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Recovering Surface Fluxes on the Boundary of the Domain from Pointwise Measurements

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Abstract. Inverse problems of recovering surface fluxes on the boundary of a domain from pointwise observations are considered. The prolbem is not well-posed in the Hadamard sense. Sharp conditions on the data ensuring existence and uniqueness of solutions in Sobolev classes are exposed and the numerical method relying on the finite element method in the space variables and a finite difference method in time is constructed. The results of numerical experiments are quite satisfactory and the procedure is stable under small perturbations.

Keywords: inverse problem, surface flux, convection-diffusion, pointwise measurement, Tikhonov's regularization.

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Introduction

Under consideration is the parabolic equation

$$Mu = u_t - Lu = u_t - div(c(x,t)\nabla u) + \vec{a}(x,t)\nabla u + a_0(x,t)u = f, \tag{1}$$

where $c = diag(c_1(t,x), \dots, c_n(t,x))$ is a diagonal matrix with strictly positive continuous entries, $(t,x) \in Q = (0,T) \times G$, $\vec{a}(x,t) = (a_1(x,t), \dots, a_n(x,t))^T$, $\nabla u = \Big(\dfrac{\partial u}{\partial x_1}, \dots, \dfrac{\partial u}{\partial x_n}\Big)^T$, $n = 2, 3$, and G is a domain in $\mathbb{R}^n$ with boundary Γ. The equation (1) is furnished with the initial-boundary conditions

$$Bu|_S = g(t,x) \ \ (S = (0,T) \times \Gamma), \quad u|_{t=0} = u_0(x), \tag{2}$$

where $Bu = \sum\limits_{i=1}^{n} \nu_i c_i u_{x_i} + \sigma(t,x)u$, with ν being the outward unit normal to Γ, and with the overdetermination conditions

$$u(t,b_i) = \psi_i(t) \ (i = 1, 2, \dots, r), \tag{3}$$

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where $\{b_i\}_{i=1}^r$ is a collection of points lying in G. It is possible that $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$, Γ_0, Γ_1 are open subsets of Γ, and the condition (2) is given in the form

$$Bu|_{S_0} = g(t,x),\ u|_{S_1} = g_1(t,x)\ \ (S_i = (0,T) \times \Gamma_i,\ i = 0,1),\ \ u|_{t=0} = u_0(x). \tag{4}$$

Assume that $g(t,x) = \sum\limits_{i=1}^{r} \alpha_i(t)\Phi_i(x)$ for some known functions Φ_j, the problem consists in recovering both a solution to (1) satisfying (2), (3) (or (4), (3)) and functions α_j, $j = 1, 2, \ldots, r$. Note that any function g can be approximated by the sums of this form for a suitable choice of basis functions Φ_i.

Inverse problems of recovering the boundary regimes are classical. They arise in many different problems of mathematical physics, in particular, in the heat and mass transfer theory, diffusion, filtration (see [1–3]), and ecology [4–9].

A particular attention is payed to numerical solution of the problems (1)–(3) and close to them. Most of the methods are based on reducing the problems to optimal control ones and minimization of the corresponding quadratic functionals (see, for instance, [10–16]). However, it is possible that these functionals can have several local minima (see Section 3.3 in [17]) and the problem is not always well-posed. Describe some articles, where pointwise measurements are employed as additional data. Numerical determination of constant fluxes in the case of $n = 2$ is described in [11]. Similar results are presented in [18] for $n = 1$. The three-dimensional problem of recovering constant fluxes of green house gases is discussed in [4], but numerical results are presented only in the one-dimensional case. In [5] (see also [6]) the method of recovering a constant surface flux relying on the approach developed in [19] is described, where special solutions to the adjoint problem are employed (see also [7,8]). The surface fluxes depending on t are recovered in [3,14,20,21] in the case of $n = 1$, and in [13,23–25] in the case of $n > 1$. The flux depending on time and spatial variables is reconstructed in [16,26]. The case of flux depending on space variables is discussed in [25]. In this article the flux is sought a finite segment of a series with the use of piecewise linear basis of the finite element method. In literature, there are results in the case in which additional Dirichlet data are given on a part of the boundary and the flux is reconstructed with the use of these data on another part of the boundary (see [27]). The article [15] is devoted to the recovering of the flux $h(t,x)f(x)$ (the function $f(x)$ is unknown) with the use of final or integral overdetermination data. There is a limited number of theoretical results devoted to the problem (1)–(3). If the points $\{b_i\}_{i=1}^r$ are interior points of G then the problem is ill-posed and this fact was observed in many articles (see [28]).

In this article we describe some new theoretical results (see [29]) as applied to this problem, expose a new algorithm of calculating the flux based on our theoretical arguments and describe the results of numerical experiments. The method relies on the finite element method in the space variables and the finite difference method in time. The number of summands in the reprentatation of the function g depends on the number of measurements. The results of numerical experiments are quite satisfactory and the procedure is stable under small perturbations.

1. Preliminaries

The notations of the Sobolev spaces $W_p^s(G)$ and $W_p^s(Q)$ are conventional (see [30, 31]). Given an interval $J = (0,T)$, put $W_p^{s,r}(Q) = W_p^s(J; L_p(G)) \cap L_p\big(J; W_p^r(G)\big)$ and $W_p^{s,r}(S) = W_p^s(J; L_p(\Gamma)) \cap L_p\big(J; W_p^r(\Gamma)\big)$ [30]. By the norm of a vector, we mean the sum of the norms of

its coordinates. Denote by $B_\delta(b)$ the ball of radius δ centered at b. The symbol $\rho(X,Y)$ stands for the distance between the sets X, Y.

The definition of the inclusion $\Gamma \in C^s$, $s \geqslant 1$, can be found in [31, Chapter 1]. The coefficients of the equation (1) are assumed to be real. We consider an elliptic operator L, i.e., there exists a constant $\eta_0 > 0$ such that $c_i(t,x) \geqslant \eta_0$ for all $(t,x) \in Q$ and $i = 1, \dots, n$.

2. Recovering of the heat flux

Under consideration is the conventional heat and mass transfer model (1). We take $G = \Omega \times (0, Z)$, with $\Omega = (0, X)$ for $n = 2$ and Ω is a bounded domain with smooth boundary ($\partial\Omega \in C^2$) for $n = 3$. Let $\Gamma_0 = \{x \in \Gamma : \ x_n = 0\} = \{(0, x') : \ x' \in \Omega\}$ ($x' = (x_1, \dots, x_{n-1})$) and let $S_0 = (0,T) \times \Gamma_0$. The problem is to find a solution to the equation (1) and the function $g = \sum\limits_{i=1}^{r} \alpha_i(t)\Phi_i(x)$ such that

$$u(b_i, t) = \psi_i(t), \ i = 1, 2, \dots, r, \ b_i \in G, \tag{5}$$

$$u|_{t=0} = u_0(x), \quad c_n u_{x_n}|_{S_0} = g(t,x), \quad u|_{S \setminus S_0} = 0. \tag{6}$$

One or more boundary conditions on $S \setminus S_0$ can be changed. This inverse problem arises in the problem of evaluation of the greenhouse gases emission from wetlands (see [4]).

We now expose some consequences of the results in [29]. Despite the fact that they refer to the model case when c is the identity matrix and the remaining coefficients are independent of t, they are rather sharp and we think that they are valid in more general situation as well. Moreover, the conditions on the data below are actually used in the numerical algorithm. We consider the model problem

$$u_t + Lu = f(t,x), \quad Lu = -\Delta u + \sum_{i=1}^{n} a_i(x) u_{x_i} + a_0(x)u, \tag{7}$$

$$u|_{t=0} = u_0(x), \quad u_{x_n}|_{S_0} = g(t,x), \quad u|_{S \setminus S_0} = 0, \tag{8}$$

$$u(t, b_i) = \psi_i(t) \ (i = 1, 2, \dots, r). \tag{9}$$

As before, the problem consists in recovering both a solution to (7) satisfying (8) and (9) and functions α_i, $i = 1, 2, \dots, r$, characterizing the function $g = \sum\limits_{i=1}^{r} \alpha_i(t)\Phi_i(x)$. We assume that

$$b_i \in K = \{x \ \in G : \ x_n < \rho(x, \Gamma \setminus \Gamma_0)\}. \tag{10}$$

Let $b_i' = (b_{i\,1}, \dots, b_{i\,n-1}, 0)$, where b_{ij} is the j-th coordinate of the point b_i. It is naturally to assume that $b_i' \neq b_j'$ for $i \neq j$. Let G_δ be the δ-neighborhood about the points b_i' $(i = 1, 2, \dots, r)$. Denote $\Gamma_\delta = G_\delta \cap \Gamma_0$. Our conditions for the data have the form

$$a_i \in W_\infty^2(G) \ (i = 1, \dots, n), \ a_0 \in L_\infty(G), \tag{11}$$

$$u_0(x) \in W_2^1(G), \ f \in L_2(Q), \tag{12}$$

$$\Phi_i(x') \in W_2^{1/2}(\Gamma_0), \ supp\, \Phi_i \subset \Omega, \tag{13}$$

there exists $\delta_0 > 0, \ \delta_0 < \min_i \rho(b_i, \Gamma \setminus \Gamma_0)$ such that

$$\Phi_i(x) \in W_2^1(\Gamma_{\delta_0}) \text{ for } \ n = 2, \quad \Phi_i(x) \in W_2^2(\Gamma_{\delta_0}) \text{ for } \ n = 3, \ i = 1, \dots, r, \tag{14}$$

$$a_0 \in W_\infty^1(G_{\delta_0} \cap G). \tag{15}$$

Under the conditions (11), (12), there exists a unique solution w_0 to the problem (7), (8), where $g = 0$, such that $w_0 \in W_2^{1,2}(Q)$ (see [33]). Changing the variables $w = u - w_0$, we obtain the simpler problem

$$w_t + Lw = 0, \quad w_{x_n}|_{S_0} = g(t,x), \quad w|_{S\setminus S_0} = 0, \quad w|_{t=0} = 0, \tag{16}$$

$$w(b_i,t) = \psi_i(t) - w_0(t,b_i) = \tilde{\psi}_i(t), \ i = 1,2,\dots,r. \tag{17}$$

We assume that the functions $\tilde{\psi}_i(t)$ admit the representations

$$\tilde{\psi}_i(t) = \int_0^t V_{\delta_i}(t-\tau)\psi_{0i}(\tau)d\tau, \quad \psi_{0i} \in \tilde{W}_2^{n/4}(0,T) \ (n = 2,3), \tag{18}$$

where $V_\gamma(t) = \dfrac{e^{-\gamma^2/4t}}{4\pi t}$ for $n = 2$ and $V_\gamma = \dfrac{\gamma e^{-\gamma^2/4t}}{2\sqrt{\pi}t^{3/2}}$ for $n = 3$. Denote by Ψ the matrix with the entries $\Psi_{ij} = \Phi_j(b_i')$ $(i,j = 1,2,\dots,r)$ and assume that

$$det\,\Psi \neq 0. \tag{19}$$

Theorem 1. *Assume that the conditions* (10)–(14), (18), (19), *and* (15) *for* $n = 3$ *hold. Then there exists a unique solution to the problem* (7)–(9) *such that* $u \in W_2^{1,2}(Q)$, $\alpha_i(t) \in W_2^{1/4}(0,T)$ $(i = 1,2,\dots,r)$.

Proof. The claim results from Theorem 5 in [29]. First of all, we note that in [29] $\Gamma \in C^2$. Nevertheless, the arguments of the proof remain valid since $W_2^{1,2}(Q)$-solvability of the boundary value problem (7), (8) holds. The well-posedness condition from [29] is reduced to the condition (19). The condition (10) ensures that the sets $\{b \in \Gamma : \ \rho(b_j,\Gamma) = |b_j - b|\}$ consist of one point $b_j' \in \Gamma_0$ and the conditions (10), (13), (14), (18), (19) guarantee the fulfillment of other conditions of Theorem 5 in [29].

Note that the condition (18) is sharp and cannot be weakened. □

3. Numerical algorithm

Describe the numerical algorithm. Consider the case of $n = 2$. We employ FEM (the finite element method). We need to find the functions $\{\alpha_i(t)\}$. As for the functions Φ_i, we can use the piecewise linear basis of FEM, in this case we obtain a piecewise linear approximation of g. Sometimes, it is better to use smoother function. We use some analog of the FEM basis. Define a collection of numbers $x_1^1 < x_1^2 < \dots, x_1^r$. Let $x_1^0 = \varepsilon > 0$, $x_1^{r+1} = X - \varepsilon$, with ε a sufficiently small parameter. Let $\delta_i = (x_1^{i+1} - x_1^{i-1})/2$, $i = 1,2,\dots,r$. Assign

$$\Phi_i(x_1) = \begin{cases} \dfrac{1}{2}\left(1 + \cos\left(\dfrac{\pi}{\delta_i}\left(x_1 - \dfrac{x_1^{i-1} + x_1^{i+1}}{2}\right)\right)\right), \ x_1 \in \left[x_1^{i-1}, x_1^{i+1}\right] \\ 0, \ x_1 \notin \left[x_1^{i-1}, x_1^{i+1}\right] \end{cases} \in W_\infty^2(0,X) \tag{20}$$

for $i = 1,2,\dots,r$. Make an additional change of variables

$$v = u - \Phi, \quad \Phi = \sum_{i=1}^r \psi_i(t)\frac{x_1^2(x_1 - X)^2\prod_{j\neq i}(x_1 - x_1^j)}{(x_1^i)^2(x_1^i - X)^2\prod_{j\neq i}(x_1^i - x_1^j)}\frac{(Z - x_2)}{(Z - x_2^i)}. \tag{21}$$

The function v is a solution to the problem

$$Mv = f - M\Phi = f_0, \quad v(b_i, t) = 0, \ i = 1, 2, \ldots, r, \ v|_{t=0} = u_0(x) - \Phi(0, x), \tag{22}$$

$$c_2 v_{x_2}|_{S_0} = g(t, x_1) - g_0(t, x_1) = \tilde{g}, \quad v|_{S \setminus S_0} = 0, \ g_0 = c_2(t, x_1, 0)\Phi_{x_2}(t, x_1, 0). \tag{23}$$

Describe the method. Construct a triangulation of the domain G and the corresponding basis $\{\varphi_i\}_{i=1}^N$ of FEM. Denote the nodes by $\{y_i\}$. We look for an approximate solution in the form $v = \sum\limits_{i=1}^{N} C_i(t)\varphi_i$. For convenience, we assume that the points $b_i = (b_1^i, b_2^i)$ $(i = 1, 2, \ldots, r)$ agree with the nodes $y_{N-r+1}, \ldots, y_N$. The functions $C_i(t)$, $i = 1, 2, \ldots, N$, are a solution to the system

$$M\vec{C}_t + K\vec{C} = -\vec{F} + \vec{f}_0, \quad \vec{C} = (C_1, C_2, \ldots, C_N)^T, \tag{24}$$

where

$$\vec{F} = \left(\int_0^X g(t, x_1)\varphi_1(x_1, 0)\, dx_1, \ldots, \int_0^X g(t, x_1)\varphi_N(x_1, 0)\, dx_1 \right)^T,$$

and the coordinates of the vector $\vec{f}_0$ are of the form

$$f_i = (f_0(t, x), \varphi_i) + \int_0^X g_0(t, x_1)\varphi_i(x_1, 0)\, dx_1, \quad (f_0(t, x), \varphi_i) = \int_G f_0(t, x)\varphi_i\, dx.$$

The matrices M and K have the entries $M_{ij} = (\varphi_i, \varphi_j) = \int\limits_G \varphi_i(x)\varphi_j(x)\, dx$ and

$$K_{jk} = (c_1(t, x)\varphi_{kx_1}, \varphi_{jx_1}) + (c_2(t, x)\varphi_{kx_2}, \varphi_{jx_2}) + (a(t, x)\nabla\varphi_k, \varphi_j) + (a_0(t, x)\varphi_k, \varphi_j),$$

respectively. We have that $\vec{C}(0) = v_0$. A solution to the system (24) is defined by the finite difference method. Define the step in time $\tau = T/m$ and replace (24) with the system

$$M\frac{\vec{C}_{i+1} - \vec{C}_i}{\tau} + K_{i+1}\vec{C}_{i+1} = -\vec{F}_{i+1} + \vec{f}_{i+1}, \quad \vec{C}_i = (C_i^1, \ldots, C_i^N)^T, \ i = 0, 1, 2, \ldots, m-1, \tag{25}$$

where $C_i^k \approx C_k(\tau i)$, $\vec{F}_i \approx \vec{F}(\tau i)$, $\vec{f}_i = \vec{f}_0(\tau i)$, $K_i = K(\tau i)$. The system (25) can be written as follows:

$$R_{i+1}\vec{C}_{i+1} = -\tau\vec{F}_{i+1} + \tau\vec{f}_{i+1} + M\vec{C}_i, \ C_i^k = C_k(\tau i), \ \vec{C}_i = (C_i^1, \ldots, C_i^N)^T, \ i = 0, 1, 2, \ldots, m-1, \tag{26}$$

where $R_{i+1} = M + \tau K_{i+1}$. Assign $\vec{\alpha}_i = (\alpha_i^1, \ldots, \alpha_i^r)^T$, $\vec{\alpha}_i \approx \vec{\tilde{\alpha}}(\tau i)$, $\alpha_i^k \approx \tilde{\alpha}_k(i\tau)$.

In view of (22), we must have $C_k^{N-r+i} = 0$ $(i = 1, 2, , \ldots, r)$. Assign $C_0^k = v_0(b_k)$ $(k = 1, \ldots, N)$. The numbers α_0^k are solutions to the system

$$\sum_{i=1}^{r} \alpha_0^i \Phi_i(x_1^k) = c_2(0, x_1, 0)u_{0x_2}(b_k'). \tag{27}$$

In dependence of smoothness of a solution we can require the consistency conditions

$$\sum_{i=1}^{r} \alpha_0^i \Phi_i(x_1) = c_2(0, x_1, 0)u_{0x_2}(x_1, 0), \ \forall x_1 \in (0, X),$$

with α_0^i a solution to the system (27). But they are not necessary, for example, for solutions $u \in W_2^{1,2}(Q)$. We also assume that

$$\det\{\Phi_i(b_1^k)\}_{k,i=1}^r \neq 0, \ \psi_i(t) \neq 0 \ \forall t, i. \tag{28}$$

Assume that we have found the vectors $\vec{\alpha}_i$, $\vec{C}_i$. We seek the quantity $\vec{C}_{i+1}$ as a solution to the system

$$R_{i+1}\vec{C}_{i+1} = -\tau B\vec{\alpha}_{i+1} + \tau\vec{f}_{i+1} + M\vec{C}_i, \tag{29}$$

where $N \times r$-matrix B has the entries $b_{kj} = \int\limits_0^X \Phi_j(x_1)\varphi_k(x_1, 0)\, dx_1$ $(j = 1, 2, \ldots, r,\ k = 1, \ldots, N)$. The vector $\vec{\alpha}_{i+1}$ is determined from the system

$$\tau B_{i+1}\vec{\alpha}_{i+1} = \tau\Phi_0 R_{i+1}^{-1}\vec{f}_{i+1} + \Phi_0 R_{i+1}^{-1}M\vec{C}_i \tag{30}$$

where the matrix $B_{i+1} = \Phi_0 R_{i+1}^{-1}B^{i+1}$ of dimension $r \times r$, where Φ_0 is a $r \times N$-matrix whose first $N - r$ columns are occupied by zeros and and the last r columns is the identity matrix of dimension $r \times r$. The matrix B_i can be singular (with small elements). To improve the convergence, we employ the Tikhonov regularization. So we replace the system (29) with the system

$$\tau(B_{i+1}^* B_{i+1} + \varepsilon)\vec{\alpha}_{i+1} = \tau B_{i+1}^*\Phi_0 R_{i+1}^{-1}\vec{f}_{i+1} + B_{i+1}^*\Phi_0 R_{i+1}^{-1}M\vec{C}_i,\ \varepsilon > 0, \tag{31}$$

where B_{i+1}^* is the adjoint matrix.

4. Program implementation and results of numerical experiments

In this section, we analyze the results of numerical experiments for several groups of input data. We will consider the dependence of accuracy of determining the coefficients α_i and the function u on the number N of points of the triangulation grid, the number of the overdetermination points b_i and the distance l between them. The coefficients in (1) are defined as follows: $a_0 = 1/(t + 1)$, $a_1 = x$, $a_2 = y$, $c_1 = x + 2$, $c_2 = y + 2$. Characteristics of the computer: Processor: Intel(R) Xeon(R) CPU E5-2678 v3 @ 2.50GHz (2 processors); RAM: 64.0 GB; System type: Windows 10 Pro 64-bit operating system.

First of all, we construct some test data. To define test functions, we construct a solution u to the direct problem (1), (6) with the known boundary condition (6) and the function g depending on the known functions Φ_i and α_i. Next, we take a collection of points b_i and determine the data (5). Solving the inverse problem (1), (5), (6), we find a solution u and the functions $\{\alpha_i\}$. Comparing given function $\{\alpha_i\}$ and obtained after calculations, we can estimate the convergence of the algorithm. To abbreviate the exposition, only graphs of the functions constructed and the results of calculating the parameters α_i will be presented.

Each experiment includes sequential steps:

- Setting the number and coordinates of overdetermination points and the functions α_i;
- Initialization of the domain for constructing a solution to the direct and inverse problems;
- Definition of service arrays of points;
- Solving the direct problem (1), (6);
- Construction of the functions Φ_i and the auxiliary function Φ;
- Solving the inverse problem (22)–(23), restoring the solution u and the function α_i.

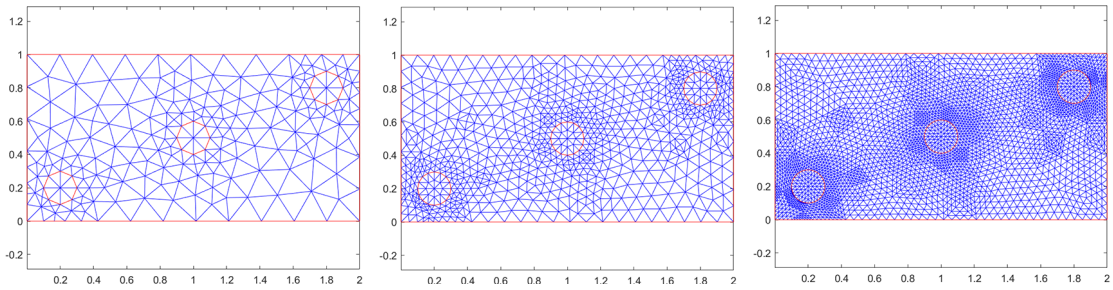
Present the software implementation for the first group of data, for the rest we will present only pivot tables.

For the first group of experiments, we take $r = 3$. The overdetermination points b_i have the coordinates: $(0.2; 0.2)$, $(1; 0.5)$, and $(1.8; 0.8)$. We take $\alpha_1 = t + 2$, $\alpha_2 = (t - 2)^2$, and $\alpha_3 = (t + 1)^3$.

Experimentally, it was found that the change in the number of grid points in time m practically does not affect the accuracy of the calculations, so we take it equal to 100. It was also found that with an increase in the number of time points m, it is necessary to decrease the regularization parameter ε, for example, for $m = 200$, you need to take $\varepsilon \leqslant 10^{-10}$, otherwise the algorithm will diverge. For all groups of experiments, we take the parameter $\varepsilon = 10^{-7}$.

1) As the domain of constructing the solution to the problem (1), (5), (6), we take a rectangle with sides $A = 2$ and $B = 1$ located along the axes x_1 and x_2, respectively. The lower left corner of the rectangle is at the point $(0; 0)$, we will use this domain for all groups of experiments. Let's add to the domain r circles with radii $R = 0.1$ and centered at the points b_i.

Using Delaunay triangulation, we get the first mesh Z_0 with 214 nodes. The new grids are obtained by dividing each triangle of the previous grid into 4 parts, we get $Z_1 = 812$ and $Z_2 = 3163$, the Fig. 1.



a) b) c)

Fig. 1. Zone with nodes a) $Z_0 = 214$; b) $Z_1 = 812$; c) $Z_2 = 3163$

2) Further, after constructing the triangulation mesh, it is necessary to determine the collections of indices of points, including the points b_i.

3) The time step is defined as $\tau = T/m$. To solve the direct problem (1), (6), we define the right-hand side $f = 1$ (see (1)), the initial condition $u_0 = 1$ and boundary function g assuming that α_i are known. The functions Φ, Φ_i, and the respective function g are constructed in accord with the formulas from the previous section (see (21), (20)). Note that with these almost arbitrary initial data, the consistency conditions at $t = 0, x_2 = 0$ are not fulfilled. This gives rise a large oscillation of a solution at $t = 0$. So, it is necessary to cut off a part of the solution that has a large error at the initial time points which arise in the calculations. One more variant which was used is to define the time shift variable as $\tau_s = 20 \cdot T/m$. It is necessary to extend the time line by changing the start point to $-\tau_s \cdot T/m$. With the shift in time, we get $m + \tau_s + 1$ time points. This stage is not obligatory.

4) A solution to the direct problem (1), (6) is defined by the formulas of the previous section, except for the equation (26) which is replaced with

$$\vec{C}_{i+1} = (M_{i+1} + \tau K_{i+1})^{-1} \cdot (-\tau \vec{G}_{i+1} + \tau \vec{F}_{i+1} + M_{i+1}\vec{C}_i),\ C_i^k = C_k(\tau i), \tag{32}$$

where $\vec{C}_j = (C_j^1, \ldots, C_j^{N-lp-tp-rp})^T,\ j = 0, 1, 2, \ldots, m + \tau_s$.

5) We calculate the functions $M\Phi$, f_0 (see (22)), and the first time derivative of the data $\psi_t^i = (\psi_i((j+1)\tau) - \psi_i(j\tau))/\tau$.

6) For further analysis of the results of solving the problem (22)–(23) and restoring the solution u, we introduce the following quantities that describe the calculation errors: the parameter

$\varepsilon_\alpha = \max_i(\max_j |\alpha^j(i\tau) - \alpha_i^j|)$, where the numbers α_i^j are the results of calculations, $j = 1, \ldots, r$; $\varepsilon_u = \max_{i,j} |u_{i,j} - u(y_i, \tau j)|$ is the error in calculating the concentration of a pollutant, where $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, m$. Let T_τ be the total running time of the algorithm, including the time to solve the direct problem, in seconds. The calculation results for three previously defined grids are presented on the Fig. 2.

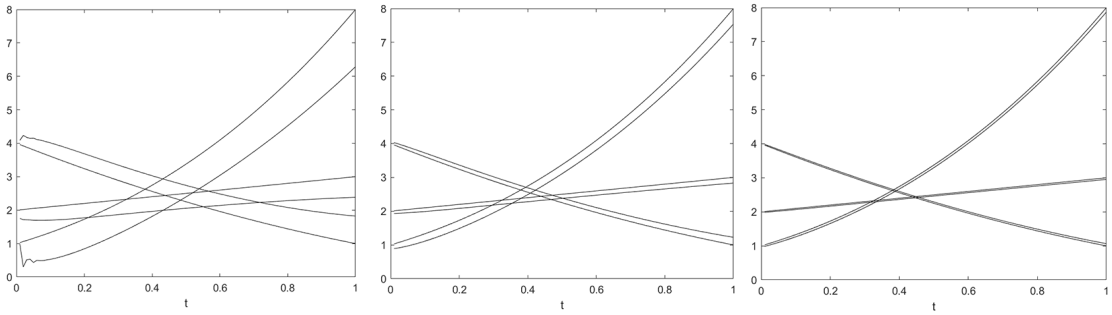


Fig. 2. The results of calculations of functions α_i on the grids a) Z_0; b) Z_1; c) Z_2

It is quite natural that an increase in the number of nodes leads to an increase in the accuracy of calculations. In this case, the calculation error ε_α, ε_u and the calculation time T_τ for three grids, respectively, are equal to $(1.7116, 0.0996, 74)$, $(0.4589, 0.0285, 238)$, $(0.1306, 0.0082, 1052)$. As is easily seen, the error is inversely proportional to the number of nodes.

Even in the case of the grids Z_0 and Z_1, solutions obtained repeat the profile of the desired solution. In this case, taking into account the increasing computation time, in subsequent experiments we will use Z_0.

For the second group of experiments, we take one overdetermination point and the function $\alpha_1 = \log(t + 1)$. The other data are the same.

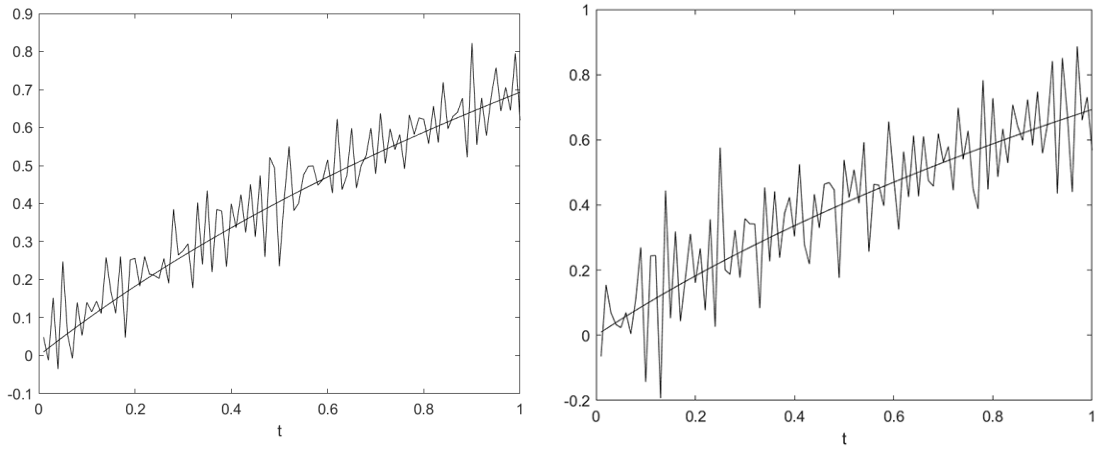
We present a summary table indicating a different number and coordinates of overdetermination points, the functions α_i, received errors, and calculation time, Tab. 1.

Table 1. Summary table

No	b_i	ε_α	ε_u	τ_s
1	(0.5;0.3)	0.0303	0.0037	35.7
2	(0.5;0.5)	0.0384	0.0044	35.8
3	(0.5;0.7)	0.0572	0.0056	35.4
4	(1;0.3)	0.0248	0.0041	38.1
5	(1;0.5)	0.0315	0.0053	39.2
6	(1;0.7)	0.0475	0.0065	37.9
7	(1.5;0.3)	0.0374	0.0059	37.6
8	(1.5;0.5)	0.0443	0.0078	36
9	(1.5;0.7)	0.0679	0.0098	36.7

According to the results obtained, it can be seen that, despite the use of the grid Z_0, the solutions are quite accurate. The error increases as the distance from the lower bound increases, which corresponds to the theoretical results (Theorem 1).

For the next part of the experiments, we will add random noise to each point of the array of the right-hand side vector, the noise value will be denoted by $n_z(i,j)$. Thus we get $f(i,j) = f(i,j) \cdot (1 + n_z(i,j))$, with $f(i,j)$ the right-hand side in the system, the results are presented in Fig. 3. The coordinates of overdetermination point (0.5; 0.3) and all other parameters are the same.



a) b)

Fig. 3. Result of calculations of the function α_1 on a grid with noise a) $n_z = 25\%$; b) $n_z = 50\%$

Despite the introduced noise, the algorithm still shows good convergence, the calculation errors are ε_α, ε_u: (0.19, 0.009) and (0.35, 0.017), respectively.

For the third experimental group, form a table with data for two points with the required functions $\alpha_1 = (t-2)^2$ and $\alpha_2 = \log(t+1)$. The remaining data are the same.

Table 2. Summary table

No	b_i	ε_α	ε_u	τ_s
1	(0.5; 0.3), (1; 0.3)	0.074	0.0067	48.6
2	(0.5; 0.3), (1; 0.5)	0.094	0.0087	50.8
3	(0.5; 0.3), (1; 0.7)	0.149	0.0154	51.7
4	(0.5; 0.3), (1.5; 0.3)	0.058	0.005	52.4
5	(0.5; 0.3), (1.5; 0.5)	0.083	0.0063	53.1
6	(0.5; 0.3), (1.5; 0.7)	0.145	0.011	52.5
7	(0.5; 0.5), (1; 0.3)	0.048	0.0047	50.6
8	(0.5; 0.5), (1; 0.5)	0.077	0.0069	50.7
9	(0.5; 0.5), (1; 0.7)	0.129	0.0097	49.7
10	(0.5; 0.5), (1.5; 0.3)	0.038	0.0046	54.5
11	(0.5; 0.5), (1.5; 0.5)	0.069	0.0052	53.6
12	(0.5; 0.5), (1.5; 0.7)	0.12	0.0083	52.5
13	(0.5; 0.7), (1; 0.3)	0.097	0.0086	49.9
14	(0.5; 0.7), (1; 0.5)	0.072	0.0057	50.1
15	(0.5; 0.7), (1; 0.7)	0.067	0.005	49.5
16	(0.5; 0.7), (1.5; 0.3)	0.054	0.0037	49.9

According to the data obtained, it is possible to confirm the conclusion made earlier that the distance between the points does not affect the accuracy of the calculation. Also, an increase in the number of overdetermination points and the unknown functions α_i increases the calculation error.

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Восстановление потока на границе области по точечным замерам

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Аннотация. Мы рассматриваем обратные задачи восстановления поверхностных потоков на границе области по точечным замерам. Задача некорректна по Адамару. Мы описываем точные условия, гарантирующие существование и единственность решений в пространствах Соболева и строим численный метод, основанный на методе конечных элементов и методе конечных разностей по времени. Представлены результаты численных экспериментов, которые вполне удовлетворительны и процедура устойчива по отношению к малым возмущениям.

Ключевые слова: поверхностный поток, регуляризация Тихонова, обратная задача, точечное переопределение, конвекция-диффузия.