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Symmetries of Linear and Nonlinear Partial Differential Equations

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Abstract. Higher symmetries and operator symmetries of linear partial differential equations are considered The higher symmetries form a Lie algebra, and operator ones form an associative algebra. The relationship between these symmetries is established. New symmetries of two-dimensional stationary equations of gas dynamics are found.

Keywords: higher symmetries, operator symmetries, gas dynamics equations.

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Introduction

It is well known that symmetries play a crucial role in finding solutions of differential equations. The theory of point symmetries is well described in numerous monographs and textbooks [1–3]. A large number of examples of invariant and partially invariant solutions were presented [4,5]. One can say that theory of point transformations is well developed. Some generalizations of the Lie theory have been proposed.The most successful advances include the theory of higher symmetries of nonlinear equations and operator symmetries of linear equations [2,6–8]. However, the use of higher symmetry operators is complicated because transformations constructed using these operators act in infinite-dimensional spaces and they are represented by formal series [2]. As a result, it is difficult to determine analogs of invariant solutions with respect to such transformations.

Modified definitions of admitted operators and operator symmetries for linear systems of partial differential equations are introduced in this paper. It is easily verified that operator symmetries form an associative algebra with respect to ordinary multiplication and a Lie algebra with commutator multiplication. It is proved that some admitted operator corresponds to each operator symmetry. It turns out that symmetries of linear equations can be transformed into symmetries of nonlinear equations in some cases. Here, as an example, a system of two equations that describes plane, steady, irrotational gas flows is considered [9, 10]. Using hodograph transformation, a linear system is obtained. The admitted operators of this system give rise to an infinite series of symmetries of nonlinear gas dynamics equations.

1. Symmetries

Let us consider the matrix differential operator

$$L = \sum_{|\alpha|\geqslant 0}^{k} A_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}, \tag{1}$$

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where $\alpha = (\alpha_1, \dots, \alpha_n)$, A_α are $m \times m$ matrices depending on $x = (x_1, \dots, x_n)$. The operator defines a system of linear partial differential equations

$$Lu = 0, \tag{2}$$

where $u = (u^1, \dots, u^m)$ is a set of unknown with respect to x functions.

Further, the operator of total derivative [1,2] with respect to x_i $(i = 1, \dots, n)$ is denoted by D_{x_i}. The expression D^α means the product of operators $D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$.

Definition 1. *System (2) admits the operator in canonical form*

$$X = \sum_{j=1}^{m} \eta_j \frac{\partial}{\partial u^j} + \sum_{\substack{1 \leqslant j \leqslant m \\ \alpha \in \mathbb{N}^n}} D^\alpha \eta_j \frac{\partial}{\partial u^j_\alpha}, \tag{3}$$

if there is a matrix differential operator M such that the equality

$$L\eta = MLu, \tag{4}$$

is true, where $u = (u^1, \dots, u^m)$ is a set of arbitrary smooth functions of x, and $\eta = (\eta_1, \dots, \eta_m)$. Relation (4) is called the defining equation.

The above definition differs from the standard one [2,3]. Obviously, condition (4) is sufficient for the classical invariance of system (2). One can shown that it is necessary condition but it is not needed here.

Remark. If system of partial differential equations $L(u) = 0$ is nonlinear system then condition (4) must be replaced by the following condition

$$XL(u) = ML(u)$$

.

It follows from (4) that if u is a solution of system (2) then $\tilde{u} = \eta$ is also solution of this system. Thus the transformation $x \longrightarrow x, \quad u \longrightarrow \eta$ acts on solutions of system (2). Such transformations are called L-symmetries.

Proposition 1. *If $\eta^1, \dots, \eta^p$ are solutions of the defining equations*

$$L\eta^k = M_k Lu, \qquad k = 1, \dots p, \tag{5}$$

then

$$x \longrightarrow x, \quad u \longrightarrow \sum_{k=1}^{p} c_k \eta^k, \qquad c_k \in \mathbb{R} \tag{6}$$

is the L-symmetry of equation (2).

Indeed, since functions η^k satisfy (5) then the equality

$$L(\sum_{k=1}^{p} c_k \eta^k) = (\sum_{k=1}^{p} c_k M_k) Lu.$$

is true due to linearity of the operators. This means that transformation (6) is L-symmetry.

Proposition 2. *The set of L-symmetries of system (2) forms a monoid with the composition operation.*

This immediately follows from the fact that symmetries act on solutions of the system and, therefore, the composition of two L-symmetries of system (2) is an L-symmetry. Moreover, the identity transformation is also a symmetry.

The second method of introducing symmetries of linear equations is described in [7,8]. Some modified versions of definitions are provided below.

Definition 2. *Let differential operator (1) be given. The differential operator S is called the operator symmetry of equation (2) if there is a differential operator $\mathcal{D}$ such that*

$$LS = \mathcal{D}L. \tag{7}$$

It is assumed that S is not a polynomial in L.

Obviously the operator symmetry S acts on solutions of equation (2), i.e., transforms solutions into solutions.

Proposition 3. *Let $\mathcal{S}_1, \mathcal{S}_2$ be two operator symmetries of system (2). Then*

$$b_1\mathcal{S}_1 + b_2\mathcal{S}_2, \quad \mathcal{S}_1\mathcal{S}_2, \quad \mathcal{S}_1\mathcal{S}_2 - \mathcal{S}_2\mathcal{S}_1,$$

also operator symmetries for any $b_1, b_2 \in \mathbb{R}$.

Proof. By condition the equalities

$$L\mathcal{S}_1 = \mathcal{D}_1 L, \qquad L\mathcal{S}_2 = \mathcal{D}_2 L.$$

are true. It follows that

$$L(b_1\mathcal{S}_1 + b_2\mathcal{S}_2) = b_1 L\mathcal{S}_1 + b_2 L\mathcal{S}_2 = b_1\mathcal{D}_1 L + b_2\mathcal{D}_2 L = (b_1\mathcal{D}_1 + b_2\mathcal{D}_2)L,$$

$$L\mathcal{S}_1\mathcal{S}_2 = \mathcal{D}_1 L\mathcal{S}_2 = \mathcal{D}_1\mathcal{D}_2 L,$$

$$L(\mathcal{S}_1\mathcal{S}_2 - \mathcal{S}_2\mathcal{S}_1) = \mathcal{D}_1\mathcal{D}_2 L - \mathcal{D}_2\mathcal{D}_1 L = (\mathcal{D}_1\mathcal{D}_2 - \mathcal{D}_2\mathcal{D}_1)L. \qquad \square$$

Remark. If the commutator of operators $\mathcal{S}_1, \mathcal{S}_2$ is introduced according to the well-known formula $[\mathcal{S}_1, \mathcal{S}_2] = \mathcal{S}_1\mathcal{S}_2 - \mathcal{S}_2\mathcal{S}_1$ then the last expression in the proof is rewritten as

$$L[\mathcal{S}_1, \mathcal{S}_2] = [\mathcal{D}_1, \mathcal{D}_2]L.$$

Corollary. The set of operator symmetries of system (2) forms an associative algebra over $\mathbb{R}$ with respect to ordinary multiplication and a Lie algebra with respect to commutator multiplication.

Proposition 4. *If S is an operator symmetry of equation (2), and $u = (u^1, \dots, u^m)$ is a set of smooth functions then $\eta = Su$ is a solution to the defining equation that generates L-symmetry.*

By assumption, there exists an operator $\mathcal{D}$ that satisfies equality (7). Applying to u the operators on the left and right sides of (7), equality (4) is obtained in which $\eta = Su$ and $M = \mathcal{D}$.

2. Symmetries of stationary gas dynamics equations

The L-symmetries introduced above can be applied to some nonlinear equations. As an example, let us consider the well-known system of stationary equations

$$u_y - v_x = 0, \quad (u^2 - c^2)u_x + 2uvu_y + (v^2 - c^2)v_y = 0, \tag{8}$$

that describes flat steady irrotational flows of compressible fluid [9,10]. Here u, v are components of the velocity vector, c is the speed of sound, expressed from the Bernoulli integral

$$u^2 + v^2 + I(c^2) = const.$$

Writing equations (8) in terms of hodograph variables, system of linear equations [9]

$$x_v - y_u = 0, \quad (u^2 - c^2)y_v - 2uvx_v + (v^2 - c^2)x_u = 0, \tag{9}$$

is obtained for two unknown functions x, y that depend on u, v. It is not difficult to see [9] that both systems admit the following rotation, scaling and translation operators:

$$-v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v} - y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\ , \quad x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\ , \quad \frac{\partial}{\partial x}\ , \quad \frac{\partial}{\partial y}\ .$$

The rotation operator admitted by the system (9) in canonical form has the form

$$(-y + vx_u - ux_v)\frac{\partial}{\partial x} + (x + vy_u - uy_v)\frac{\partial}{\partial y}\ .$$

Therefore, according to Proposition 1, the transformation

$$\tilde{u} = u\ , \quad \tilde{v} = v\ , \quad \tilde{x} = -y + vx_u - ux_v\ , \quad \tilde{y} = x + vy_u - uy_v$$

acts on solutions of linear system (9) and it is an L-symmetry of this system. Using three other symmetry operators, L-symmetry of the form

$$\tilde{u} = u, \quad \tilde{v} = v \tag{10}$$

$$\tilde{x} = c_1(-y + vx_u - ux_v) + c_2x + c_3, \quad \tilde{y} = c_1(x + vy_u - uy_v) + c_2y + c_4, \tag{11}$$

is obtained, where c_i are arbitrary constants.

In order to obtain symmetries of gas dynamics equations (8), it is necessary to express the derivatives x_u, x_v, y_u, y_v in terms of the derivatives u_x, u_y, v_x, v_y. Using the hodograph transformation, it is easy to find these derivatives [9]

$$x_u = v_y/J, \quad x_v = -u_y/J, \quad y_u = -v_x/J, \quad y_v = u_x/J,$$

where $J = \dfrac{\partial(u, v)}{\partial(x, y)}$ is the Jacobian of functions u, v. Thus, the formulas of transformations (10), (11) have the following form

$$\tilde{u} = u, \quad \tilde{v} = v \tag{12}$$

$$\tilde{x} = c_1\left(-y + \frac{vv_y + uu_y}{J}\right) + c_2x + c_3, \quad \tilde{y} = c_1\left(x - \frac{vv_x + uu_x}{J}\right) + c_2y + c_4. \tag{13}$$

The last expressions determine the transformation of solutions of system (8) back into solutions of this system.

Combination of 10), (11) and

$$\hat{u} = \tilde{u}, \quad \hat{v} = \tilde{v}$$

$$\hat{x} = b_1(-\tilde{y} + \tilde{v}\tilde{x}_u - \tilde{u}\tilde{x}_v) + b_2\tilde{x} + b_3, \quad \hat{y} = c_1(\tilde{x} + \tilde{v}\tilde{y}_u - \tilde{u}\tilde{y}_v) + b_2\tilde{y} + b_4,$$

gives a new second-order symmetry of system (9).

One can obtain symmetries of system (9) of arbitrary order by means of compositions. Thus, an infinite series of symmetries of the system in the hodograph variables arises. An infinite series of symmetries of the gas dynamics equations (8) are obtained by recalculating the corresponding derivatives.

Conclusion

Using the found symmetries, it is possible to construct solutions from known ones. For example, if scale-invariant solutions [9] of system (8) are taken then new solutions can be generated using formulas (12), (13). The first works devoted to new types of symmetries have appeared recently [11, 12]. This approach requires further development and construction of new examples.

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Симметрии линейных и нелинейных уравнений с частными производными

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Аннотация. Рассматриваются операторы высших и операторных симметрий линейных уравнений с частными производными. Операторы высших симметрий образуют алгебру Ли, а операторные - ассоциативную алгебру. Устанавливается связь между этими симметриями. Найдены новые симметрии двумерных стационарных уравнений газовой динамики.

Ключевые слова: высшие симметрии, операторные симметрии, уравнения газовой динамики.