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On Generalized Voigt Function and its Associated Properties

Ulfat Ansari[†]
Musharraf Ali[*]
Department of Mathematics, Gandhi Faiz-E-Aam College
Shahjahanpur-242001, India
Affiliated to Mahatma Jyotiba Phule Rohilkhand University
Bareilly-243006, India
Mohd Ghayasuddin[‡]
Department of Mathematics, Integral University
Centre Shahjahanpur-242001, India

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Abstract. In the present manuscript, we aim to present a new type of the generalized Voigt function, and investigate its series representations. By using the series representations of our function, we also point out some generating relations associated with the Kampé de Fériet function, Srivastava's triple hypergeometric series, confluent hypergeometric functions of one and two variables, and generalized hypergeometric function. Furthermore, two interesting recurrence relations of our introduced Voigt function are also indicated.

Keywords: Voigt function, Wright function, Kampé de Fériet function, Srivastava's triple hypergeometric series.

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1. Introduction and preliminaries

The well-known Voigt functions $K(x_1, x_2)$ and $L(x_1, x_2)$ have occurred in a wide variety of problems in physics such as astrophysical spectroscopy, transfer of radiation in heated atmosphere and also in the theory of neutron reactions.

The integral representations of these two functions (due to Reiche [11]) are given as follows:

$$K(x_1, x_2) = \frac{1}{\sqrt{\pi}} \int_0^\infty \exp\left(- x_2 t - \frac{1}{4} t^2 \right) \cos(x_1 t) dt \tag{1.1}$$

and

$$L(x_1, x_2) = \frac{1}{\sqrt{\pi}} \int_0^\infty \exp\left(- x_2 t - \frac{1}{4} t^2 \right) \sin(x_1 t) dt \tag{1.2}$$

$$(x_1 \in \Re; x_2 \in R^+).$$

Afterwards, Srivastava and Miller [13] presented the following interesting extension of these Voigt functions:

$$V_{\mu,\nu}(x_1, x_2) = \sqrt{\frac{x_1}{2}} \int_0^\infty t^\mu \exp\left(- x_2 t - \frac{1}{4} t^2 \right) J_\nu(x_1 t) dt \tag{1.3}$$

[†]ansari.ulfat96@gmail.com http://orcid.org/0009-0006-9977-4285
[*]drmusharrafali@gmail.com http://orcid.org/0000-0001-9791-3217
[‡]ghayas.maths@gmail.com http://orcid.org/0000-0001-8346-4078

$$(x_1, x_2 \in R^+; \Re(\mu+\nu) > -1),$$

where $J_\nu(x_1)$ denotes the familiar Bessel function [12, p.109, eq.(3)].

It is well-known that $J_{-\frac{1}{2}}(x_1) = \sqrt{\frac{2}{\pi x_1}} \cos x_1$ and $J_{\frac{1}{2}}(x_1) = \sqrt{\frac{2}{\pi x_1}} \sin x_1$.

Thus, we have

$$K(x_1, x_2) = V_{\frac{1}{2},-\frac{1}{2}}(x_1, x_2) \text{ and } L(x_1, x_2) = V_{\frac{1}{2},\frac{1}{2}}(x_1, x_2). \tag{1.4}$$

In continuation of this study, Klusch [7] replaced the number $\frac{1}{4}$ before t^2 in (1.3) by a variable to propose the following slightly more generalization of the function in (1.3):

$$\Omega_{\mu,\nu}(x_1, x_2, x_3) = \sqrt{\frac{x_1}{2}} \int_0^\infty t^\mu \exp(-x_2 t - x_3 t^2) J_\nu(x_1 t) dt \tag{1.5}$$

$$(x_1, x_2, x_3 \in R^+; \Re(\mu+\nu) > -1).$$

It is easy to see that

$$\Omega_{\mu,\nu}\left(x_1, x_2, \frac{1}{4}\right) = V_{\mu,\nu}(x_1, x_2). \tag{1.6}$$

Furthermore, various generalizations of the Voigt function have been introduced and investigated by a number of authors (see for details, [15, 3, 9, 4] and the references cited therein).

The classical Wright function $W_{a,b}(x_1)$ is defined by (see [8], see also [6, 10])

$$W_{a,b}(x_1) = \sum_{n\geqslant 0} \frac{1}{\Gamma(b+an)} \frac{(x_1)^n}{n!}, \tag{1.7}$$

$$(b \in \mathbb{C}, a > -1).$$

In 2015, EI-Shahed and Salem [2] introduced the following extension of above Wright function:

$$W_{a,b}^{c,d}(x_1) = \sum_{n\geqslant 0} \frac{(c)_n}{(d)_n\ \Gamma(b+an)} \frac{(x_1)^n}{n!} \tag{1.8}$$

$$(a \in \Re,\ b, c, d \in \mathbb{C}, a > -1, d \neq 0, -1, -2, \cdots, \text{ with } x_1 \in \mathbb{C} \text{ and } |x_1| < 1 \text{ with } a = -1).$$

Clearly, on setting $c = d$ in (1.8), we easily get the function given in (1.7).

Also, we have the following relation between the classical Bessel function and classical Wright function (see [5]):

$$J_\nu(x_1) = \left(\frac{x_1}{2}\right)^\nu W_{1,\nu+1}\left(-\frac{x_1^2}{4}\right)$$

or

$$W_{1,\nu+1}\left(-\frac{x_1^2}{4}\right) = \left(\frac{2}{x_1}\right)^\nu J_\nu(x_1). \tag{1.9}$$

Hence, we can also define here the relation between generalized Wright function and classical Bessel function as follows:

$$W_{1,\nu+1}^{c,c}\left(-\frac{x_1^2}{4}\right) = \left(\frac{2}{x_1}\right)^\nu J_\nu(x_1) \text{ or } W_{1,\nu+1}^{d,d}\left(-\frac{x_1^2}{4}\right) = \left(\frac{2}{x_1}\right)^\nu J_\nu(x_1). \tag{1.10}$$

In this paper, we aim to introduce a new generalization of the Voigt function associated with the generalized Wright function $W_{a,b}^{c,d}(x_1)$ given in (1.8). Also, we investigate several properties of this generalized Voigt function such as series representations, generating relations and recurrence relations.

2. Generalized Voigt function and its series representations

In this section, we introduce a new type of the generalized Voigt function and its series representations by making use of series manipulation and integral transform techniques.

Definition 2.1. *Let* $x_1, x_2, x_3 \in R^+$, $a \in \Re$, $b, c, d \in \mathbb{C}$, $a \geqslant 1$, $d \neq 0, -1, -2, \cdots$, *and* $\Re(\mu+\nu) > -1$. *Then the generalized Voigt function* $\Upsilon_{\mu,\nu}^{(a,b,c,d)}(x_1, x_2, x_3)$ *is defined by*

$$\Upsilon_{\mu,\nu}^{(a,b,c,d)}(x_1, x_2, x_3) = \left(\frac{x_1}{2}\right)^{\nu+\frac{1}{2}} \int_0^\infty t^{\mu+\nu} \exp(-x_2 t - x_3 t^2)\, W_{a,b}^{c,d}\left(-\frac{x_1^2 t^2}{4}\right) dt, \tag{2.1}$$

where $W_{a,b}^{c,d}(z)$ *is the generalized Wright function given in* (1.8).

Remark 2.2. **(i)** *If we set* $a = 1$, $b = \nu + 1$ *and* $c = d$ *in* (2.1), *and by using* (1.10), *we easily get*

$$\Upsilon_{\mu,\nu}^{(1,\nu+1,c,c)}(x_1, x_2, x_3) = \Omega_{\mu,\nu}(x_1, x_2, x_3) \text{ or } \Upsilon_{\mu,\nu}^{(1,\nu+1,d,d)}(x_1, x_2, x_3) = \Omega_{\mu,\nu}(x_1, x_2, x_3). \tag{2.2}$$

(ii) *Further, on setting* $a = 1, b = \nu + 1, c = d$ *and* $x_3 = \frac{1}{4}$ *in* (2.1), *and by using* (1.10), *we find that*

$$\Upsilon_{\mu,\nu}^{(1,\nu+1,c,c)}\left(x_1, x_2, \frac{1}{4}\right) = V_{\mu,\nu}(x_1, x_2) \text{ or } \Upsilon_{\mu,\nu}^{(1,\nu+1,d,d)}\left(x_1, x_2, \frac{1}{4}\right) = V_{\mu,\nu}(x_1, x_2). \tag{2.3}$$

(iii) *It is easy to find from* (2.2) *and* (2.3) *that*

$$\Upsilon_{\frac{1}{2},-\frac{1}{2}}^{\left(1,\frac{1}{2},c,c\right)}\left(x_1, x_2, \frac{1}{4}\right) = K(x_1, x_2) \quad \text{or} \quad \Upsilon_{\frac{1}{2},-\frac{1}{2}}^{\left(1,\frac{1}{2},d,d\right)}\left(x_1, x_2, \frac{1}{4}\right) = K(x_1, x_2), \tag{2.4}$$

and

$$\Upsilon_{\frac{1}{2},\frac{1}{2}}^{\left(1,\frac{3}{2},c,c\right)}\left(x_1, x_2, \frac{1}{4}\right) = L(x_1, x_2) \quad \text{or} \quad \Upsilon_{\frac{1}{2},\frac{1}{2}}^{\left(1,\frac{3}{2},d,d\right)}\left(x_1, x_2, \frac{1}{4}\right) = L(x_1, x_2). \tag{2.5}$$

Theorem 2.3. *Let* $x_1, x_2, x_3 \in R^+$; $b, c, d \in \mathbb{C}$, $a(\geqslant 1) \in \Re$, $d \neq 0, -1, -2, \cdots$, *and* $\Re(\mu+\nu) > -1$. *Then the generalized Voigt function in* (2.1) *has the following representation:*

$$\begin{aligned}\Upsilon_{\mu,\nu}^{(a,b,c,d)}(x_1, x_2, x_3) =& \\ =&\frac{x_1^{\nu+\frac{1}{2}}}{2^{\nu+\frac{3}{2}} x_3^A\, \Gamma(b)} \left\{ \Gamma(A) F_{0:a+1;1}^{1:1;0} \left[\begin{matrix} A: & c; & ; \\ -: & \Delta(a;b),\ d; & \frac{1}{2}; \end{matrix} \ -\frac{x_1^2}{4a^a x_3}, \ \frac{x_2^2}{4x_3} \right] - \right. \\ &\left. - \frac{x_2}{\sqrt{x_3}} \Gamma\left(A + \frac{1}{2}\right) F_{0:a+1;1}^{1:1;0} \left[\begin{matrix} A+\frac{1}{2}: & c; & -; \\ -: & \Delta(a;b),\ d; & \frac{3}{2}; \end{matrix} \ -\frac{x_1^2}{4a^a x_3}, \ \frac{x_2^2}{4x_3} \right] \right\},\end{aligned} \tag{2.6}$$

where $A = \dfrac{\mu+\nu+1}{2}$, $\Delta(a;b)$ *abbreviates the array of 'a' parameters* $\dfrac{b}{a}, \dfrac{b+1}{a}, \cdots, \dfrac{b+a-1}{a}$, *and* $F_{g:h;k}^{p:q;r}$ *denotes the well-known Kampé de Fériet function (see [14, p.63, eq.(16)]).*

Proof. Expressing the exponential function $\exp(-x_2 t)$ and generalized Wright function $W_{a,b}^{c,d}\left(-\frac{x_1^2 t^2}{4}\right)$ in their respective series on the right-hand side of (2.1), and interchanging the order of summations and integration, which is guaranteed under the conditions, we get

$$\Upsilon_{\mu,\nu}^{(a,b,c,d)}(x_1,x_2,x_3) = \\ = \left(\frac{x_1}{2}\right)^{\nu+\frac{1}{2}} \sum_{n\geqslant 0}\sum_{m\geqslant 0} \frac{(c)_n}{(d)_n\ \Gamma(b+an)} \frac{\left(-\frac{x_1^2}{4}\right)^n}{n!} \frac{(-x_2)^m}{m!} \int_0^\infty t^{\mu+\nu+2n+m}\ e^{-x_3 t^2} dt. \tag{2.7}$$

It is easy to see from the Euler's Gamma function that

$$\int_0^\infty t^\lambda\ e^{-x_3 t^2} dt = \frac{1}{2} x_3^{-\left(\frac{\lambda+1}{2}\right)} \Gamma\left(\frac{\lambda+1}{2}\right) \tag{2.8}$$

$$(\Re(x_3) > 0; \Re(\lambda) > -1).$$

Applying (2.8) to the integral in (2.7), we find that

$$\Upsilon_{\mu,\nu}^{(a,b,c,d)}(x_1,x_2,x_3) = \frac{x_1^{\nu+\frac{1}{2}}}{2^{\nu+\frac{3}{2}} x_3^A} \sum_{n\geqslant 0}\sum_{m\geqslant 0} \frac{(c)_n}{(d)_n\ \Gamma(b+an)} \frac{\left(-\frac{x_1^2}{4x_3}\right)^n}{n!} \frac{(-x_2)^m}{m!} \Gamma\left(A+n+\frac{m}{2}\right)(x_3)^{-\frac{m}{2}}.$$

Now separating the m-series into its even and odd terms, and by using the result (see [14])

$$\Gamma(b+an) = \Gamma(b)\ a^{an} \left(\frac{b}{a}\right)_n \left(\frac{b+1}{a}\right)_n \left(\frac{b+2}{a}\right)_n \cdots \left(\frac{b+a-1}{a}\right)_n,$$

we arrive at

$$\Upsilon_{\mu,\nu}^{(a,b,c,d)}(x_1,x_2,x_3) = \frac{x_1^{\nu+\frac{1}{2}}}{2^{\nu+\frac{3}{2}} x_3^A\ \Gamma(b)} \times \\ \times \left\{ \Gamma(A) \sum_{n\geqslant 0}\sum_{m\geqslant 0} \frac{(A)_{n+m}\ (c)_n}{\left(\frac{b}{a}\right)_n \left(\frac{b+1}{a}\right)_n \cdots \left(\frac{b+a-1}{a}\right)_n (d)_n \left(\frac{1}{2}\right)_m} \frac{\left(-\frac{x_1^2}{4a^a x_3}\right)^n}{n!} \frac{\left(\frac{x_2^2}{4x_3}\right)^m}{m!} - \right. \\ \left. - \frac{x_2}{\sqrt{x_3}} \Gamma\left(A+\frac{1}{2}\right) \sum_{n\geqslant 0}\sum_{m\geqslant 0} \frac{\left(A+\frac{1}{2}\right)_{n+m} (c)_n}{\left(\frac{b}{a}\right)_n \left(\frac{b+1}{a}\right)_n \cdots \left(\frac{b+a-1}{a}\right)_n (d)_n \left(\frac{3}{2}\right)_m} \frac{\left(-\frac{x_1^2}{4a^a x_3}\right)^n}{n!} \frac{\left(\frac{x_2^2}{4x_3}\right)^m}{m!} \right\}, \tag{2.9}$$

which, upon using the definition of Kampé de Fériet function [14, p. 63, eq. (16)], yields our claimed representation. □

Theorem 2.4. *Let* $q,\ w,\ x_3,\ x_3 - s - t + \frac{x_1 t}{s} \in R^+;\ b, c, d \in \mathbb{C},\ a(\geqslant 1) \in \Re,\ d \neq 0, -1, -2, \cdots,$ *and* $\Re(\mu+\nu) > -1$*. Then the generalized Voigt function in* (2.1) *with a slightly changed variable has the following representation:*

$$\Upsilon_{\mu,\nu}^{(a,b,c,d)}\left(q, w, x_3 - s - t + \frac{x_1 t}{s}\right) = \frac{q^{\nu+\frac{1}{2}}}{2^{\nu+\frac{3}{2}} x_3^A\ \Gamma(b)} \sum_{i=-\infty}^{\infty}\sum_{j=0}^{\infty} \frac{\left(\frac{s}{x_3}\right)^i}{i!} \frac{\left(\frac{t}{x_3}\right)^j}{j!} \left\{ \Gamma(A+i+j) \times \right. \tag{2.10}$$

$$\times F^{(3)} \left[\begin{matrix} A+i+j :: -;\ -;\ - : & c;\ -;\ -j; \\ - :: -;\ -;\ - : & \Delta(a;b),\ d;\ \frac{1}{2};\ i+1; \end{matrix}\ -\frac{q^2}{4a^a x_3}, \frac{w^2}{4x_3}, \frac{x_1}{x_3} \right] - \frac{w}{\sqrt{x_3}} \Gamma\left(A+i+j+\frac{1}{2}\right) \times$$

$$\times F^{(3)}\left[\begin{array}{c} A+i+j+\frac{1}{2} :: -;\ -;\ -: \quad c;\ -;\ -j; \\ - \quad :: -;\ -;\ -: \quad \Delta(a;b),\ d;\ \frac{3}{2};\ i+1; \end{array} -\frac{q^2}{4a^a x_3},\ \frac{w^2}{4x_3},\ \frac{x_1}{x_3}\right]\Bigg\},$$

where $A=\dfrac{\mu+\nu+1}{2}$ *and* $F^{(3)}[x_1,x_2,x_3]$ *denotes the well-known Srivastava's triple hypergeometric series (see [14, p. 69, eq. (39)]).*

Proof. We begin by recalling the following known result given by Srivastava et al. [16, p. 8, eq. (1.3)]:

$$\exp\left(s+t-\frac{x_1 t}{s}\right)=\sum_{i=-\infty}^{\infty}\sum_{j\geqslant 0}\frac{s^i}{i!}\frac{t^j}{j!}\ {}_1F_1[-j;\ i+1;\ x_1], \tag{2.11}$$

where ${}_1F_1[\ \alpha;\ \beta;\ x_1]$ is the confluent hypergeometric function (see [12, p. 123, eq. (1)]).

On replacing s,t and x_1 by $s\eta^2, t\eta^2$ and $x_1\eta^2$, respectively, and multiplying both sides of the resulting identity by $\eta^{\mu+\nu}\ \exp(-w\eta-x_3\eta^2)\ W_{a,b}^{c,d}\left(-\dfrac{q^2\eta^2}{4}\right)$, and integrating both sides of the last resulting identity with respect to η from 0 to ∞, we obtain

$$\begin{aligned}\int_0^\infty \eta^{\mu+\nu}\exp\left[-w\eta-\left(x_3-s-t+\frac{x_1 t}{s}\right)\eta^2\right] W_{a,b}^{c,d}\left(-\frac{q^2\eta^2}{4}\right)d\eta=\sum_{i=-\infty}^{\infty}\sum_{j\geqslant 0}\frac{s^i}{i!}\frac{t^j}{j!}\times \\ \times\int_0^\infty \eta^{\mu+\nu+2i+2j}\ \exp(-w\eta-x_3\eta^2)\ W_{a,b}^{c,d}\left(-\frac{q^2\eta^2}{4}\right){}_1F_1[\ -j;\ i+1;\ x_1\eta^2]d\eta.\end{aligned} \tag{2.12}$$

On comparing (2.1) and (2.12), we get

$$\begin{aligned}\Upsilon_{\mu,\nu}^{(a,b,c,d)}\left(q,w,x_3-s-t+\frac{x_1 t}{s}\right)=\left(\frac{q}{2}\right)^{\nu+\frac{1}{2}}\sum_{i=-\infty}^{\infty}\sum_{j\geqslant 0}\frac{s^i}{i!}\frac{t^j}{j!}\times \\ \times\int_0^\infty \eta^{\mu+\nu+2i+2j}\ \exp(-w\eta-x_3\eta^2)\ W_{a,b}^{c,d}\left(-\frac{q^2\eta^2}{4}\right){}_1F_1[\ -j;\ i+1;\ x_1\eta^2]d\eta.\end{aligned} \tag{2.13}$$

Now using the series representations of exponential function $\exp(-w\eta)$ and generalized Wright function $W_{a,b}^{c,d}\left(-\dfrac{q^2\eta^2}{4}\right)$ and then by applying the following known results [1, p. 337, eq. (9)]:

$$\int_0^\infty x_1^{s-1}e^{-\alpha x_1^2}{}_1F_1[a;\ b;\ \beta x_1^2]dx_1=\frac{1}{2}\ \alpha^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right){}_2F_1\left[a,\ \frac{s}{2};\ b;\ \frac{\beta}{\alpha}\right]$$

$$(\Re(s)>0;\Re(\alpha)>\max\{0,\Re(\beta)\}),$$

we arrive at

$$\begin{aligned}\Upsilon_{\mu,\nu}^{(a,b,c,d)}\left(q,w,x_3-s-t+\frac{x_1 t}{s}\right)=\frac{q^{\nu+\frac{1}{2}}}{2^{\nu+\frac{3}{2}}\ x_3^A}\sum_{i=-\infty}^{\infty}\sum_{j\geqslant 0}\frac{\left(\frac{s}{x_3}\right)^i}{i!}\frac{\left(\frac{t}{x_3}\right)^j}{j!}\sum_{k=0}^{\infty}\frac{(-w)^k}{k!}\times \\ \times\sum_{l=0}^{\infty}\frac{(c)_l}{(d)_l\Gamma(b+al)}\frac{\left(-\frac{q^2}{4x_3}\right)^l}{l!}\frac{1}{x_3^{\frac{k}{2}}}\Gamma\left(A+i+j+l+\frac{k}{2}\right){}_2F_1\left[-j,\ A+i+j+l+\frac{k}{2};\ i+1;\frac{x_1}{x_3}\right].\end{aligned} \tag{2.14}$$

Now expanding ${}_2F_1$ in its defining series (see [14, p. 29, eq. (4)]), and separating the resulting series into even and odd terms with respect to the summation index k, and arranging the last resulting multiple series into the Srivastava's triple hypergeometric series $F^{(3)}[x_1,x_2,x_3]$, we arrive at the right-hand side of (2.10). This completes the proof. □

3. Generating relations

Here, by using the results given in the previous section, we derive some interesting generating relations.

Theorem 3.1. *Let* $q, w, x_3, x_3 - s - t + \frac{x_1 t}{s} \in R^+;\ b, c, d \in \mathbb{C},\ a(\geqslant 1) \in \Re,\ d \neq 0, -1, -2, \cdots,$ *and* $\Re(\mu+\nu) > -1$. *Then the following generating relation holds true:*

$$\left(\frac{x_3}{Z}\right)^A \left\{ \Gamma(A) F^{1:1;0}_{0:a+1;1} \left[\begin{matrix} A: & c; & -; \\ -: & \Delta(a;b),\ d; & \frac{1}{2}; \end{matrix} \ -\frac{q^2}{4a^a Z},\ \frac{w^2}{4Z} \right] - \frac{w}{\sqrt{Z}} \Gamma\left(A+\frac{1}{2}\right) \times \right. \tag{3.1}$$

$$\left. \times F^{1:1;0}_{0:a+1;1} \left[\begin{matrix} A+\frac{1}{2}: & c; & -; \\ -: & \Delta(a;b),\ d; & \frac{3}{2}; \end{matrix} \ -\frac{q^2}{4a^a Z},\ \frac{w^2}{4Z} \right] \right\} = \sum_{i=-\infty}^{\infty} \sum_{j\geqslant 0} \frac{(\frac{s}{x_3})^i}{i!} \frac{(\frac{t}{x_3})^j}{j!} \times$$

$$\times \left\{ \Gamma(A+i+j) F^{(3)} \left[\begin{matrix} A+i+j :: -;\ -;\ -: & c;\ -;\ -j; \\ - :: -;\ -;\ -: & \Delta(a;b),\ d;\ \frac{1}{2};\ i+1; \end{matrix} \ -\frac{q^2}{4a^a x_3}, \frac{w^2}{4x_3}, \frac{x_1}{x_3} \right] - \frac{w}{\sqrt{x_3}} \times \right.$$

$$\left. \times\ \Gamma\left(A+i+j+\frac{1}{2}\right) F^{(3)} \left[\begin{matrix} A+i+j+\frac{1}{2} :: -;\ -;\ -: & c;\ -;\ -j; \\ - :: -;\ -;\ -: & \Delta(a;b),\ d;\ \frac{3}{2};\ i+1; \end{matrix} \ -\frac{q^2}{4a^a x_3}, \frac{w^2}{4x_3}, \frac{x_1}{x_3} \right] \right\},$$

where $Z = x_3 - s - t + \dfrac{x_1 t}{s}$, $F^{p:q;r}_{g:h;k}$ *is the Kampé de Fériet function [14, p. 63, eq. (16)] and* $F^{(3)}[x_1, x_2, x_3]$ *is the Srivastava's triple hypergeometric series [14, p. 69, eq. (39)].*

Proof. Expanding the left-hand side of (2.10) with the aid of (2.6) is seen to prove the result here. □

Corollary 3.2. *Let the conditions of Theorem 3.1 be satisfied. Then the following generating relation holds true:*

$$\left(\frac{x_3}{Z}\right)^A \left\{ \Gamma(A)\ {}_1F_1\left[A; \frac{1}{2}; \frac{w^2}{4Z}\right] - \frac{w}{\sqrt{Z}} \Gamma\left(A+\frac{1}{2}\right)\ {}_1F_1\left[A+\frac{1}{2}; \frac{3}{2}; \frac{w^2}{4Z}\right] \right\} =$$

$$= \sum_{i=-\infty}^{\infty} \sum_{j\geqslant 0} \frac{(\frac{s}{x_3})^i}{i!} \frac{(\frac{t}{x_3})^j}{j!} \left\{ \Gamma(A+i+j)\ \Psi_1\left[A+i+j,\ -j;\ i+1,\ \frac{1}{2};\ \frac{x_1}{x_3},\ \frac{w^2}{4x_3}\right] - \right. \tag{3.2}$$

$$\left. -\frac{w}{\sqrt{x_3}} \Gamma\left(A+i+j+\frac{1}{2}\right)\ \Psi_1\left[A+i+j+\frac{1}{2},\ -j;\ i+1,\ \frac{3}{2};\ \frac{x_1}{x_3},\ \frac{w^2}{4x_3}\right] \right\},$$

where ${}_1F_1[\alpha;\beta;x_1]$ *is the confluent hypergeometric function of one variable [12, p.123, eq.(1)] and* $\Psi_1[\alpha,\beta;\gamma,\delta;x_1,x_3]$ *is the confluent hypergeometric function of two variables [14, p.59, eq.(41)].*

Proof. Taking $q \to 0$ in (3.1) is seen to yield the desired result (3.2). □

Corollary 3.3. *Let the condition of Theorem 3.1 be satisfied. Then the following generating relation holds true:*

$$\left(\frac{x_3}{Z}\right)^A {}_2F_{a+1} \left[\begin{matrix} A,\ c; \\ \Delta(a;b),\ d; \end{matrix} \ -\frac{q^2}{4a^a Z} \right] = \sum_{i=-\infty}^{\infty} \sum_{j\geqslant 0} \frac{(\frac{s}{x_3})^i}{i!} \frac{(\frac{t}{x_3})^j}{j!} (A)_{i+j} \times$$

$$\times\ F^{1:1;1}_{0:a+1;1} \left[\begin{matrix} A+i+j: & c;\ -j; \\ -: & \Delta(a;b),\ d;\ i+1; \end{matrix} \ -\frac{q^2}{4a^a x_3},\ \frac{x_1}{x_3} \right], \tag{3.3}$$

where ${}_2F_{a+1}$ denotes the generalized hypergeometric function [14, p. 42, eq. (1)].

Proof. This corollary can be established with the help of (3.1) by putting $w = 0$. □

Corollary 3.4. *Let the condition of Theorem 3.1 be satisfied. Then we have:*

$$\left(\frac{x_3}{Z}\right)^A {}_2F_{a+1}\left[\begin{matrix} A,\ c; \\ \Delta(a;b),\ d; \end{matrix} \quad -\frac{q^2}{4a^a Z}\right] = \sum_{i=-\infty}^{\infty}\sum_{j\geqslant 0} \frac{(\frac{s}{x_3})^i}{i!}\frac{(\frac{t}{x_3})^j}{j!}(A)_{i+j} \times \times {}_2F_{a+1}\left[\begin{matrix} A+i+j,\ c; \\ \Delta(a;b),\ d; \end{matrix} \quad -\frac{q^2}{4a^a x_3}\right]. \tag{3.4}$$

Proof. On setting $x_1 = 0$ in (3.3), we easily get our claimed result(3.4). □

4. Recurrence relations

In this section, we establish the following recurrence relations for our introduced Voigt function.

Theorem 4.1. *The following recurrence relations for our generalized Voigt function $\Upsilon_{\mu,\nu}^{(a,b,c,d)}(x_1,x_2,x_3)$ holds true:*

$$\Upsilon_{\mu,\nu}^{(a,b,c,c+2)} + c\ \Upsilon_{\mu,\nu}^{(a,b,c+1,c+1)} - (c+1)\ \Upsilon_{\mu,\nu}^{(a,b,c,c+1)} = 0 \tag{4.1}$$

and

$$\Upsilon_{\mu,\nu}^{(a,b-1,c,d)} + (1-b)\ \Upsilon_{\mu,\nu}^{(a,b,c,d)} + \frac{ac}{4d}x_1^2\ \Upsilon_{\mu+2,\nu}^{(a,a+b,c+1,d+1)} = 0. \tag{4.2}$$

Proof. We have the following recurrence relation of the generalized Wright function (see [2, p. 8, eq. (72)]):

$$W_{a,b}^{c,c+2}(z) + c\ W_{a,b}^{c+1,c+1}(z) = (c+1)\ W_{a,b}^{c,c+1}(z). \tag{4.3}$$

From above relation, we can easily arrive at

$$\left(\frac{x_1}{2}\right)^{\nu+\frac{1}{2}} \int_0^\infty t^{\mu+\nu}\exp(-x_2t - x_3t^2)\ W_{a,b}^{c,c+2}\left(-\frac{x_1^2t^2}{4}\right)dt + + c\left(\frac{x_1}{2}\right)^{\nu+\frac{1}{2}} \int_0^\infty t^{\mu+\nu}\exp(-x_2t - x_3t^2) \times W_{a,b}^{c+1,c+1}\left(-\frac{x_1^2t^2}{4}\right)dt = = (c+1)\left(\frac{x_1}{2}\right)^{\nu+\frac{1}{2}} \int_0^\infty t^{\mu+\nu}\exp(-x_2t - x_3t^2)\ W_{a,b}^{c,c+1}\left(-\frac{x_1^2t^2}{4}\right)dt. \tag{4.4}$$

By applying (2.1) in (4.4), we receive our needed result (4.1).

Similarly, the other recurrence relation (4.2) can be established with the help of the following recurrence relation of $W_{a,b}^{c,d}(z)$ (see [2, p. 9, eq. (74)]):

$$W_{a,b-1}^{c,d}(z) + (1-b)W_{a,b}^{c,d}(z) = \frac{ac}{d}z\ W_{a,a+b}^{c+1,d+1}(z).$$

□

5. Concluding remarks

In the present study, we have defined a new type of the generalized Voigt function by making use of the generalized Wright function. We have also studied various interesting and useful properties (for example, series representations involving Kampé de Fériet function $F^{p:q;r}_{g:h;k}$ and Srivastava's triple hypergeometric series $F^{(3)}[x_1, x_2, x_3]$, generating relations and recurrence relations) of our proposed Voigt function.

In this section, we shortly discuss about two interesting variations in the integral representation of our introduced Voigt function $\Upsilon^{(a,b,c,d)}_{\mu,\nu}$.

The generalized Wright function $W^{c,d}_{a,b}(z)$ have the undermentioned relations with the Fox H-Function $H^{m,n}_{r,s}$ and Fox Wright hypergeometric function ${}_p\Psi_q$ (see [2, p.4])):

$$W^{c,d}_{a,b}(z) = \frac{\Gamma(d)}{\Gamma(c)} H^{1,1}_{1,3}\left[-z \Bigg/ \begin{matrix} (1-c,1) \\ (0,1),\ (1-b,a),\ (1-d,1) \end{matrix} \right] \tag{5.1}$$

and

$$W^{c,d}_{a,b}(z) = \frac{\Gamma(d)}{\Gamma(c)} {}_1\Psi_2\left[\begin{matrix} (c,1); \\ (d,1),\ (b,a); \end{matrix}\ z \right]. \tag{5.2}$$

Therefore, by using (5.1) and (5.2), we can propose two interesting variations in the integral representation of our generalized Voigt function $\Upsilon^{(a,b,c,d)}_{\mu,\nu}$ as follows:

$$\Upsilon^{(a,b,c,d)}_{\mu,\nu}(x_1, x_2, x_3) = \frac{\Gamma(d)}{\Gamma(c)} \left(\frac{x_1}{2}\right)^{\nu+\frac{1}{2}} \times$$
$$\times \int_0^\infty t^{\mu+\nu} \exp(-x_2 t - x_3 t^2) H^{1,1}_{1,3}\left[\frac{x_1^2 t^2}{4} \Bigg/ \begin{matrix} (1-c,1) \\ (0,1),\ (1-b,a),\ (1-d,1) \end{matrix} \right] dt \tag{5.3}$$

and

$$\Upsilon^{(a,b,c,d)}_{\mu,\nu}(x_1, x_2, x_3) =$$
$$= \frac{\Gamma(d)}{\Gamma(c)} \left(\frac{x_1}{2}\right)^{\nu+\frac{1}{2}} \int_0^\infty t^{\mu+\nu} \exp(-x_2 t - x_3 t^2)\, {}_1\Psi_2\left[\begin{matrix} (c,1); \\ (d,1),\ (b,a); \end{matrix}\ -\frac{x_1^2 t^2}{4} \right] dt. \tag{5.4}$$

The authors declare that there is no conflict of interest.
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Об обобщенной функции Фойгта и связанных с ней свойствах

Ульфат Ансари
Мушарраф Али
Факультет математики, колледж Ганди Фаиз-И-Аам
Шахджаханпур-242001, Индия
Университет Махатмы Джотибы Пхуле Рохилкханда
Барейли-243006, Индия
Мохд Гаясуддин
Факультет математики, Интегральный университет
Центр Шахджаханпур-242001, Индия

Аннотация. В настоящей статье мы стремимся представить новый тип обобщенной функции Фойгта и исследовать ее рядовые представления. Используя рядовые представления нашей функции, мы также указываем некоторые порождающие соотношения, связанные с функцией Кампе де Ферието, тройным гипергеометрическим рядом Шриваставы, конфлюэнтными гипергеометрическими функциями одной и двух переменных и обобщенной гипергеометрической функцией. Кроме того, также указаны два интересных рекуррентных соотношения нашей введенной функции Фойгта.

Ключевые слова: функция Фойгта, функция Райта, функция Кампе де Ферье, тройной гипергеометрический ряд Шриваставы.