EDN: YBTFJT
УДК 517.5

Finding Power Sums of Zeros of an Entire Function of Finite Order of Growth

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Received 10.11.2023, received in revised form 30.12.2023, accepted 04.03.2024

Abstract. Formulas are given for finding power sums of zeros to a negative power for entire functions of finite order of growth.

Keywords: power sum of zeros, entire function of finite order of growth.

Citation: D. Djumabaev, Finding Power Sums of Zeros of an Entire Function of Finite Order of Growth, J. Sib. Fed. Univ. Math. Phys., 2024, 17(3), 427–430. EDN: YBTFJT.

Let the function $f(z)$ be of the form

$$f(z) = 1 + \sum_{k=1}^{\infty} b_k z^k, \quad b_0 = 1,$$

is an entire function of finite growth order and having zeros of the numbers
$\alpha_1,\ \alpha_2,\ \ldots, \alpha_n, \ldots$ (each root is counted as many times as its multiplicity). There can be a finite or infinite number of zeros. We will arrange them in ascending order of modules $0 < |\alpha_1| \leqslant |\alpha_2| \leqslant \ldots \leqslant |\alpha|_n \leqslant \ldots$.

Let us recall the Hadamard expansion for such functions (see, for example, [1, Chapter 8, Theorem 8.2.4], [2, Chapter 7]).

Theorem 1. *If $f(z)$ is an entire function of finite order ρ, then*

$$f(z) = z^s e^{Q(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\alpha_n}\right) e^{\frac{z}{\alpha_n} + \frac{z^2}{2\alpha_n^2} + \ldots + \frac{z^p}{p\alpha_n^p}}, \tag{1}$$

where $Q(z)$ is a polynomial whose degree q is not higher than ρ, s is the multiplicity of zero f at point 0 and the number $p \leqslant \rho$.

The infinite product in (1) converges absolutely and uniformly in $\mathbb{C}$. (Recall that a certain sequence of holomorphic functions converges uniformly in the open set U, if it converges uniformly on every compact set in U.)

In what follows we assume that $f(0) = 1$. We will write the polynomial $Q(z)$ in the form

$$Q(z) = \sum_{j=1}^{q} d_j z^j.$$

Here $d_0 = 0$, since $f(0) = 1$.

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Expression

$$\Phi(z)=\prod_{n=1}^{\infty}\left(1-\frac{z}{\alpha_n}\right)e^{\frac{z}{\alpha_n}+\frac{z^2}{2\alpha_n^2}+\ldots+\frac{z^p}{p\alpha_n^p}} \tag{2}$$

is called *the canonical product*, and the number p is *the genus of the canonical product. The genus of the entire function* $f(z)$ is the number $\max\{q,p\}$. If we denote by ρ' — the order of the canonical product (2), then $\rho=\max\{q,\rho'\}$.

Consider the series

$$\sum_{n=1}^{\infty}\frac{1}{|\alpha_n|^{\gamma}}. \tag{3}$$

The infimum of positive numbers γ for which the series (3) converges is called the *index of convergence* of zeros of the canonical product $\Phi(z)$.

It is well known (see, for example, [1, Chapter 8, paragraph 8.2.5], [2, Chapter 7]) that the index of convergence of zeros of a canonical product is equal to its order ρ' .

Therefore *power sums of zeros to a negative power*

$$\sigma_k=\sum_{n=1}^{\infty}\frac{1}{\alpha_n^k},\quad k\in\mathbb{N},$$

are absolutely convergent series for $k>\rho'$, i.e. and for $k>\rho$. It is also known that $\rho'-1\leqslant p\leqslant\rho'$ (see, for example, [1]).

In what follows, we will consider power sums with positive integer exponents k.

For polynomials, the recurrent formulas of Newton and Waring are well known, connecting the usual power sums of the roots of a polynomial and its coefficients (see, for example, [3–5]).

Now we will connect the integrals in the formula

$$\int_{\gamma_r}\frac{1}{z^k}\frac{df}{f} \tag{4}$$

and power sums of zeros σ_k. Here

$$\gamma_r=\{z:\ |z|=r\},r>0\},$$

Let's express this integral in terms of power sums of zeros using Hadamard's formula. Let us restrict ourselves to the case when $s=0$.

In a sufficiently small neighborhood of zero we have (by Hadamard's formula (1))

$$\varphi(z)=\ln f(z)=Q(z)+\sum_{n=1}^{\infty}\ln\left[\left(1-\frac{z}{\alpha_n}\right)e^{P_n(z)}\right],$$

where $P_n(z)=\dfrac{z}{\alpha_n}+\dfrac{z^2}{2\alpha_n^2}+\ldots+\dfrac{z^p}{p\alpha_n^p}$.

The series for $\varphi(z)$ converges uniformly and absolutely in a sufficiently small neighborhood of zero, since the zeros of α_j are delimited from zero.

Let's find the integrals in (4) for each term. Obviously

$$\frac{1}{2\pi i}\int_{\gamma_r}\frac{1}{z^k}\cdot dQ(z)=\begin{cases}kd_k & \text{при}\quad 1\leqslant k\leqslant q,\\ 0 & \text{при}\quad k>q.\end{cases}$$

Let's transform the expression

$$d\ln\left[\left(1-\frac{z}{\alpha_n}\right)e^{P_n(z)}\right]=\frac{d\left[\left(1-\frac{z}{\alpha_n}\right)e^{\frac{z}{\alpha_n}+\frac{z^2}{2\alpha_n^2}+\cdots+\frac{z^p}{p\alpha_n^p}}\right]}{\left(1-\frac{z}{\alpha_n}\right)e^{\frac{z}{\alpha_n}+\frac{z^2}{2\alpha_n^2}+\cdots+\frac{z^p}{p\alpha_n^p}}}=$$

$$= \frac{d\left(1-\frac{z}{\alpha_n}\right)e^{\frac{z}{\alpha_n}+\frac{z^2}{2\alpha_n^2}+\cdots+\frac{z^p}{p\alpha_n^p}} + \left(1-\frac{z}{\alpha_n}\right)e^{\frac{z}{\alpha_n}+\frac{z^2}{2\alpha_n^2}+\cdots+\frac{z^p}{p\alpha_n^p}} d\left(\frac{z}{\alpha_n}+\frac{z^2}{2\alpha_n^2}+\cdots+\frac{z^p}{p\alpha_n^p}\right)}{\left(1-\frac{z}{\alpha_n}\right)e^{\frac{z}{\alpha_n}+\frac{z^2}{2\alpha_n^2}+\cdots+\frac{z^p}{p\alpha_n^p}}} =$$

$$= \frac{d\left(1-\frac{z}{\alpha_n}\right)}{\left(1-\frac{z}{\alpha_n}\right)} + d\left(\frac{z}{\alpha_n}+\frac{z^2}{2\alpha_n^2}+\cdots+\frac{z^p}{p\alpha_n^p}\right) =$$

$$= \frac{dz}{z-\alpha_n} + \left(\frac{1}{\alpha_n}+\frac{z}{\alpha_n^2}+\cdots+\frac{z^{p-1}}{\alpha_n^p}\right)dz =$$

$$= \frac{dz}{z-\alpha_n} + \frac{1}{\alpha_n}\left[\frac{\left(\frac{z^p}{\alpha_n^p}-1\right)}{\left(\frac{z}{\alpha_n}-1\right)}\right]dz = \frac{dz}{z-\alpha_n} + \frac{(z^p-\alpha_n^p)dz}{\alpha_n^{p-1}(z-\alpha_n)} = \frac{z^p dz}{\alpha_n^p(z-\alpha_n)}.$$

Then

$$\frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{\gamma_r}\frac{d\left[\left(1-\frac{z}{\alpha_n}\right)e^{\frac{z}{\alpha_n}+\frac{z^2}{2\alpha_n^2}+\cdots+\frac{z^p}{p\alpha_n^p}}\right]}{z^k\left(1-\frac{z}{\alpha_n}\right)e^{\frac{z}{\alpha_n}+\frac{z^2}{2\alpha_n^2}+\cdots+\frac{z^p}{p\alpha_n^p}}} = \frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{\gamma_r}\frac{z^{p-k}dz}{\alpha_n^p(z-\alpha_n)} =$$

$$= \begin{cases} 0, \quad \text{если} \quad k \leqslant p, \\ -\sum\limits_{n=1}^{\infty}\frac{1}{\alpha_n^k} = -\sigma_k, \quad \text{если} \quad k > p. \end{cases}$$

Thus the equality is true if $q \leqslant p$, then

$$\frac{1}{2\pi i}\int_{\gamma_r}\frac{1}{z^k}\frac{df}{f} = \begin{cases} kd_k \quad \text{при} \quad k \leqslant q, \\ 0 \quad \text{при} \quad q < k \leqslant p, \\ -\sigma_k \quad \text{при} \quad k > p. \end{cases}$$

Let $q > p$, then

$$\frac{1}{2\pi i}\int_{\gamma_r}\frac{1}{z^k}\frac{df}{f} = \begin{cases} kd_k \quad \text{for} \quad k \leqslant p, \\ kd_k - \sigma_k \quad \text{for} \quad p < k \leqslant q, \\ -\sigma_k \quad \text{for} \quad k > q. \end{cases}$$

Thus we obtain the statement

Theorem 2. *For $q \leqslant p$ the following equalities are true:*

$$\frac{1}{2\pi i}\int_{\gamma_r}\frac{1}{z^k}\frac{df}{f} = \begin{cases} kd_k \quad \textit{for} \quad k \leqslant q, \\ 0 \quad \textit{при} \quad q < k \leqslant p, \\ -\sigma_k \quad \textit{for} \quad k > p. \end{cases}$$

For $q > p$ the following equalities are true:

$$\frac{1}{2\pi i}\int_{\gamma_r}\frac{1}{z^k}\frac{df}{f} = \begin{cases} kd_k \quad \textit{for} \quad k \leqslant p, \\ kd_k - \sigma_k \quad \textit{for} \quad p < k \leqslant q, \\ -\sigma_k \quad \textit{for} \quad k > q. \end{cases}$$

This theorem generalizes Proposition 1.4.1 from [6].

Corollary 1. *The equality is true*

$$\frac{1}{2\pi i}\int_{\gamma_r}\frac{1}{z^k}\frac{df}{f} = -\sigma_k, \quad \text{если} \quad k > \rho.$$

Corollary 2. *The formulas are valid*

$$\sigma_k = -\frac{(-1)^{k-1}}{b_0^k}\begin{vmatrix} b_1 & b_0 & 0 & \dots & 0 \\ 2b_2 & b_1 & b_0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ kb_k & b_{k-1} & b_{k-2} & \dots & b_1 \end{vmatrix} \quad \text{при} \quad k > \rho. \tag{5}$$

To prove it, it is enough to multiply the second column in the formula (5) by b_1, the third by b_2, etc., then add them to the first column.

These formulas relate the power sums σ_k and the Taylor coefficients of the function f.

Example 1. Consider the function

$$f(z) = \cos z \cdot e^z.$$

This is a first order growth function ($\rho = 1$). Then

$$f(z) = \cos z \cdot e^z = 1 + z + \frac{z^3}{3!} + \dots.$$

Then from (5) we obtain $\sigma_2 = 1$, which corresponds to the known equalities.

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Нахождение степенных сумм нулей целых функций конечного порядка роста

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Аннотация. Получены формулы для нахождения степенных сумм нулей в отрицательной степени целых функций конечного порядка роста.

Ключевые слова: степенные суммы нулей, целая функция конечного порядка роста.