, where it is assumed that the flow of an ideal fluid is the limiting flow of a viscous fluid when the viscosity tends to zero. In this case, the vorticity in the plane case is equal to a constant, in the axisymmetric case $\omega(z,r) = \omega_0 r$, ω_0 is a constant $[1,2,4]$. Respectively

$$\Delta\Psi(x,y) = \omega_0, \quad L\Psi(z,r) = \omega(z,r)r = \omega_0 r^2. \tag{3}$$

In this case, the M. A. Lavrentiev scheme of plane flows with n vortex zones $[1,2,5]$ for axisymmetric flows with n effectively inviscid vortex zones in unbounded and bounded regions can be formulated as follows: given a flow region D with a boundary Γ, numbers $\omega_i,\ i = 1,\dots,n$. The value of the flow function $\Psi(z,r)$ on the boundary Γ of the region D or its behavior at infinity is specified. It is required to construct disjoint flow zones $D_i,\ \bigcup D_i = D$ and find in the region D a continuously differentiable flow function $\Psi(z,r)$, which in each zone D_i satisfies the equation $L\Psi = \omega_i r^2$. At all points of the boundaries Γ_i of zones B_i not belonging to the boundary Γ of area D, it is equal to zero. The possibility of the existence of zones in which the values of ω_i coincide or $\omega_i = 0$ cannot be excluded. In the latter case, the flow in the zone D_i is potential.

Note that, taking into account corrections associated with viscosity, M. A. Lavrentyev, using a plane flow model with three flow zones in a deep trench (two vortex zones with constant vorticities $\pm\omega$, and in the third — potential flow), substantiated the unacceptability burial of radioactive residues in ocean depressions $[1,2]$.

The formulated problem with n vortex zones is nonlinear, and here an important role is played by the consideration of model problems, the results of which can be used in the formulation and solution of general problems. This will be seen when solving problem (31).

Let us formulate a simple property related to the geometry of zones D_i and the signs of ω_i.

Property 1. *If the boundaries of the zones D_i, D_j are zero streamlines and $\omega_i\omega_j > 0$ then the two zones have no common point which can be touched with circles both from the region D_i and the region D_j.*

Let there be such a point M^*. Since the flow function $\Psi(z,r)$ vanishes at the boundaries of the zones $D_i,\ D_j$, then in the case $\omega_i > 0,\ \omega_j > 0$ the function $\Psi(z,r)$ at this point attains its maximal value in the zones $D_i,\ D_j$, for $\omega_i < 0,\ \omega_j < 0$ smallest. In such a situation, the derivatives of the solution at the point M^* along the external normals from the zones $D_i,\ D_j$ are of the same sign [6], which contradicts the continuous differentiability of the solution when passing through the common boundary of the zones.

Using the known relation

$$L(r^2U(z,r)) = r^2L^*U(z,r), \quad L^*U(z,r) = \frac{\partial^2 U(z,r)}{\partial z^2} + \frac{\partial^2 U(z,r)}{\partial r^2} + \frac{3}{r}\frac{\partial U(z,r)}{\partial r},$$

to obtain solutions to the equation $L\Psi = \omega_0 r^2$, in (3), it is convenient to pass to the equation $L^*U = \omega_0$, after replacing $\Psi(z,r) = r^2U(z,r)$.

In the equation $L^*U = \omega_0$ it is already possible to look for a solution depending only on R ($R^2 = r^2 + z^2$), $U(z,r) = U(R)$. In this case

$$L^*U(R) = \frac{\partial^2 U(R)}{\partial R^2} + \frac{4}{R}\frac{\partial U(R)}{\partial R} = \omega_0.$$

Its solution is the function

$$U(R) = \frac{\omega_0}{10}R^2 + \frac{c}{R^3} + d, \; R \neq 0, \tag{4}$$

c, d — arbitrary constants. Note that $\left(\frac{\partial^2}{\partial R^2} + \frac{4}{R}\frac{\partial}{\partial R}\right)\frac{1}{R^3} = 0.$

After returning to the function $\Psi(z,r) = r^2\left(\frac{\omega_0}{10}R^2 + \frac{c}{R^3} + d\right)$ we have a solution to the equation $L\ Psi(z,r) = \omega_0 r^2$.

For further purposes, let us formulate what can be verified by direct differentiation:

Property 2. *Let* $\Psi_i = r^2\left(\frac{\omega_i}{10}R^2 + \frac{c_i}{R^3} + d_i\right)$. *If the constants* $c_i, d_i,\ c_j, d_j$ *are such that the functions* $\Psi_i,\ \Psi_j$ *vanish for* $R = a$, *then the condition for their continuous differentiability for* $R = a$ *is written in the form*

$$\frac{1}{10}\left(2\omega_i a - \frac{c_i}{a^4}\right) = \frac{1}{10}\left(2\omega_j a - \frac{c_j}{a^4}\right).$$

1. Hill vortex with n vortex spherical layers

Let us consider the possibility of the existence in the entire space of an axisymmetric flow with n vortex zones with a given geometry of the vortex zones: ($D_1: R \leqslant a_1$, $D_i: a_{i-1} \leqslant R \leqslant a_i$, $a_1 > 0$, $a_{i-1} < a_i$, $i = 2, \ldots, n$). In the zone ($D_{n+1}: R > a_n$, $\omega_{n+1} = 0$) the flow is potential.

For a given flow case, the problem can be written in analytical form

$$L\Psi(z,r) = \begin{cases} \omega_1 r^2 & \text{if } R < a_1, \\ \omega_i r^2 & \text{if } a_{i-1} < R < a_i, i = 2, \ldots, n, \\ 0 & \text{if } R > a_n, \end{cases} \tag{5}$$

given that

$$\Psi|_{R=a_i} = 0, \; i = 1, \ldots, n, \quad \lim_{R\to\infty} \frac{\Psi}{r^2} = A > 0. \tag{6}$$

Given such a geometry of zones, according to Property 1, the signs of numbers ω_i must alternate if none of them is zero,

In accordance with (4), we look for a solution to problem (5), (6) in the form

$$\Psi(z,r) = \begin{cases} \frac{r^2}{10}\omega_1(R^2 - a_1^2) & \text{if } 0 \leqslant R \leqslant a_1, \\ \frac{r^2}{10}(\omega_i R^2 + \frac{c_i}{R^3} + d_i) & \text{if } a_{i-1} \leqslant R \leqslant a_i, i = 2, \ldots, n, \\ Ar^2(1 - \frac{a_n^3}{R^3}) & \text{if } R \geqslant a_n. \end{cases}$$

Satisfying the boundary conditions (6) and the continuous differentiability of the solution when passing through the boundaries of the zones, in accordance with Property 2, we obtain the system

$$\omega_i a_{i-1}^2 + \frac{c_i}{a_{i-1}^3} + d_i = 0, \quad \omega_i a_i^2 + \frac{c_i}{a_i^3} + d_i = 0, \ i = 2, \dots, n, \tag{7}$$

$$2\omega_1 a_1 = 2\omega_2 a_1 - \frac{3c_2}{a_1^4}, \quad 2\omega_i a_i - \frac{3c_i}{a_i^4} = 2\omega_{i+1} a_i - \frac{3c_{i+1}}{a_i^4}, \ i = 2, \dots, n-1, \tag{8}$$

$$2\omega_n a_n - \frac{3c_n}{a_n^4} = \frac{30A}{a_n}. \tag{9}$$

From (7–9)

$$\omega_i(a_{i-1}^2 - a_i^2) + c_i\left(\frac{1}{a_{i-1}^3} - \frac{1}{a_i^3}\right) = 0, \ i = 2, \dots, n-1, \tag{10}$$

$$c_2 = \frac{2(\omega_2 - \omega_1)}{3} a_1^5, \quad c_{i+1} = \frac{2}{3}(\omega_{i+1} - \omega_i) a_i^5 + c_i, \quad c_1 = 0, \ i = 2, \dots, n-1, \tag{11}$$

$$c_n = \frac{2\omega_n}{3} a_n^5 - 10 A a_n^3. \tag{12}$$

From (11)

$$\sum_{j=2}^{i} c_j = \frac{2}{3} \sum_{j=2}^{i} (\omega_j - \omega_{j-1}) a_{j-1}^5 + \sum_{j=2}^{i} c_{j-1}, \quad c_i = \frac{2}{3} \sum_{j=2}^{i} (\omega_j - \omega_{j-1}) a_{j-1}^5, \ i = 2, \dots, n. \tag{13}$$

Let us prove a property of system (10), (11), which will be used further.

Property 3. *If the signs ω_i, $i = 1, \dots, n$, alternate, then $a_{i-1} = t_{i-1} a_i$, $i = 2, \dots, n$ $0 < t_{i-1} < 1$ is a root of the equation*

$$t^4 + t^3 + t^2 + \gamma_{i-1} t + \gamma_{i-1} = 0, \ \left(\gamma_{i-1} = \frac{3\omega_i}{2(\bar{\omega}_{i-1} - \omega_i)}\right), \tag{14}$$

$$\bar{\omega}_i = (1 - t_{i-1}^5)\omega_i + t_{i-1}^5 \bar{\omega}_{i-1}, \ \bar{\omega}_1 = \omega_1, \ c_i = \frac{2(\omega_i - \bar{\omega}_{i-1})}{3} t_{i-1}^5 a_i^5, \ i = 2, \dots, n, \tag{15}$$

$$sign(\bar{\omega}_i) = sign(\omega_i), \ i = 1, \dots, n. \tag{16}$$

We assume $i = 2$. Considering $\bar{\omega}_1 = \omega_1, \ c_2 = \dfrac{2(\omega_2 - \bar{\omega}_1)}{3} a_1^5$, from (10)

$$\omega_2(a_1^2 - a_2^2) + \frac{2(\omega_2 - \bar{\omega}_1)}{3} a_1^5 \left(\frac{1}{a_1^3} - \frac{1}{a_2^3}\right) = 0.$$

Hence $a_1 = a_2$, or

$$\omega_2(a_1 + a_2) + \frac{2(\bar{\omega}_1 - \omega_2)}{3} a_1^2 \frac{a_2^2 + a_2 a_1 + a_1^2}{a_2^3} = 0. \tag{17}$$

d Equation (17) is homogeneous. Denoting $t_1 = \dfrac{a_1}{a_2} < 1, \ \gamma_1 = \dfrac{3\omega_2}{2(\bar{\omega}_1 - \omega_2)}, \ \omega_1 \neq \omega_2$, we obtain an equation for finding the value t_1

$$t^4 + t^3 + t^2 + \gamma_1 t + \gamma_1 = 0. \tag{18}$$

For the existence of a root $0 < t < 1$ it is necessary $\gamma_1 < 0$. From the resulting equation

$$\gamma_1(t) = -\left(t^3 + t - 1 + \frac{1}{t+1}\right), \quad 0 \leqslant t \leqslant 1, \quad \gamma_1(0) = 0, \quad \gamma_1(1) = -\frac{3}{2} < 0,$$

$$\gamma_1'(t) = -(3t^2 + 1 - \frac{1}{(t+1)^2}).$$

For $t > 0$, $\gamma_1'(t) < 0$, the function $\gamma_1(t)$ decreases monotonically as $t \geqslant 0$. Hence, for

$$-\frac{3}{2} < \frac{3\omega_2}{2(\bar{\omega}_1 - \omega_2)} < 0 \tag{19}$$

equation (18) has a single root t_1 on the interval (0,1). Note that inequality (19) is satisfied, if ω_1, ω_2 have different signs. We got $a_1 = t_1 a_2$. For $\omega_2 = \omega_1$, equation (17) implies $\omega_2 = 0$, and hence $\omega_1 = 0$, or $a_2 = a_1$. Zones D_1, D_2 are combined into one — the number of zones with such geometry should be reduced by one when setting the problem.

For what follows we set $i = 3$. From relation (13) it follows

$$c_3 = \frac{2}{3}(\omega_3 - ((1 - t_1^5)\omega_2 + t_1^5\bar{\omega}_1))a_2^5 = \frac{2}{3}(\omega_3 - \bar{\omega}_2)a_2^5.$$

From (17), similarly to the case $i = 2$, we obtain equation (14) with $\gamma_2 = \dfrac{3\omega_3}{2(\bar{\omega}_2 - \omega_3)}$.

Let us show that $sign(\bar{\omega}_2) = sign(\omega_2)$. Let's write it down

$$sign(\bar{\omega}_2) = sign(\lambda(t_1)(1 - t_1^5) + t_1^5)sign(\bar{\omega}_1), \quad \lambda(t_1) = \frac{\omega_2}{\bar{\omega}_1} = \frac{2\gamma_1}{3 + 2\gamma_1}. \tag{20}$$

From equation (14) with $\gamma_{i-1} = \gamma_2$ we express γ_2 and after substitution into (20)

$$\lambda(t_1)(1 - t_1^5) + t_1^5 = \frac{2(t_1^4 + t_1^3 + t_1^2)}{2(t_1^4 + t_1^3 + t_1^2) - 3t_1 - 3}(1 - t_1^5) + t_1^5 = -t_1^2\frac{3t_1^3 + 6t_1^2 + 4t_1 + 2}{2t_1^3 + 4t_1^2 + 6t_1 + 3} < 0.$$

We got $sign(\bar{\omega}_2) = -sign(\bar{\omega}_1) = -sign(\omega_1) = sign(\omega_2)$. Since ω_1, ω_2 by assumption have different signs. This implies that inequality (19) holds for $\omega_3,\ \omega_2$ of different signs. Then $a_2 = t_2 a_3$.

Increasing i successively by one, similar to the previous one, we obtain $a_{i-1} = t_{i-1}a_i$ and $c_i = \dfrac{2(\omega_i - \bar{\omega}_{i-1})}{3}a_{i-1}^5 = \dfrac{2(\omega_i - \bar{\omega}_{i-1})}{3}t_{i-1}^5 a_i^5,\ sign(\bar{\omega}_i) = sign(\omega_i)$.

Let us return to the problem under consideration (5), (6). Let in some zone $D_i,\ \omega_i = 0$, and in the zone $D_{i-1},\ \omega_{i-1} \neq 0,\ i = 2, 3, \ldots, n$, The function $\Psi(z, r)$ and its partial derivatives in the zone D_i are identically equal to zero, which contradicts the inequality of the normal derivative from the zone D_{i-1} to zero at points $R = a_{i-1}$ of the common boundary of the zones $D_{i-1},\ D_i$, since the function $\Psi(z, r)$ in the zone D_{i-1} at boundary points $R = a_{i-1}$ takes either the largest or smallest value, depending on the sign of ω_{i-1}[6]. Further continuing these arguments, successively decreasing the index i, and then successively increasing it, we arrive at all $\omega_i = 0,\ i = 1.2, \ldots, n$.

Thus, a flow with the considered geometry of vortex zones cannot have a single internal zone with potential flow, and the alternation of signs of ω_i in zones D_i is a necessary condition for the existence of a solution to the problem under consideration.

Let the signs of ω_i alternate in the statement of the problem. Substituting $c_n = \dfrac{2(\omega_n - \bar{\omega}_{n-1})}{3}t_{n-1}^5 a_i^5$ (Property 3) into (12), we obtain the equation for finding a_n

$$\bar{\omega}_n a_n^2 = 15A, \quad \bar{\omega}_n = (1 - t_{n-1}^5)\omega_n + t_{n-1}^5\bar{\omega}_{n-1}.$$

Requiring $\omega_n > 0$, in the problem statement by Property 3, we obtain $sign(\bar{\omega}_n) = sign(\omega_n) > 0$. Then

$$a_n = \sqrt{\frac{15A}{(1 - t_{n-1}^5)\omega_n + t_{n-1}^5\bar{\omega}_{n-1}}}.$$

Next, a_i are determined inversely through a_n, $a_{i-1} = t_{i-1}a_i$, $i = n, n-1, \dots, 2$.

Note that if $\omega_n > 0$, is required, ω_1 must be less than zero when n is even and $\omega_1 > 0$ when n is odd.

Thus, we have obtained that in space, within the framework of an ideal fluid, it is possible to move a liquid sphere of radius a_n, streamlined around by a potential flow, inside which there are n vortex zones with vorticities $\omega_i r$, with alternating signs ω_i, at $\omega_n > 0$.

Let us write down the solution to problem (5), (6) (the signs of ω_i alternate, $\omega_n > 0$).

$$\Psi(z,r) = \begin{cases} \dfrac{r^2}{10}\omega_1\big(R^2 - a_1^2\big) & \text{if } 0 \leqslant R \leqslant a_1, \\ \dfrac{r^2}{10}\omega_i\big(R^2 - a_i^2\big) + \dfrac{2(\omega_i - \bar{\omega}_{i-1})}{3} t_{i-1}^5 a_i^5 \left(1 - \left(\frac{a_1}{R}\right)^3\right) & \text{if } a_{i-1} \leqslant R \leqslant a_i,\ i = 2, \dots, n, \\ Ar^2\left(1 - \dfrac{a_n^3}{R^3}\right) & \text{if } R \geqslant a_n. \end{cases}$$

For $n = 1$ (one vortex zone with $\omega > 0$) we have the spherical Hill vortex, known in hydrodynamics [7], in plasma theory after "spherical plasmoid" [8]

$$\Psi(z,r) = \begin{cases} \omega r^2(R_0^2 - R^2) & \text{if } 0 \leqslant R \leqslant R_0, \\ Ar^2\left(1 - \dfrac{R_0^3}{R^3}\right) & \text{if } R \geqslant R_0, \end{cases}$$

R_0, ω, A are related by the relation $\omega = \dfrac{15A}{R_0^2}$. The Hill vortex represents a liquid sphere moving in the direction of the OZ axis in a potential flow around it with a speed of $\frac{A}{2}$ at infinity, inside which there is a vortex motion with a vorticity of ωr. It was shown in [9] that in the vicinity of the spherical Hill vortex there is no other axisymmetric vortex with one vortex zone, which differs little from it.

Note that the resulting solution to problem (5), (6) describes a natural axisymmetric generalization of the Hill vortex with n vortex zones. This structure of the vortex flow can be called a composite spherical Hill vortex.

Let us write the flow function for a composite Hill vortex with two vortex zones ($\omega_1 < 0$, $\omega_2 > 0$)

$$\Psi(z,r) = \begin{cases} \dfrac{\omega_1}{10} r^2(R^2 - a_1^2) & \text{if } R \leqslant a_1, \\ \dfrac{r^2}{10}\omega_2(R^2 - a_2^2) + \dfrac{3(\omega_2 - \omega_1)}{2} a_1^5 \left(1 - \dfrac{a_2^3}{R^3}\right) & \text{if } a_1 \leqslant R \leqslant a_2, \\ Ar^2\left(1 - \dfrac{a_2^3}{R^3}\right) & \text{if } R \geqslant a_2, \end{cases}$$

$$a_2 = \sqrt{\frac{15A}{((1 - t_1^5)\omega_2 + t_1^5\omega_1)}},\ a_1 = a_2 t_1 = \sqrt{\frac{15A}{((1 - t_1^5)\omega_2 + t_1^5\omega_1)}} t_1. \tag{21}$$

It is important to note that we wrote the general problem of axisymmetric flow in space with n vortex zones in analytical form for a specific particular case of the geometry of vortex zones in the form of spherical layers, assigning each zone its own vorticity from a given set of vorticities $\omega_i r$. Since the existence of such a solution requires only that the vorticities alternate signs in adjacent zones (the numerical values of ω_i determine the radii of the layers) and $\omega_n > 0$, then the specified ω_i must have $\dfrac{n}{2}$ positive and $\dfrac{n}{2}$ negative ω_i, if n is even, and $\dfrac{n+1}{2}$ positive, $\dfrac{n-1}{2}$ negative ω_i, if n is odd.

Thus, given a set of vorticities with the properties specified above, there is the possibility of the existence of $\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!$ in space if n is even and $\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!$ if n is odd, composite Hill vortices with n vortex zones in the form of spherical layers.

Let us note an interesting fact that, along with the composite Hill vortex with two vortex zones at $\omega_1 < 0,\ \omega_2 > 0$ of radius a_2 (21), there is a Hill vortex with the same radius a_2, but with one vortex zone with vorticity $\omega(z,r) = \big((1-t_1^5)\omega_2 + t_1^5\omega_1\big)r$ with the same value A.

Let's consider the inverse problem. Given a Hill vortex with a given value A and one vortex zone of radius R_0. It is required to find a composite Hill vortex with two vortex zones with the same values $R_0,\ A$.

In accordance with (21), we arrive at the problem of finding the numbers $\omega_1 < 0,\ \omega_2 > 0$ satisfying the relation

$$(1-t^5)\omega_2 + t^5\omega_1 = \omega = \frac{15A}{R_0^2},\ 0 < t < 1,$$

where t is an implicit function of $\omega_1,\ \omega_2$, given by equation (18) with $\gamma_1 = \dfrac{3\omega_2}{2(\omega_1-\omega_2)}$.

From $\gamma_1 = \dfrac{3\omega_2}{2(\omega_1-\omega_2)}$ we write $\omega_2 = \dfrac{2\gamma}{3+2\gamma_1}\omega_1$, and then from $(1-t^5)\omega_2 + t_1^5\omega_1 = \omega$, we get

$$\omega_1 = \frac{3+2\gamma_1}{2\gamma_1+3t^5}\omega, \quad \omega_2 = \frac{2\gamma_1}{2\gamma_1+3t^5}\omega, \quad \omega = \frac{15A}{R_0^2}. \tag{22}$$

From equation (18) we find $\gamma_1 = -\dfrac{t^4+t^3+t^2}{t+1}$, and after substitution into (22), we find

$$\omega_2 = \frac{-2(t^2+t+1)}{(t-1)(3t^3+6t^2+4t+2)}\omega > 0, \quad \omega_1 = -\frac{2t^3+4t^2+6t+3}{(3t^3+6t^2+4t+2)t^2}\omega < 0,\ 0<t<1.$$

Next, setting $t,\ 0<t<1$, arbitrarily, we find $\omega_1,\ \omega_2$, and then using formulas (21) for $t_1 = t$ the values $a_1,\ a_2$. By construction $a_2 = R_0$. Note that due to the arbitrariness of the value of $t,\ 0<t<1$, the inverse problem under consideration has an infinite number of solutions.

2. Flow in a sphere with vortex spherical layers

Let us consider the possibility of axisymmetric flow in a sphere of radius R_0 with a given geometry of n vortex zones in the form of spherical layers ($D_1 : R \leqslant a_1,\ D_i : a_{i-1} \leqslant R \leqslant a_i$, $i = 2,\dots,n$) and with one selected zone ($D_{n+1} : a_n \leqslant R \leqslant R_0$), adjacent to the boundary $R = R_0$, only in which vorticity can become zero, i.e. the flow may be potential. This flow design for $\omega_{n+1} = 0$ is an analogue of a composite Hill vortex in a sphere.

Just as in point 1. the problem can be written in analytical form

$$L\Psi(z,r) = \begin{cases} \omega_1 r^2 & \text{if } R < a_1, \\ \omega_i r^2 & \text{if } a_{i-1} < R < a_i,\ i = 2,\dots,n, \\ \omega_{n+1} r^2 & \text{if } a_n < R < R_0, \end{cases}$$

given that

$$\Psi|_{R=a_i} = 0,\ i = 1,\dots,n, \quad \Psi|_{R=R_0} = A > 0. \tag{23}$$

In accordance with (4), we look for a solution to the problem in the form

$$\Psi(z,r)=\begin{cases}\dfrac{r^2}{10}\omega_1(R^2-a_1^2), \quad 0\leqslant R\leqslant a_1,\\ \dfrac{r^2}{10}(\omega_i R^2+\dfrac{c_i}{R^3}+d_i), \quad a_{i-1}\leqslant R\leqslant a_i,\ i=2,\dots,n,\\ \dfrac{r^2}{10}\left(\omega_{n+1}(R^2-a_n^2)+\dfrac{(10A-\omega_{n+1}(R_0^2-a_n^2))R_0^3}{(R_0^3-a_n^3)}\left(1-\dfrac{a_n^3}{R^3}\right)\right), \quad a_n\leqslant R\leqslant R_0.\end{cases}$$

Here, the boundary conditions are satisfied in the zone D_1 with $R=a_1$, in the zone D_n with $R=a_n$, in the zone D_{n+1} with $R=a_n,\ R=R_0$. Satisfying the remaining boundary conditions (23) and the continuous differentiability of the solution when passing through the boundaries of the zones, we obtain system (10–12), in which equation (12) should be replaced by the equation

$$\frac{1}{10}\left(2\omega_n a_n-\frac{3c_n}{a_n^4}\right)=\frac{1}{10}\left(2\omega_{n+1}a_n+\frac{3R_0^3(10A-\omega_{n+1}(R_0^2-a_n^2))}{(R_0^3-a_n^3)a_n}\right). \tag{24}$$

In accordance with Property 3, the signs of ω_i must alternate and $a_{n-1}=t_{n-1}a_n,\ c_n=\dfrac{2(\omega_n-\bar{\omega}_{n-1})}{3}t_{n-1}^5a_n^5$. Taking this into account, from (24) the equation for determining the value of a_n follows

$$a_n^5-R_0^3a_n^2\left(1-\frac{3\omega_{n+1}}{2(\bar{\omega}_n-\omega_{n+1})}\right)+\frac{3R_0^3(10A-\omega_{n+1}R_0^2)}{2(\bar{\omega}_n-\omega_{n+1})}=0. \tag{25}$$

We set $\omega_n>0,\ \omega_{n+1}\leqslant 0$. Then $\bar{\omega}_n>0$ and $\dfrac{3R_0^3(10A-\omega_{n+1}R_0^2)}{2(\bar{\omega}_n-\omega_{n+1})}>0$. Consider the function

$$f(a_n)=a_n^5-R_0^3a_n^2\left(1-\frac{3\omega_{n+1}}{2(\bar{\omega}_n-\omega_{n+1})}\right)+\frac{3R_0^3(10A-\omega_{n+1}R_0^2)}{2(\bar{\omega}_n-\omega_{n+1})}.$$

We have $f(0)>0,\ f(R_0)>0$. At point $a_n^*=\left(\dfrac{2}{5}\left(1-\dfrac{3\omega_{n+1}}{2(\bar{\omega}_n-\omega_{n+1})}\right)\right)^{\frac{1}{3}}R_0,\ f'(a_n^*)=0$. It is checked that if $\omega_n,\ \omega_{n+1}$ have different signs, then $0<\left(\dfrac{2}{5}\left(1-\dfrac{3\omega_{n+1}}{2(\bar{\omega}_n-\omega_{n+1})}\right)\right)^{\frac{1}{3}}R_0<R_0$ and $f''(a_n^*)>0$. So at point a_n^* the function $f(a_n)$ has a minimum.

Demanding $f(a_n^*)\leqslant 0$, we obtain the condition under which equation (25) on the interval $0<a_n<R_0$ has a root (in the case of a strict inequality, there are two roots)

$$\frac{10A}{R_0^2}\leqslant\frac{2(\bar{\omega_n}-\omega_{n+1})}{3}\left(\frac{3}{5}\left(\frac{2}{5}\right)^{\frac{2}{3}}(1-\gamma_n)^{\frac{5}{3}}+\gamma_n\right),\ \gamma_n=\frac{3\omega_{n+1}}{2(\bar{\omega}_n-\omega_{n+1})}. \tag{26}$$

From Property 3. it follows $-\dfrac{3}{2}<\gamma_n<0$.

On the interval $(-\frac{3}{2},0)$ the function $F(\gamma_n)=\dfrac{3}{5}\left(\dfrac{2}{5}\right)^{\frac{2}{3}}(1-\gamma_n)^{\frac{5}{3}}+\gamma_n$ is positive, monotonically increasing and $0<F(\gamma_n)<\dfrac{3}{5}\left(\dfrac{2}{5}\right)^{\frac{2}{5}},\ F\left(-\dfrac{3}{2}\right)=0,\ F(0)=\dfrac{3}{5}\left(\dfrac{2}{5}\right)^{\frac{2}{5}}$. Taking into account that $\bar{\omega}_n-\omega_{n+1}>0$, we found that the right part of the inequality in (26) is greater than zero.

Note that the right-hand side of condition (26) does not depend on A and R_0, therefore condition (26) is satisfied. In the case of strict inequality in condition (26), there are two solutions. If condition (26) is not met, there is no solution.

Let us write down condition (26) when in the zone B_{n+1}, adjacent to the boundary of the ball $R = R_0$, the flow is potential ($\omega_{n+1} = 0$)

$$\frac{A}{\bar{\omega}_n R_0^2} \leqslant \frac{1}{25}\Big(\frac{2}{5}\Big)^{\frac{2}{3}}.$$

Let us write down the solutions to the problem ($\omega_n > 0,\ \omega_{n+1} \leqslant 0$, signs of $\omega_i,\ i \leqslant n$ alternate)

$$\Psi(z,r) = \begin{cases} \dfrac{r^2}{10}\omega_1(R^2 - a_1^2), \quad 0 \leqslant R \leqslant a_1, \\ \dfrac{r^2}{10}\omega_i(R^2 - a_i^2) + \dfrac{2(\omega_i - \bar{\omega}_{i-1})}{3} t_{i-1}^5 a_i^5 \Big(1 - \Big(\frac{a_1}{R}\Big)^3\Big), \quad a_{i-1} \leqslant R \leqslant a_i, i = 2, \dots, n, \\ \dfrac{r^2}{10}\omega_{n+1}(R^2 - a_n^2) + \dfrac{(10A - \omega_{n+1}(R_0^2 - a_n^2))R_0^3}{(R_0^3 - a_n^3)}\Big(1 - \dfrac{a_n^3}{R^3}\Big), \quad a_n \leqslant R \leqslant R_0, \end{cases}$$

$a_{i-1} = t_{i-1}^5 a_n$, t_{i-1} — root of equation (14), corresponding to the $i-1$ zone, a_n — root of equation (25).

Let us note an interesting fact: if a flow with a given number of vortex zones in space exists, for example, a composite spherical Hill vortex, and in it the geometry of the layers is determined uniquely, then a similar flow in the sphere does not always exist, and if it does exist, then two different geometries are possible spherical layers.

Let us consider the possibility of the existence of two zones at $\omega_1 \leqslant 0,\ \omega_2 > 0$. We will need this model example later. For this case, equation (25) for finding the value a_1 takes the form

$$f(a_1) = a_1^5 - R_0^3 a_1^2\Big(1 - \frac{3\omega_2}{2(\omega_1 - \omega_2)}\Big) + \frac{3R_0^3(10A - \omega_2 R_0^2)}{2(\omega_1 - \omega_2)} = 0 \tag{27}$$

We have $f(0) = \dfrac{3R_0^3(10A - \omega_2 R_0^2)}{2(\omega_1 - \omega_2)}$, $f(R_0) = \dfrac{15R_0^3 A}{(\omega_1 - \omega_2)} < 0$. At point $a_1^* = \Big(\dfrac{2}{5}\Big(1 - \dfrac{3\omega_2}{2(\omega_1 - \omega_2)}\Big)\Big)^{\frac{1}{3}} R_0$ its only extremum is the minimum, since $f''(a_1^*) = \dfrac{6\omega_1 - 15\omega_2}{\omega_1 - \omega_2} > 0$. And only for $f(0) > 0$ does equation (27) have a root on the interval $(0, R_0)$, and this root is unique. The condition $f(0) > 0$ is satisfied for $\omega_2 > \dfrac{10A}{R_0^2}$. For $\omega_1 = 0$ in each meridian plane in the zone $R \leqslant a_1^*$ the flow function $\Psi(z,r)$ is identically equal to zero.

We found that in a sphere with $\omega_2 > \dfrac{10A}{R_0^2}$ it is possible for two vortex zones with $\omega_1 \leqslant 0$, $\omega_2 > 0$, and zones with the considered geometry are calculated uniquely.

It is obvious that problem (5), (6) for $\omega_n > 0,\ \omega_{n+1} = 0$ is a generalization of the problem of M. A. Gol'dshchik [1, 10] in M. A. Lavrentiev scheme of plane flow of an ideal fluid in the model case of an axisymmetric flow with $n+1$ vortex zones.

For $\omega_1 \leqslant 0,\ \omega_2 > 0$ its formulation is an extension of the problem of plane motion of an ideal fluid in the field of Coriolis forces [1, 11] to the axisymmetric case, as well as on the model principle.

3. Vortex flows in an arbitrary limited axisymmetric region

Let D be an arbitrary bounded region adjacent to the axis $r = 0$ in variables $z, r,\ r \geqslant 0$. Its boundary Γ consists of a smooth curve σ in the upper half-plane $r > 0$ and the segment $[\alpha, \beta]$ of the axis $z = 0$, $\alpha < 0, \beta > 0$. The curve σ adjoins the points α, β at angles different from zero and π respectively. Let us write the boundary condition for the flow function

$$\Psi|_\Gamma = \varphi(s)r^2 \geqslant 0. \tag{28}$$

Since the flow region and boundary function are arbitrary, assumptions about the geometry of vortex zones, as was done for flow in a sphere or in all space, are problematic. In this regard, at the first stage a difficult problem arises in the analytical formulation of the problem. It is natural to begin the study for a flow with two vortex zones.

For the analytical formulation of the problem in this case, the formulation of two dual problems by M. A. Gol'dshtik [1, 11] is well suited. This has already been discussed when constructing flows in a sphere. Thus, in a flat bounded domain D, it is required to find continuously differentiable solutions to problems ($\omega_1 > 0$, $\varphi(s) \geqslant 0$)

$$\Delta\Psi(x,y) = \begin{cases} \omega_1 & \text{if } \Psi < 0, \\ 0 & \text{if } \Psi > 0, \end{cases} \quad \Delta\Psi(x,y) = \begin{cases} \omega_1 & \text{if } \Psi > 0, \\ 0 & \text{if } \Psi \leqslant 0, \end{cases} \quad \Psi|_S = \varphi(s) \geqslant 0. \tag{29}$$

They define flows with two zones, vortex and potential.

In accordance with these problems, to obtain a flow with two vortex zones in the axisymmetric case, we come to two also dual problems, written in analytical form ($\omega_1 > 0, \omega_2 \leqslant 0$)

$$L\Psi(x,y) = \begin{cases} \omega_1 r^2 & \text{if } \Psi < 0, \\ \omega_2 r^2 & \text{if } \Psi > 0, \end{cases} \quad \Psi|_\Gamma = \varphi(s)r^2 \geqslant 0, \tag{30}$$

$$L\Psi(x,y) = \begin{cases} \omega_1 r^2 & \text{if } \Psi > 0, \\ \omega_2 r^2 & \text{if } \Psi < 0, \end{cases} \quad \Psi|_\Gamma = \varphi(s)r^2 \geqslant 0. \tag{31}$$

Let's consider problem (30). A function $\Psi_0(z,r)$ satisfying the equation $L\Psi_0(z,r) = \omega_2 r^2$ and boundary condition (30) in the domain D is positive in the domain D, and therefore is trivial solution to this problem. At $\omega_2 = 0$ the flow is potential in the entire region D, at $\omega_2 < 0$ the entire region D is a vortex zone. In [12], the existence of a nontrivial solution was proven.

Let us observe that the possibility of the existence of a second nontrivial solution with two vortex zones, which exists in a model problem in a sphere, is a difficult, independent mathematical problem. For the plane case with $\omega_2 = 0$, the existence of a nontrivial solution (flow with a vortex and potential zone) was proven in [10, 13, 14], and the existence of a second nontrivial solution in [5, 15]. For $\omega_2 \neq 0$ the existence of a nontrivial solution was proven in [16].

Let's consider problem (31). Note that its solution cannot take negative values in the region D. We assume that at some point $M^* \subset D$, $\Psi(M^*) < 0$. From the boundary condition in (31) it follows that there is a subdomain $D^* \subset D$ on the boundary of which $\Psi^* = 0$, and inside it $L\Psi(z,r) = \omega_2 r^2 \leqslant 0$. Hence $\Psi \geqslant 0$ in D^*. We obtain a contradiction.

From problem (31) we move on to the problem: we need to find a continuously differentiable non-negative solution to the problem

$$L\Psi(z,r) = \omega_1 r^2 \text{ if } \Psi(z,r) > 0, \ \Psi|_\Gamma = r^2\varphi(s) \geqslant 0. \tag{32}$$

To construct a solution to the problem $L\Psi(z,r) = \omega r^2 f(z,r)$, $\Psi|_\Gamma = r^2\varphi(s)$ it is convenient to go to the variables $z = \xi$, $r = 2\sqrt{t}$, after which $L\Psi(z,r) = S\Psi = t\Psi_{tt} + \Psi_{\xi\xi} = 4\omega t f(\xi, 2\sqrt{t})$, $L^*U(z,r) = S^*U = tU_{tt} + U_{\xi\xi} + 2U_t = \omega f(\xi, 2\sqrt{t})$. In the variables ξ, t we obtained the equations ($S(tU) = tS^*U$) degenerate on the boundary of the region at $t = 0$, which are well studied in [17–19]. For example, for the equation $S\Psi = 0$ the usual formulation of the Dirichlet problem is correct, for the equation $S^*U = 0$ for the considered domain D the modified formulation is correct — the solution is specified only on the curve σ and the solution is sought in class of functions bounded at $r \to 0$ [17]. Note that in the case under consideration such a solution is continuous up to $r = 0$ and extreme values are reached at σ[19].

For the operator L^* there is a fundamental solution[20]

$$E(z, r, z_1, r_1) = \frac{4}{\pi} \int_0^{\pi} [(z - z_1)^2 + r^2 + r_1^2 - 2rr_1 \cos\beta]^{-\frac{3}{2}} \sin^2 \beta d\beta,$$

which has a logarithmic singularity for $r,\ r_1 > 0$

$$E(z, r, z_1, r_1) = -\frac{2}{\pi}(rr_1)^{-\frac{3}{2}} \ln((z - z_1)^2 + (r - r_1)^2) + \Phi(z, r, z_1, r_1),$$

$\Phi(z, r, z_1, r_1)$ is a regular function.

Using Green's formula [18, 19] with $v = W,\ u = rG(z, z_1, r, r_1)$

$$\iint\limits_D (uSv - vS^*u) d\xi_1 dt_1 = \oint\limits_\Gamma (vu_{\xi_1} - v_{\xi_1} u) dt_1 - (t_1 v u_{t_1} - t_1 v_{t_1} u - vu) d\xi_1,$$

we obtain a representation of the solution to the problem $LW(z, r) = \omega r^2 f(z, r),\ W|_\Gamma = 0$ in the form

$$W(z, r) = -\frac{\omega}{8} r^2 \iint\limits_D r_1^3 f(z_1, r_1) G(z, z_1, r, r_1) dz_1 dr_1. \tag{33}$$

Here $G(z, z_1, r, r_1)$ is the Green's function for the problem $L^*U = \omega f(z, r),\ U|_\sigma = \varphi(s)$ (the solution is bounded for $r \to 0$), which is standardly constructed using the fundamental solution $E(z, r, z_1, r_1)$. $G(z, z_1, r, r_1) = E(z, r, z_1, r_1) - G_1(z, z_1, r, r_1)$, where $G_1(z, z_1, r, r_1)$ in variables $\xi_1 \neq \xi,\ t_1 \neq t$ solution of problem $S^*G_1 = 0,\ G|_\sigma = -E|_\sigma$ bounded at $t_1 \to 0$. From the above extremum principle for the equation $S^*U = 0$ it follows that for $z \neq z_1, r \neq r_1$ the Green's function $G(z, z_1, r, r_1) > 0$ in $D \bigcup (\alpha, \beta)$,

It is important to note that function (33) has all the properties of a logarithmic potential in the D domain, since the Green's function by construction has a logarithmic singularity inside the D domain.

Let's return to problem (32).

To prove the existence of a solution to this problem that goes to zero at points in the region D, consider the sequence of problems

$$L\Psi_n(z, r) = \omega_1 r^2 th(n\Psi_n(z, r)), \quad \Psi_n|_\Gamma = r^2\varphi(s) \geqslant 0. \tag{34}$$

Just as before, it is easy to show that $\Psi_n \geqslant 0$.

Problem (34) is equivalent to the integral equation

$$\Psi_n(z, r) = -\frac{\omega_1}{8} r^2 \iint\limits_D r_1^3 th(n\Psi_n(z_1, r_1)) G(z, z, r, r_1) dz_1 dr_1 + \Psi_0(z, r). \tag{35}$$

Similarly [1, 11, 12], taking into account the properties of the integral (33) with the introduced Green's function as a logarithmic potential, using Schauder's theorem, we establish the existence for each $n > 0$ of a solution $\Psi_n \geqslant 0$ continuous in $\overline{D}$ of the integral equation (35), and by

Arzel's theorem, the compactness of the sequence of solutions $\Psi_n(z,r)$ in the space of functions continuously differentiable in the domain D. Note that the solution to problem (34) is unique, which follows from $\dfrac{\partial th(n\Psi_n)}{\partial \Psi_n} = \dfrac{n}{ch^2(n\Psi_n)} > 0$. Let the subsequence $\Psi_{n_k}(z,r)$ converge to a continuously differentiable function $\Psi(z,r) \geqslant 0$.

Further, repeating the proof from [1, 11, 12], it is established that the limit function is a solution to problem (34)

Here it is taken into account that for the right side of the equation in (34)

$$\lim_{n_k\to\infty} th(n_k\Psi_{n_k}(z,r)) = 1 \quad \text{if} \quad \Psi(z,r) > 0.$$

Let us obtain the condition under which the resulting solution goes to zero in the region D. Under the assumption that $\Psi(z,r) > 0$ at all points of the region D, it follows from equation (35)

$$\Psi(z,r) = \Psi_0(z,r) - \frac{\omega_1}{8} r^2 \iint\limits_D r_1^3 G(z,r,z_1,r_1)dz_1 r_1. \tag{36}$$

Let D_0 be a semicircle ($r \geqslant 0$) of the largest radius R_0 that can be inscribed in the region D (we can assume that its center is at the origin of coordinates $z = 0, r = 0$) and $C = \max(\varphi(s)r^2)$.

For the model case $D = D_0,\ r^2\varphi(s) = Cr^2$ in the second paragraph, if we go to the notation of problem (31), redesignating ω_2 by ω_1, ω_1 on ω_2, it is found that if the inequality $\omega_1 > \dfrac{10C}{R_0^2}$ is satisfied, the problem (31) under consideration has a solution $\Psi_{D_0}(z,r)$, which in the semicircle $D_a \subset D_0,\ r^2 + z^2 \leqslant a^2,\ r \geqslant 0,\ a < R_0$ is identically equal to zero, and in $D_0 \setminus D_a$ is greater than zero. The value a is the root of equation (27) at $A = C,\ \omega_1 = 0$.

The function $\Psi_{D_0}(z,r)$ for this case can be written as

$$\Psi_{D_0}(z,r) = C - \frac{\omega_1}{8} r^2 \iint\limits_{D_0\setminus D_a} r_1^3 G_{R_0}(z,r,z_1,r_1)dz_1 r_1.$$

$G_{R_0}(z,r,z_1,r_1)$, the Green's function introduced in the work for the region D_0.

Let us represent the function $\Psi(z,r)$ (36) in D_a in the form

$$\begin{aligned}\Psi(z,r) = (\Psi_0(z,r) - C) + \Bigg(C - \frac{\omega_1}{8} r^2 \iint\limits_{D_0\setminus D_a} r_1^3 G_{R_0}(z,r,z_1,r_1)dz_1 r_1\Bigg) + \\ + \frac{\omega_1}{8} r^2 \iint\limits_{D_0\setminus D_a} r_1^3 \big(G_{R_0}(z,r,z_1,r_1) - G(z,r,z_1,r_1)\big)dz_1 dr_1 - \\ - \frac{\omega_1}{8} r^2 \iint\limits_{D\setminus D_0} r_1^3 G(z,r,z_1,r_1)dz_1 dr_1 - \frac{\omega_1}{8} r^2 \iint\limits_{D_a} r_1^3 G(z,r,z_1,r_1)dz_1 dr_1.\end{aligned} \tag{37}$$

In the circle D_a, the expression in the second bracket of equality (37) is equal to zero, the remaining terms on the right side are negative. So, the function $\Psi(z,r)$ is negative in $D_a \subset D$, which contradicts the assumption that it is positive in the entire domain D. We found that for $\omega_1 > \dfrac{10C}{R_0^2}$ the function $\Psi(z,r)$ goes to zero in the domain $D_a \in D$.

Further, similarly to works [1, 11, 12], it is possible to prove that the problem (31) under consideration has a unique solution and under the condition that the boundary set σ on which the boundary function $\varphi(s)$ is nonzero is connected , the set on which the solution is positive is a region.

Thus, it was established that with $\omega_1 > \dfrac{10C}{R_0^2}$ and with the above requirement on the boundary function $\varphi(s)$, in the region D a flow is possible, which in some region vortex with vorticity $\omega_1 r$, and in addition to it the flow function $\Psi(z, r)$ equals zero - the fluid is motionless.

Note that in the work the problem of the possibility of the existence of vortex axisymmetric flows in a limited area was considered with only two vortex zones. Therefore, it is natural to continue the study of the existence of flows with n $(n > 2)$ vortex zones with questions of their non-uniqueness, which occurs with model flows in a sphere.

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Осесимметрические потоки идеальной жидкости с эффективно невязкими вихревыми зонами

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Аннотация. В работе сформулирована модель осесимметрического течения идеальной жидкости с n эффективно невязкими вихревыми зонами, обобщающая известную модель М. А. Лаврентьева о склейке вихревых и потенциальных течений в плоском случае. Показана возможность в рамках такой модели существования в пространстве жидкой сферы, обтекаемой потенциальным осесимметрическим потоком, состоящей из n шаровых слоев осесимметрических вихревых течений. Этот модельный пример обобщает известный в гидродинамике сферический вихрь Хилла с одной вихревой зоной. Такое вихревое течение с n шаровыми слоями также возможно и в сфере, причем в отличие от течения в пространстве, такое течение неединственно. Рассмотрена задача об осесимметрическом вихревом течении в ограниченной области, по постановке обобщающая плоское течение идеальной жидкости в поле кориолисовых сил.

Ключевые слова: идеальная жидкость, вихревые течения, сферический вихрь Хилла.