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Multivalued Δ-symmetric Covariant Results in Bipolar Metric Spaces

G. N. V. Kishore*
Department of Engineering Mathematics & Humanities
Sagi Rama Krishnam Raju Engineering College
Andhra Pradesh, India

B. Srinuvasa Rao†
Department of Mathematics
Dr.B.R.Ambedkar University
Andhra Pradesh, India

D. Ram Prasad‡
Department of Mathematics
Nalla Malla Reddy Engineering college
Divya nagar, Ghatkesar mandal
Telangana, India

Stojan Radenovic§
Faculty of Mechanical Engineering
University of Belgrade
Beograd, Serbia

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Abstract. In this paper, we proved some coupled fixed point theorems for Hybrid pair of mappings by using Δ-symmetric covariant mappings in bipolar metric spaces. Also we give some examples which supports our results.

Keywords: Δ-symmetric covariant mapping, Hybrid Pair of mappings, Coupled fixed point, bipolar metric spaces.

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1. Introduction and preliminaries

In 1922, S. Banach [4] introduced the notion of Banach contraction principle. It was extended by Nadler [13] for multivalued mappings and some results related with generalization of various directions (see [1–18]).

Very recently, in 2016 Mutlu and Gürdal [11] introduced the notion of Bipolar metric spaces, which is one of generalizations metric spaces. Also they investigated some fixed point and coupled fixed point results on this space, see [11, 12].

*gnvkishore@srkrec.ac.in, kishore.apr2@gmail.com
†srinivasabagathi@gmail.com
‡ramprasadmphil09@gmail.com
§radens@beotel.net

In this paper, we proved some coupled fixed point theorems for multivalued maps. Also we provide examples, which supports our main results.

First we recall some basic definitions and results.

Definition 1.1 ([11]). *Let Γ and Θ be a two non-empty sets. Suppose that $d : \Gamma \times \Theta \to [0, +\infty)$ be a mapping satisfying the following properties :*

(π_0) *If $d(\sigma, \tau) = 0$ then $\sigma = \tau$ for all $(\sigma, \tau) \in \Gamma \times \Theta$,*

(π_1) *If $\sigma = \tau$ then $d(\sigma, \tau) = 0$ for all $(\sigma, \tau) \in \Gamma \times \Theta$,*

(π_2) *If $d(\sigma, \tau) = d(\tau, \sigma)$ for all $\sigma, \tau \in \Gamma \cap \Theta$.*

(π_3) *If $d(\sigma_1, \tau_2) \leqslant d(\sigma_1, \tau_1) + d(\sigma_2, \tau_1) + d(\sigma_2, \tau_2)$ for all $\sigma_1, \tau_2 \in \Gamma$, $\tau_1, \tau_2 \in \Theta$.*

Then the mapping d is called a Bipolar-metric on the pair (Γ, Θ) and the triple (Γ, Θ, d) is called a Bipolar-metric space.

Definition 1.2 ([11]). *Assume (Γ_1, Θ_1) and (Γ_2, Θ_2) as two pairs of sets and a function as $\Psi : \Gamma_1 \cup \Theta_1 \rightrightarrows \Gamma_2 \cup \Theta_2$ is said to be a covariant map. If $\Psi(\Gamma_1) \subseteq \Gamma_2$ and $\Psi(\Theta_1) \subseteq \Theta_2$, and denote this with $\Psi : (\Gamma_1, \Theta_1) \rightrightarrows (\Gamma_2, \Theta_2)$. And the mapping $\Psi : \Gamma_1 \cup \Theta_1 \rightrightarrows \Gamma_2 \cup \Theta_2$ is said to be a contravariant map. If $\Psi(\Gamma_1) \subseteq \Theta_2$ and $\Psi(\Theta_1) \subseteq \Gamma_2$, and write $\Psi : (\Gamma_1, \Theta_1) \leftrightarrows (\Gamma_2, \Theta_2)$. In particular, if d_1 and d_2 are bipolar metric on (Γ_1, Θ_1) and (Γ_2, Θ_2), respectively, we some time use the notation $\Psi : (\Gamma_1, \Theta_1, d_1) \rightrightarrows (\Gamma_2, \Theta_2, d_2)$ and $\Psi : (\Gamma_1, \Theta_1, d_1) \leftrightarrows (\Gamma_2, \Theta_2, d_2)$.*

Definition 1.3 ([11]). *Assume (Γ, Θ, d) be a bipolar metric space. A point $\xi \in \Gamma \cup \Theta$ is termed as a left point if $\xi \in \Gamma$, a right point if $\xi \in \Theta$ and a central point if both. Similarly, a sequence $\{\sigma_n\}$ on the set Γ and a sequence $\{\tau_n\}$ on the set Θ are called a left and right sequence respectively. In a bipolar metric space, sequence is the simple term for a left or right sequence. A sequence $\{\xi_n\}$ as considered convergent to a point ξ, if and only if $\{\xi_n\}$ is a left sequence, ξ is a right point and $\lim_{n\to+\infty} d(\xi_n, \xi) = 0$; or $\{\xi_n\}$ is a right sequence, ξ is a left point and $\lim_{n\to+\infty} d(\xi, \xi_n) = 0$. A bisequence $(\{\sigma_n\}, \{\tau_n\})$ on (Γ, Θ, d) is sequence on the set $\Gamma \times \Theta$. If the sequence $\{\sigma_n\}$ and $\{\tau_n\}$ are convergent, then the bisequence $(\{\sigma_n\}, \{\tau_n\})$ is said to be convergent. $(\{\sigma_n\}, \{\tau_n\})$ is Cauchy sequence, if $\lim_{n,m\to+\infty} d(\sigma_n, \tau_m) = 0$. In a bipolar metric space, every convergent Cauchy bisequence is biconvergent. A bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.*

2. Methods / Experimental Section

2.1. Coupled Fixed Point Theorems via Δ-symmetric covariant mapping

Definition 2.1. *Let $\Psi : (\Gamma \times \Theta) \cup (\Theta \times \Gamma) \to CL(\Gamma \cup \Theta)$ be a given covariant mapping. We say that Ψ is a Δ-symmetric covariant mapping if and only if $(\sigma, \tau) \in \Delta$ implies $\Psi(\sigma, \tau)\Re\Psi(\tau, \sigma)$*

Definition 2.2. *Let (Γ, Θ, d) be a bipolar metric spaces, $\sigma \in \Gamma, \tau \in \Theta$ and $\Psi : (\Gamma \times \Theta, \Theta \times \Gamma) \rightrightarrows CL(\Gamma, \Theta)$ be a covariant mapping. An element (σ, τ) is said to be a coupled fixed point of $\Psi : (\Gamma \times \Theta) \cup (\Theta \times \Gamma) \to CL(\Gamma \cup \Theta)$ if $\sigma \in \Psi(\sigma, \tau)$ and $\tau \in \Psi(\tau, \sigma)$.*

Theorem 2.3. *Let (Γ, Θ, d) be an complete bipolar metric space endowed with a partial order $\preceq$. Suppose that Δ is non empty, that is there exists $(\sigma, \rho) \in \Delta$. Let $F : (\Gamma \times \Theta, \Theta \times \Gamma) \rightrightarrows CL(\Gamma, \Theta)$ be a Δ-symmetric covariant mapping and consider that $f : \Gamma \times \Theta \to [0, +\infty)$ as*

$$f(\sigma, \rho) = D(\sigma, F(\varrho, \tau)) + D(F(\tau, \varrho), \rho) \text{ for all } \sigma, \tau \in \Gamma \text{ and } \rho, \varrho \in \Theta \tag{2.1}$$

is lower semi-continuous and there exists a mapping $\psi : [0, +\infty) \to (0, 1)$ *satisfying*

$$\lim_{r \to t^+} \sup \psi(r) < 1 \ for\ each\ t \in [0, +\infty). \tag{2.2}$$

Assume that for any $(\sigma, \rho) \in \Delta$ *there exist* $x \in F(\sigma, \rho)$ *and* $y \in F(\rho, \sigma)$ *satisfying*

$$\sqrt{\psi(f(\sigma, \rho))}[d(\sigma, y) + d(x, \rho)] \leqslant f(\sigma, \rho) \tag{2.3}$$

such that

$$f(x, y) \leqslant \psi(f(\sigma, \rho))[d(\sigma, y) + d(x, \rho)]. \tag{2.4}$$

Then $F : (\Gamma \times \Theta) \cup (\Theta \times \Gamma) \to CL(\Gamma \cup \Theta)$ *has a coupled fixed point. That is there exists* $(\alpha, \beta) \in (\Gamma \times \Theta) \cup (\Theta \times \Gamma)$ *such that* $\alpha \in F(\alpha, \beta)$ *and* $\beta \in F(\beta, \alpha)$.

Proof. Since by the definitions of ψ we have $\psi(f(\sigma, \rho)) < 1$ and $\psi(f(\tau, \varrho)) < 1$ for each $\sigma, \tau \in \Gamma$, $\rho, \varrho \in \Theta$, it follows that for any $(\sigma, \rho), (\tau, \varrho) \in \Gamma \times \Theta$, there exist $u \in F(\sigma, \rho)$, $v \in F(\rho, \sigma)$, $l \in F(\tau, \varrho)$ and $m \in F(\varrho, \tau)$ such that

$$\sqrt{\psi(f(\sigma, \rho))}d(\sigma, m) \leqslant D(\sigma, F(\varrho, \tau)) \text{ and } \sqrt{\psi(f(\tau, \varrho))}d(\tau, v) \leqslant D(\tau, F(\rho, \sigma))$$

$$\sqrt{\psi(f(\sigma, \rho))}d(l, \rho) \leqslant D(F(\tau, \varrho), \rho) \text{ and } \sqrt{\psi(f(\tau, \varrho))}d(u, \varrho) \leqslant D(F(\sigma, \rho), \varrho).$$

Therefore, for each $(\sigma, \rho), (\tau, \varrho) \in \Gamma \times \Theta$, there exist $u \in F(\sigma, \rho)$, $v \in F(\rho, \sigma)$, $l \in F(\tau, \varrho)$ and $m \in F(\varrho, \tau)$ satisfying (2.3).
Let $(\sigma_0, \rho_0), (\tau_0, \varrho_0) \in \Delta$ be an arbitrary and fixed. By our assumptions (2.3) and (2.4), choose $\sigma_1 \in F(\sigma_0, \rho_0)$, $\rho_1 \in F(\rho_0, \sigma_0)$ and $\tau_1 \in F(\tau_0, \varrho_0)$, $\varrho_1 \in F(\varrho_0, \tau_0)$ such that

$$\sqrt{\psi(f(\sigma_0, \rho_0))}(d(\sigma_0, \varrho_1) + d(\tau_1, \rho_0)) \leqslant f(\sigma_0, \rho_0) \tag{2.5}$$

$$\sqrt{\psi(f(\tau_0, \varrho_0))}(d(\tau_0, \rho_1) + d(\sigma_1, \varrho_0)) \leqslant f(\tau_0, \varrho_0) \tag{2.6}$$

and

$$f(\sigma_1, \rho_1) \leqslant \psi(f(\sigma_0, \rho_0))(d(\sigma_0, \varrho_1) + d(\tau_1, \rho_0)) \tag{2.7}$$

$$f(\tau_1, \varrho_1) \leqslant \psi(f(\tau_0, \varrho_0))(d(\tau_0, \rho_1) + d(\sigma_1, \varrho_0)). \tag{2.8}$$

From (2.5) and (2.7) we obtain that

$$\begin{aligned} f(\sigma_1, \rho_1) &\leqslant \psi(f(\sigma_0, \rho_0))(d(\sigma_0, \varrho_1) + d(\tau_1, \rho_0)) \leqslant \\ &\leqslant \sqrt{\psi(f(\sigma_0, \rho_0))}\sqrt{\psi(f(\sigma_0, \rho_0))}(d(\sigma_0, \varrho_1) + d(\tau_1, \rho_0)) \leqslant \\ &\leqslant \sqrt{\psi(f(\sigma_0, \rho_0))}f(\sigma_0, \rho_0). \end{aligned} \tag{2.9}$$

From (2.6) and (2.8) we obtain that

$$\begin{aligned} f(\tau_1, \varrho_1) &\leqslant \psi(f(\tau_0, \varrho_0))(d(\tau_0, \rho_1) + d(\sigma_1, \varrho_0)) \leqslant \\ &\leqslant \sqrt{\psi(f(\tau_0, \varrho_0))}\sqrt{\psi(f(\tau_0, \varrho_0))}(d(\tau_0, \rho_1) + d(\sigma_1, \varrho_0)) \leqslant \\ &\leqslant \sqrt{\psi(f(b_0, q_0))}f(b_0, q_0). \end{aligned} \tag{2.10}$$

Since F is a Δ-symmetric covariant mapping and $(\sigma_0, \rho_0), (\tau_0, \varrho_0) \in \Delta$, we have $F(\sigma_0, \rho_0)\Re F(\rho_0, \sigma_0) \Rightarrow (\sigma_1, \rho_1) \in \Delta$ and $F(\tau_0, \varrho_0)\Re F(\varrho_0, \tau_0) \Rightarrow (\tau_1, \varrho_1) \in \Delta$.
By our assumptions (2.3) and (2.4), choose $\sigma_2 \in F(\sigma_1, \rho_1)$, $\rho_2 \in F(\rho_1, \sigma_1)$ and $\tau_2 \in F(\tau_1, \varrho_1)$, $\varrho_2 \in F(\varrho_1, \tau_1)$ such that

$$\sqrt{\psi(f(\sigma_1, \rho_1))}(d(\sigma_1, \varrho_2) + d(\tau_2, \rho_1)) \leqslant f(\sigma_1, \rho_1) \tag{2.11}$$

$$\sqrt{\psi(f(\tau_1,\varrho_1))}(d(\tau_1,\rho_2)+d(\sigma_2,\varrho_1)) \leqslant f(\tau_1,\varrho_1) \tag{2.12}$$

and

$$f(\sigma_2,\rho_2) \leqslant \psi(f(\sigma_1,\rho_1))(d(\sigma_1,\varrho_2)+d(\tau_2,\rho_1)) \tag{2.13}$$

$$f(\tau_2,\varrho_2) \leqslant \psi(f(\tau_1,\varrho_1))(d(\tau_1,\rho_2)+d(\sigma_2,\varrho_1)). \tag{2.14}$$

From (2.11) and (2.13) we obtain that

$$\begin{aligned} f(\sigma_2,\rho_2) &\leqslant \psi(f(\sigma_1,\rho_1))(d(\sigma_1,\varrho_2)+d(\tau_2,\rho_1)) \leqslant \\ &\leqslant \sqrt{\psi(f(\sigma_1,\varrho_1))}\sqrt{\psi(f(\sigma_1,\varrho_1))}(d(\sigma_1,\varrho_2)+d(\tau_2,\rho_1)) \leqslant \\ &\leqslant \sqrt{\psi(f(\sigma_1,\rho_1))}f(\sigma_1,\rho_1). \end{aligned} \tag{2.15}$$

From (2.12) and (2.14) we obtain that

$$\begin{aligned} f(\tau_2,\varrho_2) &\leqslant \psi(f(\tau_1,\varrho_1))(d(\tau_1,\rho_2)+d(\sigma_2,\varrho_1)) \leqslant \\ &\leqslant \sqrt{\psi(f(\tau_1,\varrho_1))}\sqrt{\psi(f(\tau_1,\varrho_1))}(d(\tau_1,\rho_2)+d(\sigma_2,\varrho_1)) \leqslant \\ &\leqslant \sqrt{\psi(f(\tau_1,\varrho_1))}f(\tau_1,\varrho_1) \end{aligned} \tag{2.16}$$

with $(\sigma_2,\rho_2),(\tau_2,\varrho_2) \in \Delta$. Continue in this way, we get bisequence $(\sigma_n,\rho_n),(\tau_n,\varrho_n)$ with $(\sigma_n,\rho_n),(\tau_n,\varrho_n)\in\Delta$, $\sigma_{n+1}\in F(\sigma_n,\rho_n)$, $\rho_{n+1}\in F(\rho_n,\sigma_n)$ and $\tau_{n+1}\in F(\tau_n,\varrho_n)$, $\varrho_{n+1}\in F(\varrho_n,\tau_n)$ such that for all $n \in N$, we have

$$\sqrt{\psi(f(\sigma_n,\rho_n))}(d(\sigma_n,\varrho_{n+1})+d(\tau_{n+1},\rho_n)) \leqslant f(\sigma_n,\rho_n) \tag{2.17}$$

$$\sqrt{\psi(f(\tau_n,\varrho_n))}(d(\tau_n,\rho_{n+1})+d(\sigma_{n+1},\varrho_n)) \leqslant f(\tau_n,\varrho_n) \tag{2.18}$$

and

$$f(\sigma_{n+1},\rho_{n+1}) \leqslant \psi(f(\sigma_n,\rho_n))(d(\sigma_n,\varrho_{n+1})+d(\tau_{n+1},\rho_n)) \tag{2.19}$$

$$f(\tau_{n+1},\varrho_{n+1}) \leqslant \psi(f(\tau_n,\varrho_n))(d(\tau_n,\rho_{n+1})+d(\sigma_{n+1},\varrho_n)). \tag{2.20}$$

From (2.17) and (2.19) we obtain that

$$\begin{aligned} f(\sigma_{n+1},\rho_{n+1}) &\leqslant \psi(f(\sigma_n,\rho_n))(d(\sigma_n,\varrho_{n+1})+d(\tau_{n+1},\rho_n)) \leqslant \\ &\leqslant \sqrt{\psi(f(\sigma_n,\rho_n))}\sqrt{\psi(f(\sigma_n,\rho_n))}(d(\sigma_n,\varrho_{n+1})+d(\tau_{n+1},\rho_n)) \leqslant \\ &\leqslant \sqrt{\psi(f(\sigma_n,\rho_n))}f(\sigma_n,\rho_n). \end{aligned} \tag{2.21}$$

From (2.18) and (2.20) we obtain

$$\begin{aligned} f(\tau_{n+1},\varrho_{n+1}) &\leqslant \psi(f(\tau_n,\varrho_n))(d(\tau_n,\rho_{n+1})+d(\sigma_{n+1},\varrho_n)) \leqslant \\ &\leqslant \sqrt{\psi(f(\tau_n,\varrho_n))}\sqrt{\psi(f(\tau_n,\varrho_n))}(d(\tau_n,\rho_{n+1})+d(\sigma_{n+1},\varrho_n)) \leqslant \\ &\leqslant \sqrt{\psi(f(\tau_n,\varrho_n))}f(\tau_n,\varrho_n). \end{aligned} \tag{2.22}$$

Therefore, we get

$$f(\sigma_{n+1},\rho_{n+1})+f(\tau_{n+1},\varrho_{n+1}) \leqslant \sqrt{\psi(f(\sigma_n,\rho_n))}f(\sigma_n,\rho_n)+\sqrt{\psi(f(\tau_n,\varrho_n))}f(\tau_n,\varrho_n). \tag{2.23}$$

On the other hand

$$f(\sigma_{n+1},\rho_n)+f(\tau_{n+1},\varrho_n) \leqslant \sqrt{\psi(f(\sigma_n,\rho_{n-1}))}f(\sigma_n,\rho_{n-1})+\sqrt{\psi(f(\tau_n,\varrho_{n-1}))}f(\tau_n,\varrho_{n-1}) \tag{2.24}$$

and

$$f(\sigma_n,\rho_{n+1})+f(\tau_n,\varrho_{n+1})\leqslant\sqrt{\psi(f(\sigma_{n-1},\rho_n))}f(\sigma_{n-1},\rho_n)+\sqrt{\psi(f(\tau_{n-1},\varrho_n))}f(\tau_{n-1},\varrho_n). \quad (2.25)$$

Now we prove $f(\sigma_n,\rho_n)+f(\tau_n,\varrho_n)\to 0$ as $n\to+\infty$. Suppose that $f(\sigma_n,\rho_n)+f(\tau_n,\varrho_n)>0$ for all $n\in N$, since if $f(\sigma_n,\rho_n)+f(\tau_n,\varrho_n)=0$ for some $n\in N$. Then we obtain $(D(\sigma_n,F(\varrho_n,\tau_n))+D(F(\tau_n,\varrho_n),\rho_n)+(D(\tau_n,F(\rho_n,\sigma_n))+D(F(\sigma_n,\rho_n),\varrho_n))=0$

$$\begin{aligned}
D(\sigma_n,F(\varrho_n,\tau_n))=0 &\quad \text{implies that} \quad \sigma_n\in\overline{F(\varrho_n,\tau_n)}=F(\varrho_n,\tau_n)\\
D(F(\tau_n,\varrho_n),\rho_n)=0 &\quad \text{implies that} \quad \rho_n\in\overline{F(\tau_n,\varrho_n)}=F(\tau_n,\varrho_n)\\
D(\tau_n,F(\rho_n,\sigma_n))=0 &\quad \text{implies that} \quad \tau_n\in\overline{F(\rho_n,\sigma_n)}=F(\rho_n,\sigma_n)\\
D(F(\sigma_n,\rho_n),\varrho_n)=0 &\quad \text{implies that} \quad \varrho_n\in\overline{F(\sigma_n,\rho_n)}=F(\sigma_n,\rho_n)
\end{aligned}$$

also, we have

$$\begin{aligned}
0 &\leqslant \inf\nolimits_{\varrho_n\in F(\sigma_n,\rho_n)} d(\sigma_n,\varrho_n)=\\
&= D(\sigma_n,F(\sigma_n,\rho_n))\leqslant\\
&\leqslant D(\sigma_n,F(\varrho_n,\tau_n))+D(\rho_n,F(\varrho_n,\tau_n))+D(\rho_n,F(\sigma_n,\rho_n))\leqslant\\
&\leqslant f(\sigma_n,\rho_n)+\inf\nolimits_{\rho_n\in F(\tau_n,\varrho_n)} d(\rho_n,\rho_n)\leqslant\\
&\leqslant \lim_{n\to+\infty} f(\sigma_n,\rho_n)=0.
\end{aligned}$$

Therefore, $\sigma_n=\varrho_n$ and similarly, we shows that $\tau_n=\rho_n$. Then $(\sigma_n,\rho_n)\in(\Gamma\times\Theta)\cap(\Theta\times\Gamma)$ is coupled fixed point of F. Hence theorem is proved. □

Using (2.23)–(2.25) and $\psi(t)<1$, we conclude that $\{f(\sigma_n,\rho_n)\}$ and $\{f(\tau_n,\varrho_n)\}$ are strictly decreasing bisequence of non-negative real numbers. Thus there exist $\delta\geqslant 0$ and $\lambda\geqslant 0$ such that $\lim\limits_{n\to+\infty} f(\sigma_n,\rho_n)=\delta$ and $\lim\limits_{n\to+\infty} f(\tau_n,\varrho_n)=\lambda$.
Now we will prove $\delta=\lambda=0$. Suppose that $\delta>0$ and $\lambda>0$. Letting $n\to+\infty$ in (2.23)–(2.25), we obtain

$$\begin{aligned}
\delta+\lambda &\leqslant \lim_{f(\sigma_{n+1},\rho_{n+1})\to\delta^+}\sup\sqrt{\psi(f(\sigma_{n+1},\rho_{n+1})}\delta+\lim_{f(\tau_{n+1},\varrho_{n+1})\to\lambda^+}\sup\sqrt{\psi(f(\tau_{n+1},\varrho_{n+1})}\lambda<\\
&<\delta+\lambda
\end{aligned}$$

and

$$\begin{aligned}
\delta+\lambda &\leqslant \lim_{f(\sigma_{n+1},\rho_n)\to\delta^+}\sup\sqrt{\psi(f(\sigma_{n+1},\rho_n)}\delta+\lim_{f(\tau_{n+1},\varrho_n)\to\lambda^+}\sup\sqrt{\psi(f(\tau_{n+1},\varrho_n)}\lambda<\\
&<\delta+\lambda
\end{aligned}$$

also

$$\begin{aligned}
\delta+\lambda &\leqslant \lim_{f(\sigma_n,\rho_{n+1})\to\delta^+}\sup\sqrt{\psi(f(\sigma_n,\rho_{n+1})}\delta+\lim_{f(\tau_n,\varrho_{n+1})\to\lambda^+}\sup\sqrt{\psi(f(\tau_n,\varrho_{n+1})}\lambda<\\
&<\delta+\lambda.
\end{aligned}$$

In any case which is contradiction. Hence $\delta=\lambda=0$, that is $\lim\limits_{n\to+\infty} f(\sigma_n,\rho_n)=\lim\limits_{n\to+\infty} f(\tau_n,\varrho_n)=0$.
Now we shows that (σ_n,ρ_n) and (τ_n,ϱ_n) are Cauchy bisequences in (Γ,Θ,d).
Suppose that

$\delta = \lim_{f(\sigma_{n+1},\rho_{n+1})\to 0^+} \sup \sqrt{\psi(f(\sigma_{n+1},\rho_{n+1})}$
and $\lambda = \lim_{f(\tau_{n+1},\varrho_{n+1})\to 0^+} \sup \sqrt{\psi(f(\tau_{n+1},\varrho_{n+1})}$. Then by our assumption (2.2), we have $\delta < 1$, $\lambda < 1$. Let ξ *and* ζ be such that $\delta < \xi < 1$ and $\lambda < \zeta < 1$ then there is some $n_0 \in N$ such that $\sqrt{\psi(f(\sigma_{n+1},\rho_{n+1})} < \xi$, $\sqrt{\psi(f(\tau_{n+1},\varrho_{n+1})} < \zeta$, for each $n \geqslant n_0$. Thus, from (2.23), we obtain

$$\begin{aligned} f(\sigma_{n+1},\rho_{n+1}) + f(\tau_{n+1},\varrho_{n+1}) &\leqslant \xi f(\sigma_n,\rho_n) + \zeta f(\tau_n,\varrho_n) \leqslant \\ &\leqslant \xi^2 f(\sigma_{n-1},\rho_{n-1}) + \zeta^2 f(\tau_{n-1},\varrho_{n-1}) \leqslant \\ &\vdots \\ &\leqslant \xi^{n+1-n_0} f(\sigma_{n_0},\rho_{n_0}) + \zeta^{n+1-n_0} f(\tau_{n_0},\varrho_{n_0}). \end{aligned} \tag{2.26}$$

Since $\psi(t) \geqslant b > 0$ for all $t \geqslant 0$, from (2.17), (2.18) and (2.26), we get

$$\begin{aligned} (d(\sigma_n,\varrho_{n+1}) + d(\tau_{n+1},\rho_n)) + (d(\tau_n,\rho_{n+1}) + d(\sigma_{n+1},\varrho_n)) \leqslant \\ \leqslant \frac{1}{\sqrt{b}}(\xi^{n-n_0} f(\sigma_{n_0},\rho_{n_0}) + \zeta^{n-n_0} f(\tau_{n_0},\varrho_{n_0})). \end{aligned} \tag{2.27}$$

On the other hands from (2.24) and (2.25)

$$\begin{aligned} f(\sigma_{n+1},\rho_n) + f(\tau_{n+1},\varrho_n) \quad &\leqslant \xi f(\sigma_n,\rho_{n-1}) + \zeta f(\tau_n,\varrho_{n-1}) \leqslant \\ &\leqslant \xi^2 f(\sigma_{n-1},\rho_{n-2}) + \zeta^2 f(\tau_{n-1},\varrho_{n-2}) \\ &\vdots \\ &\leqslant \xi^{n+1} f(\sigma_1,\rho_0) + \zeta^{n+1} f(\tau_1,\varrho_0) \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} (d(\sigma_n,\varrho_n) + d(\tau_{n+1},\rho_{n-1})) + (d(\tau_n,\rho_n) + d(\sigma_{n+1},\varrho_{n-1})) \leqslant \\ \leqslant \frac{1}{\sqrt{b}}(\xi^{n-n_0} f(\sigma_{n_1},\rho_{n_0}) + \zeta^{n-n_0} f(\tau_{n_1},\varrho_{n_0})) \end{aligned} \tag{2.29}$$

also

$$\begin{aligned} f(\sigma_n,\rho_{n+1}) + f(\tau_n,\varrho_{n+1}) &\leqslant \xi f(\sigma_{n-1},\rho_n) + \zeta f(\tau_{n-1},\varrho_n) \leqslant \\ &\leqslant \xi^2 f(\sigma_{n-2},\rho_{n-1}) + \zeta^2 f(\tau_{n-2},\varrho_{n-1}) \\ &\vdots \\ &\leqslant \xi^{n+1} f(\sigma_0,\rho_1) + \zeta^{n+1} f(\tau_0,\varrho_1) \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} (d(\sigma_{n-1},\varrho_{n+1}) + d(\tau_n,\rho_n)) + (d(\tau_{n-1},\rho_{n+1}) + d(\sigma_n,\varrho_n)) \leqslant \\ \leqslant \frac{1}{\sqrt{b}}(\xi^{n-n_0} f(\sigma_{n_0},\rho_{n_1}) + \zeta^{n-n_0} f(\tau_{n_0},\varrho_{n_1})). \end{aligned} \tag{2.31}$$

For each $n, m \in N$ with $n < m$, we have (27), (29) and (31)

$$
\begin{aligned}
& d(\sigma_n, \varrho_m) + d(\tau_m, \rho_n) + d(\sigma_m, \varrho_n) + d(\tau_n, \rho_m) \leqslant \\
& \leqslant (d(\sigma_n, \varrho_{n+1}) + d(\tau_{n+1}, \rho_n)) + d(\sigma_{n+1}, \varrho_n) + d(\tau_n, \rho_{n+1})) + \\
& \quad +2(d(\sigma_{n+1}, \varrho_{n+1}) + d(\tau_{n+1}, \rho_{n+1})) + \cdots + 2(d(\sigma_{m-1}, \varrho_{m-1}) + d(\tau_{m-1}, \rho_{m-1})) + \\
& \quad +(d(\sigma_{m-1}, \varrho_m) + d(\tau_m, \rho_{m-1})) + (d(\sigma_m, \varrho_{m-1}) + d(\tau_{m-1}, \rho_m)) \leqslant \\
& \leqslant \frac{1}{\sqrt{b}}(\xi^{n-n_0} f(\sigma_{n_0}, \rho_{n_0}) + \zeta^{n-n_0} f(\tau_{n_0}, \varrho_{n_0})) + \frac{2}{\sqrt{b}}(\xi^{n+1-n_0} f(\sigma_{n_1}, \rho_{n_0}) + \\
& \quad +\zeta^{n+1-n_0} f(\tau_{n_1}, \varrho_{n_0})) + \cdots + \frac{2}{\sqrt{b}}(\xi^{m+1-n_0} f(\sigma_{n_1}, \rho_{n_0}) + \\
& \quad +\zeta^{m+1-n_0} f(\tau_{n_1}, \varrho_{n_0})) + \frac{1}{\sqrt{b}}(\xi^{m-n_0} f(\sigma_{n_0}, \rho_{n_1}) + \zeta^{n-n_0} f(\tau_{n_0}, \varrho_{n_1})). \\
& \to 0 \text{ as } n, m \to +\infty.
\end{aligned}
$$

Hence, (σ_n, ρ_n) and (τ_n, ϱ_n) are Cauchy bi-sequences in (Γ, Θ, d). Since (Γ, Θ, d) is complete, there exist $\alpha, \beta \in \Gamma$ and $\gamma, \eta \in \Theta$ such that

$$
\lim_{n\to+\infty} \sigma_n = \eta, \quad \lim_{n\to+\infty} \tau_n = \gamma, \quad \lim_{n\to+\infty} \rho_n = \beta, \quad \lim_{n\to+\infty} \varrho_n = \alpha. \tag{2.32}
$$

By our assumption f is lower semi continuous. Then we have

$$
0 \leqslant f(\alpha, \gamma) = D(\alpha, F(\eta, \beta)) + D(F(\beta, \eta), \gamma) \leqslant \lim_{n\to+\infty} \inf f(\tau_n, \varrho_n) = 0.
$$

Hence $D(\alpha, F(\eta, \beta)) = 0$ and $D(F(\beta, \eta), \gamma) = 0$ which implies that $\alpha \in F(\eta, \beta)$ and $\gamma \in F(\beta, \eta)$. And similarly we can prove that $\beta \in F(\gamma, \alpha)$ and $\eta \in F(\alpha, \gamma)$.
Again from (2.32), we get

$$
d(\alpha, \eta) = d(\lim_{n\to+\infty} \varrho_n, \lim_{n\to+\infty} \sigma_n) = \lim_{n\to+\infty} d(\sigma_n, \varrho_n) = 0
$$

and

$$
d(\beta, \gamma) = d(\lim_{n\to+\infty} \rho_n, \lim_{n\to+\infty} \tau_n) = \lim_{n\to+\infty} d(\tau_n, \rho_n) = 0.
$$

Therefore, $\alpha = \eta$ and $\beta = \gamma$. Then $\alpha \in F(\alpha, \beta)$ and $\beta \in F(\beta, \alpha)$, that is
$(\alpha, \beta) \in (\Gamma \times \Theta) \cap (\Theta \times \Gamma)$ is a coupled fixed point of F. Now we prove the uniqueness, let $(\alpha^*, \beta^*) \in (\Gamma \times \Theta) \cup (\Theta \times \Gamma)$ be another coupled fixed point of F. If $(\alpha^*, \beta^*) \in (\Gamma \times \Theta)$, then we obtain

$$
0 \leqslant f(\alpha^*, \beta^*) = D(\alpha^*, F(\alpha, \beta)) + D(F(\beta, \alpha), \beta^*) \leqslant \lim_{n\to+\infty} \inf f(\sigma_n, \rho_n) = 0.
$$

Therefore, $D(\alpha^*, F(\alpha, \beta)) = 0$ and $D(F(\beta, \alpha), \beta^*) = 0$ implies $\alpha^* \in F(\alpha, \beta)$ and $\beta^* \in F(\beta, \alpha)$. So, we get $\alpha = \alpha^*$ and $\beta = \beta^*$.
Similarly, if $(\alpha^*, \beta^*) \in (\Theta \times \Gamma)$, we have $\alpha = \alpha^*$ and $\beta = \beta^*$.
Then (α, β) is a unique coupled fixed point of F.

Example 2.4. *Let* $\Gamma = \{\mathfrak{U}_m(R)/\mathfrak{U}_m(R)$ *is upper triangular matrices overR*$\}$ *and*
$\Theta = \{\mathfrak{L}_m(R)/\mathfrak{L}_m(R)$ *is lower triangular matrices over* $R\}$ *with the bipolar metric*

$$
d(\Phi, \Omega) = \sum_{i,j=1}^{m} |\phi_{ij} - \omega_{ij}|
$$

for all $\Phi = (\phi_{ij})_{m\times m} \in \mathfrak{U}_m(R)$ *and* $\Omega = (\omega_{ij})_{m\times m} \in \mathfrak{L}_m(R)$*. On the set* (Γ, Θ)*, we consider the following relation :*

$$
\Phi, \Omega \in \Gamma \cup \Theta, \Phi \preceq \Omega \Leftrightarrow \phi_{ij} \leqslant \omega_{ij}
$$

where $\leqslant$ *is usual ordering. Then clearly,* (Γ, Θ, d) *is a complete bipolar metric space and* $(\Gamma, \Theta, \preceq)$ *is a partially ordered set. And* (Γ, Θ) *has the property as in Theorem (2.3). Let* $F : (\Gamma \times \Theta, \Theta \times \Gamma) \rightrightarrows CL(\Gamma, \Theta)$ *be defined as*

$$F(\Phi, \Omega) = (\phi_{ij})_{m\times m} I_{m\times m} \ \ \forall \ (\Phi = (\phi_{ij})_{m\times m}, \quad \Omega = (\omega_{ij})_{m\times m}) \in (\Gamma \times \Theta) \cup (\Theta \times \Gamma).$$

Then

$$\begin{aligned} f(\Phi, \Omega) &= D(\Phi, F(\Omega, \Phi)) + D(F(\Phi, \Omega), \Omega) = \\ &= \inf\{d(\Phi, Y) : Y \in (\omega_{ij})_{m\times m} I_{m\times m}\} + \inf\{d(X, \Omega) : X \in (\phi_{ij})_{m\times m} I_{m\times m}\} = \\ &= d(\Phi, \Omega) + d(\Phi, \Omega) = 2d(\Phi, \Omega) = 2\sum_{i,j=1}^{m} |\phi_{ij} - \omega_{ij}|. \end{aligned}$$

Also, let $\psi : [0, +\infty) \to (0, 1)$ *by* $\psi(t) = \dfrac{t}{1+t}$ *then obviously,* $\lim_{r\to t^+} \sup \psi(r) < 1$ *for each* $t \in [0, +\infty)$ *with out loss of generality we may assume that*

$$O = (o_{ij})_{m\times m} = Y = (y_{ij})_{m\times m} \preceq \Phi = (\phi_{ij})_{m\times m}$$

and

$$O = (o_{ij})_{m\times m} = X = (x_{ij})_{m\times m} \preceq \Omega = (\omega_{ij})_{m\times m}.$$

It is obviously,

$$\sqrt{\psi(f(\Phi, \Omega))}[d(\Phi, Y) + d(X, \Omega)] \leqslant f(\Phi, \Omega)$$

such that

$$f(X, Y) \leqslant \psi(f(\Phi, \Omega))[d(\Phi, Y) + d(X, \Omega)].$$

Hence all assertions of Theorem (2.3) are satisfied and $(O_{m\times m}, O_{m\times m})$ *is the coupled fixed point of* F.

Theorem 2.5. *Let* (Γ, Θ, d) *be an complete bipolar metric space endowed with a partial order* $\preceq$. *Suppose that* Δ *is non empty, that is there exists* $(\sigma, \rho) \in \Delta$. *Let* $F : (\Gamma \times \Theta, \Theta \times \Gamma) \rightrightarrows CL(\Gamma, \Theta)$ *be a* Δ*- symmetric covariant mapping and consider that* $f : \Gamma \times \Theta \to [0, +\infty)$ *as*

$$f(\sigma, \rho) = D(\sigma, F(\varrho, \tau)) + D(F(\tau, \varrho), \rho) \ \ for\ all\ \sigma, \tau \in \Gamma\ and\ \rho, \varrho \in \Theta \tag{2.33}$$

is lower semi-continuous and there exists a mapping $\psi : [0, +\infty) \to (0, 1)$ *satisfying*

$$\lim_{r\to t^+} \sup \psi(r) < 1 \ for\ each\ t \in [0, +\infty). \tag{2.34}$$

Assume that for any $(\sigma, \rho) \in \Delta$ *there exist* $x \in F(\sigma, \rho)$ *and* $y \in F(\rho, \sigma)$ *satisfying*

$$\sqrt{\psi(d(\sigma, y) + d(x, \rho))}[d(\sigma, y) + d(x, \rho)] \leqslant D(\sigma, F(\varrho, \tau)) + D(F(\tau, \varrho), \rho) \tag{2.35}$$

such that

$$D(x, F(v, u)) + D(F(u, v), y) \leqslant \psi(d(\sigma, y) + d(x, \rho))[d(\sigma, y) + d(x, \rho)] \tag{2.36}$$

for some $v \in F(\varrho, \tau)$ *and* $u \in F(\tau, \varrho)$. *Then* $F : (\Gamma \times \Theta) \cup (\Theta \times \Gamma) \to CL(\Gamma \cup \Theta)$ *has a coupled fixed point. That is there exists* $(\alpha, \beta) \in (\Gamma \times \Theta) \cup (\Theta \times \Gamma)$ *such that* $\alpha \in F(\alpha, \beta)$ *and* $\beta \in F(\beta, \alpha)$.

Example 2.6. *Let* $\Gamma = \{\mathfrak{U}_m(R)/\mathfrak{U}_m(R)$ *is upper triangular matrices over* $R\}$ *and* $\Theta = \{\mathfrak{L}_m(R)/\mathfrak{L}_m(R)$ *is lower triangular matrices over* $R\}$ *with the bipolar metric*

$$d(\Phi, \Omega) = \sum_{i,j=1}^{m} |\phi_{ij} - \omega_{ij}|$$

for all $\Phi = (\phi_{ij})_{m\times m} \in \mathfrak{U}_m(R)$ *and* $\Omega = (\omega_{ij})_{m\times m} \in \mathfrak{L}_m(R)$. *On the set* (Γ, Θ), *we consider the following relation :*

$$\Phi, \Omega \in \Gamma \cup \Theta,\ \ \Phi \preceq \Omega \Leftrightarrow \phi_{ij} \leqslant \omega_{ij}$$

where $\leqslant$ *is usual ordering. Then clearly,* (Γ, Θ, d) *is a complete bipolar metric space and* $(\Gamma, \Theta, \preceq)$ *is a partially ordered set. Let* $F : (\Gamma \times \Theta, \Theta \times \Gamma) \rightrightarrows CL(\Gamma, \Theta)$ *be defined as*

$$\begin{aligned} F(\Phi, \Omega) &= \frac{(\phi_{ij})_{m\times m}}{3} \\ &\forall \quad (\Phi = (\phi_{ij})_{m\times m},\ \Omega = (\omega_{ij})_{m\times m}) \in (\Gamma \times \Theta) \cup (\Theta \times \Gamma) \end{aligned}$$

define $\psi : [0, +\infty) \to (0, 1)$ *by* $\psi(t) = \frac{1}{5}$. *First we shall prove that* $F(\Phi, \Omega)$ *satisfies all the conditions of Theorem (2.5). In fact it is easy to see that the mapping* $f(\Phi, \Omega) = \frac{4}{15} \sum_{i,j=1}^{m} |\phi_{ij} - \omega_{ij}|$ *is lower semi continuous. Thus for all* $(\Phi, \Omega) \in (\Gamma \times \Theta) \cup (\Theta \times \Gamma)$, *there exist* $X \in F(\Phi, \Omega) = \frac{(\phi_{ij})_{m\times m}}{3}$ *and* $Y \in F(\Omega, \Phi) = \frac{(\omega_{ij})_{m\times m}}{3}$ *such that*

$$\begin{aligned} D(\Phi, F(\Omega, \Phi)) + D(F(\Phi, \Omega), Q) &= \frac{4}{15} \left(\sum_{i,j=1}^{m} |\phi_{ij} - \omega_{ij}| \right) = \\ &= \frac{1}{5} \left(\sum_{i,j=1}^{m} |\frac{4}{3}\phi_{ij} - \frac{4}{3}\omega_{ij}| \right) = \\ &= \frac{1}{5} \left(\sum_{i,j=1}^{m} |(\phi_{ij} - \frac{1}{3}\omega_{ij}) + (\frac{1}{3}\phi_{ij} - \omega_{ij})| \right) \leqslant \\ &\leqslant \frac{1}{5} \left(\sum_{i,j=1}^{m} |\phi_{ij} - \frac{1}{3}\omega_{ij}| + \sum_{i,j=1}^{m} |\frac{1}{3}\phi_{ij} - \omega_{ij}| \right) \leqslant \\ &\leqslant \psi(d(\Phi, Y) + d(X, \Omega))[d(\Phi, Y) + d(X, \Omega)]. \end{aligned}$$

It is obviously,

$$\sqrt{\psi(d(\Phi, Y) + d(X, \Omega))}[d(\Phi, Y) + d(X, \Omega)] \leqslant D(\Phi, F(\Omega, \Phi)) + D(F(\Phi, \Omega), \Omega)$$

such that

$$D(X, F(Y, X)) + D(F(X, Y), Y) \leqslant \psi(d(\Phi, Y) + d(X, \Omega))[d(\Phi, Y) + d(X, \Omega)].$$

Hence all assertions of Theorem (2.5) are satisfied and $(O_{m\times m}, O_{m\times m})$ *is the coupled fixed point of* F.

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Многозначные Δ-симметричные ковариантные результаты в биполярных метрических пространствах

Г. Н. В. Кишор
Факультет инженерной математики и гуманитарных наук
Инженерный колледж Саги Рама Кришнам Раджу
Андхра-Прадеш, Индия

Б. Шринуваса Рао
Департамент математики
Амбедкар университет
Андхра-Прадеш, Индия

Д. Рам Прасад
Факультет математики
Инженерный колледж Налла Малла Редди
Дивья нагар, Гхаткесар мандал
Телангана, Индия

Стоян Раденович
Факультет машиностроения
Белградский университет
Белград, Сербия

Аннотация. В этой статье мы докаываем некоторые теоремы о парных фиксированных точках для гибридных пар в отображениях, использующих Δ-симметрические ковариантные отображения в биполярных метрических пространствах. Мы также даем некоторые примеры, которые основаны на наших результатах.

Ключевые слова: Δ-симметричное ковариантное отображение, гибридная пара отображений, связанная неподвижная точка, биполярные метрические пространства.