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Difference Equations and Hadamard Composition of Multiple Power Series

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Abstract. The sufficient conditions are given for the coefficients of two power series, which ensure the rationality of their Hadamard composition. Under certain additional constraints, the existence of a system of polynomial difference equations, satisfied by the coefficients of the composition, is proven.

Keywords: systems of polynomial linear equations with constant coefficients, multiple power series, Hadamard composition.

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Introduction

Let $\mathbb{Z}^n_{\geqslant 0}$ denote the set of vectors with integer non-negative coordinates, and let $\varphi(x), \psi(x) : \mathbb{Z}^n_{\geqslant 0} \to \mathbb{C}$ be functions with integer arguments, where $\mathbb{C}$ is the set of complex numbers. For power series,

$$\Phi(z) = \sum_{x\in\mathbb{Z}^n_{\geqslant 0}} \varphi(x) z^x \text{ and } \Psi(z) = \sum_{x\in\mathbb{Z}^n_{\geqslant 0}} \psi(x) z^x$$

the Hadamard composition of these power series is defined as

$$H(z) = \sum_{x\in\mathbb{Z}^n_{\geqslant 0}} \varphi(x)\psi(x) z^x. \tag{1}$$

For $n = 1$, the Hadamard theorem on multiplication of singularities states that the singular points of the composition $H(z)$ are given by the products of the singular points of the functions Φ and Ψ (see [1]), and the main tool for investigation is the integral representation, in which the composition is expressed in terms of Φ and Ψ. Note that if $\Phi(z)$ and $\Psi(z)$ are rational functions, direct calculation of the integral shows that the composition is also a rational function. However, for $n > 1$, this is no longer the case.

Example 1. $\Phi(z_1 z_2) = \dfrac{1}{1 - z_1 - z_2} = \sum\limits_{(k_1,k_2)\in\mathbb{Z}^2_{\geqslant 0}} \dfrac{(k_1+k_2)!}{k_1!k_2!} z_1^{k_1} z_2^{k_2}$, $\Psi(z_1 z_2) = \dfrac{1}{1 - z_1 z_2} = \sum\limits_{k=0}^{\infty} z_1^k z_2^k$, *then* $H(z_1 z_2) = \sum\limits_{k=0}^{\infty} \dfrac{(2k)!}{(k!)^2} (z_1 z_2)^k = \dfrac{1}{\sqrt{1 - 4 z_1 z_2}}$.

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We are interested in the question of the classes of rational functions whose Hadamard composition is a rational function (see, for example, [2]).

This paper considers the case when the coefficients of the series $\Phi(z)$ and $\Psi(z)$ satisfy systems of polynomial difference equations with constant coefficients. The main role here is played by the multidimensional analogue of the fundamental theorem of difference equations with constant coefficients [3].

Moreover, from the point of view of enumerative combinatorial analysis, the question of the system of difference equations satisfied by the product of the coefficients $\varphi(x)\psi(x)$ of the Hadamard composition of the series $\Phi(x)$ and $\Psi(x)$ is of interest.

We provide the necessary definitions and notations and formulate the main results.

Let δ_j be the shift operator with respect to the variable x_j

$$\delta_j f(x) = \delta_j f(x_1, \ldots, x_n) = f(x_1, \ldots, x_{j+1}, \ldots, x_n),$$
$$\delta = (\delta_1, \ldots, \delta_n),\ \delta^\alpha = \delta_1^{\alpha_1} \cdots \delta_n^{\alpha_n},\ \alpha \in \mathbb{Z}^n_{\geqslant 0}.$$

Consider a polynomial difference operator with constant coefficients of the form

$$Q(\delta) = \sum_{0 \leqslant \alpha \leqslant d} c_\alpha \delta^{d-\alpha}, \tag{2}$$

where $c_\alpha \in \mathbb{C}$ are some constants, and the notation $\alpha \geqslant \beta$ for multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ means that $\alpha_j \geqslant \beta_j,\ j = 1, 2, \ldots, n$.

The characteristic polynomial for the difference equation

$$Q(\delta)f(x) = 0,\ x \in \mathbb{Z}^n_{\geqslant 0} \tag{3}$$

is defined as the polynomial

$$\sum_{0 \leqslant \alpha \leqslant d} c_\alpha \delta^{d-\alpha} = Q(z), \tag{4}$$

where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n,\ z^\alpha = z_1^{\alpha_1} \cdot \cdots \cdot z_n^{\alpha_n},\ C_0 = 1,\ C_d \neq 0$.

The zeros of the polynomial Q are called characteristic roots, and the set

$$V = \{z \in \mathbb{C}^n : Q(z) = 0\}$$

of all these zeros of Q is called the characteristic set of the equation (3).

Let us consider a set of polynomials $Q = (Q_1, \ldots, Q_n)$ of the form

$$Q_i(z) = \sum_{0 \leqslant \alpha \leqslant d_i} c^i_\alpha z^{d^i - \alpha},\ i = 1, 2, \ldots, n, \tag{5}$$

where d^i are vectors from $\mathbb{Z}^n_{\geqslant 0}$. We assume that $c^i_0 = 1,\ c^i_{d^i} \neq 0$.

We denote by V_Q the set of zeros of the system of equations:

$$Q_1(z) = Q_2(z) = \ldots = Q_n(z) = 0, \tag{6}$$

which we will call the characteristic system.

In this paper, we will consider systems of difference equations of the form (3), which satisfy the following conditions:

(*) the characteristic set V_Q is discrete and the characteristic roots do not lie on coordinate planes;

(**) the roots $a \in V_Q$ of the characteristic system (6) satisfy the following properties: there exists $d_\alpha = (d_{1,\alpha}, \dots, d_{n,\alpha}) \in \mathbb{Z}^n_\geqslant$ such that:

$$\frac{\partial^\alpha Q_i}{\partial z^\alpha}(a) = 0 \ \ for \ \ 0 \leqslant \alpha \leqslant d_a - I, \ \ i = 1, \dots, n, \tag{7}$$

$$\Delta_{d_a}(z) = det \left\| \frac{\partial^{d_{l,a}} Q_i(z)}{\partial z_l^{d_{l,a}}} \right\|_{z=a} \neq 0. \tag{8}$$

In formula (8), the indices l and i take values $1, 2, \dots, n$.

For $d_a = I = (1, \dots, 1)$, these conditions are equivalent to the point $z = a$ being a simple root of the characteristic system of equations (6).

We state the main result of this paper.

Theorem. *Let $A(z) = (A_1(z), \dots, A_n(z))$ and $B(z) = (B_1(z), \dots, B_n(z))$ be two sets of polynomials and*

$$A(\delta)\varphi(x) = 0 \ and \ B(\delta)\psi(x) = 0, \ x \in \mathbb{Z}^n_\geqslant \tag{9}$$

be the corresponding systems of polynomial difference equations. If the roots of the characteristic systems V_A and V_B are discrete, do not lie on coordinate planes, and the characteristic polynomials $A(z)$ and $B(z)$ satisfy conditions (7) and (8), then:

1) *The generating function of the product $\varphi(x)\psi(x)$ of solutions to the system (9) of difference equations is rational.*

2) *If the characteristic roots of the systems of difference equations (9) are simple, then there exists a set of polynomial difference operators $R(\delta) = (R_1(\delta), \dots, R_n(\delta))$ such that the product $\varphi(x)\psi(x)$ satisfies the system of recurrence equations*

$$R_j(\delta)[\varphi(x)\psi(x)] = 0, \ j = 1, 2, \dots, n.$$

The proof and an example

The main role in proving part 1) of the theorem is played by the multidimensional version of the fundamental theorem of difference equations with constant coefficients. We state this theorem (cf. [3]).

Theorem. *Let the polynomial vector $Q(z) = (Q_1(z), \dots, Q_n(z))$ have the form $Q_j(z) = \sum_{0 \leqslant \alpha \leqslant d^i} c^i_\alpha z^{d^i - \alpha}$ and satisfy the conditions (*), (**), (7), and (8).*

For a function $f(x) = f(x_1, \dots, x_n) \to \mathbb{C}$, the following conditions are equivalent:

(i) *The generating series for $f(x)$ is a rational function of the form*

$$F(z) = \sum_{x \geqslant 0} f(x) z^x = \sum_{j=1}^m \frac{b_j(z)}{(I - \gamma_{(j)} z)^{d_{(j)}}},$$

where

$$(I - \gamma_{(j)} z)^{d_{(j)}} = (1 - \gamma_{(j),1} z_1)^{d_{(j),1}} (1 - \gamma_{(j),2} z_2)^{d_{(j),2}} \cdots (1 - \gamma_{(j),n} z_n)^{d_{(j),n}},$$

$b_j(z)$ are some polynomials of the form $\sum_{0 \leqslant \alpha < d_{(j)}} b^j_\alpha z^\alpha$, and $\gamma_{(j)}$ are the roots of the characteristic system (6).

(ii) For any $x \in \mathbb{Z}^n_{\geqslant}$, the function $f(x)$ satisfies the system of recurrence equations

$$\sum_{0 \leqslant \alpha \leqslant d^i} c^i_\alpha \delta^{d^i - \alpha} f(x) = 0, \ i = 1, 2, \dots, n, \tag{10}$$

whose characteristic roots satisfy the conditions (7) and (8).

(iii) The function $f(x)$ has the form of an exponential polynomial

$$f(x) = \sum_{j=1}^{m} P_j(x) \gamma^x_{(j)}, \tag{11}$$

where $\gamma^x_{(j)} = \gamma^{x_1}_{(j),1} \cdot \ldots \cdot \gamma^{x_n}_{(j),n}$ and $P_j(x)$ are polynomials of the form $\sum\limits_{0 \leqslant k < d_{(j)}} P^{(j)}_k x^k$.

The proof of part 2 of the theorem is based on an algorithm for constructing a system of polynomial equations given the roots (see [4]), in the case where these roots are simple. The main element of the algorithm in [4] is the following proposition.

Proposition 1. *Let $E = \{a^{(i)}\}_{i=1}^N$, where $a^{(i)} = (a^{(i)}_1, \dots, a^{(i)}_n) \in \mathbb{C}^n$ and E_i, $i = 1, 2$, is the set of zeros of the system of polynomial equations*

$$P^{(i)}_j(z) = 0, \ j = 1, 2, \dots, n,$$

where $P^{(1)}_l = P^{(2)}_l$ for $1 \leqslant l < r < n$, and the polynomials $P^{(1)}_l$ and $P^{(2)}_l$ have no common zeros. Let

$$q_j(z) = \begin{cases} P^{(1)}_j(z), & 1 \leqslant j < r, \\ P^{(1)}_j(z) P^{(2)}_j(z), & j = r, \\ P^{(1)}_r(z) P^{(2)}_j(z) + P^{(1)}_j(z) P^{(2)}_r(z), & r < j \leqslant n, \end{cases}$$

then the set of zeros of the system

$$q_j(z) = 0, \ j = 1, 2, \dots, n$$

coincides with the set $E_1 \cup E_2$.

Proof of Theorem. $\gamma_{(j)} = (\gamma_{(j),1}, \dots, \gamma_{(j),n}) \in V_A$ and $\gamma_{(j)} = (\gamma_{(j),1}, \dots, \gamma_{(j),n}) \in V_B$ are the roots of the characteristic systems $A_1(z) = \dots = A_n(z)$ and $B_1(z) = \dots = B_n(z)$, respectively. Conditions (*) and (**) of the theorem are satisfied, therefore we can use the implication $(ii) \Rightarrow (iii)$ of the multivariate version of the fundamental theorem of the theory of difference equations. For solutions $\varphi(x)$ and $\psi(x)$ of the difference equations systems, we obtain

$$\varphi(x) = \sum_j p_j(x) \gamma^x_{(j)}, \tag{12}$$

$$\psi(x) = \sum_i q_i(x) \nu^x_{(i)}, \tag{13}$$

where $\gamma^x_{(j)} = \gamma^{x_1}_{(j),1} \cdot \ldots \cdot \gamma^{x_n}_{(j),n}$, $\nu^x_{(i)} = \nu^{x_1}_{(i),1} \cdot \ldots \cdot \nu^{x_n}_{(i),n}$.

The polynomials $p_j(x)$ and $q_i(x)$ have the form

$$p_j(x) = \sum_{0 \leqslant \alpha \leqslant d} a_\alpha x^\alpha, \quad q_i(x) = \sum_{0 \leqslant \alpha \leqslant s} b_\alpha x^\alpha,$$

while the vectors $d_{(j)}$ and $s_{(i)}$ are determined by conditions (7) and (8).

Multiplying the exponential representations (12) and (13) for $\varphi(x)$ and $\psi(x)$, we obtain

$$h(x) = \varphi(x)\psi(x) = \sum_j p_j(x)\gamma_{(j)}^x \sum_i q_i(x)\nu_{(i)}^x = \sum_{j,\, i} p_j(x)q_i(x)(\gamma_{(j)}^x \cdot \nu_{(i)}^x),$$

where $(\gamma_{(j)}^x \cdot \nu_{(i)}^x) = (\gamma_{(j),1} \cdot \nu_{(i),1}, \dots, \gamma_{(j),n} \cdot \nu_{(i),n})$.

Thus $h(x)$ has an exponential representation, and due to the implication $(iii) \Rightarrow (i)$, we obtain that the generating function $H(z)$ for the product $\varphi(x) \cdot \psi(x)$ has the form

$$H(z) = \sum_{j,\, i} \frac{b_{ij}(z)}{(I - \gamma_{(j)}\nu_{(i)})^{d_{(ij)}}},$$

where $(I - \gamma_{(j)}\nu_{(i)})^{d_{(ij)}} = (1 - \gamma_{(j),1}\nu_{(i),1})^{d_{(ij)1}} \cdot \dots \cdot (1 - \gamma_{(j),n}\nu_{(i),n})^{d_{(ij)n}}$, $b_ij(z)$ are some polynomials of the form $\sum\limits_{0 \leqslant \alpha \leqslant d^{(j)} \neq s^{(i)}} b_\alpha^{ij} z^\alpha$.

To prove the second part of the theorem, we will use the fact that the roots of the characteristic systems for the difference equations $A(\delta)\varphi(x) = 0$ and $B(\delta)\psi(x) = 0$ are simple. In this case, the exponential representations for $\varphi(x)$ and $\psi(x)$ have the form

$$\varphi(x) = \sum_j p_j \gamma_{(j)}^x \text{ and } \psi(x) = \sum_i q_i \nu_{(i)}^x,$$

where p_j, q_i are constants, and the exponential representation for the product $\varphi(x) \cdot \psi(x)$ has the form $\varphi(x) \cdot \psi(x) = \sum\limits_{j,\, i} p_j q_i (\gamma_{(j)} \cdot \nu_{(i)})^x$.

This means that the difference equations system for the product $\varphi(x) \cdot \psi(x)$ (if it exists) has simple roots.

Using method from [4], we construct a system of polynomial equations

$$R_k(z) = 0,\ k = 1, 2, \dots, n,$$

whose roots are the numbers $\{\gamma_{(j)}\nu_{(i)}\}$. For polynomial difference operators $R_k(\delta)$, we have the formula:

$$R_k(\delta)((\gamma_{(j)} \cdot \nu_{(i)})^x) = (\gamma_{(j)}\nu_{(i)})^x R_k(\gamma_{(j)}\nu_{(i)}) = 0$$

for $x \in \mathbb{Z}_{\geqslant}^n$. From the multivariate version of the fundamental theorem of the theory of difference equations, by the equivalence of $(ii) \approx (iii)$, it follows that $h(x) = \varphi(x)\psi(x)$ is a solution of the difference equations system

$$R_k(z) = 0,\ k = 1, 2, \dots, n.$$

Example 2. *Let us consider two difference equations systems*

$$\begin{cases} \varphi(x_1 + 1, x_2) - \varphi(x_1, x_2) = 0 \\ \varphi(x_1, x_2 + 1) - \varphi(x_1, x_2) = 0 \end{cases} \qquad (1,1) \textit{ is the root,}$$

$$\begin{cases} \psi(x_1 + 2, x_2) - a^2\psi(x_1, x_2) = 0 \\ \psi(x_1, x_2 + 1) - a^2\psi(x_1, x_2) = 0 \end{cases} \qquad (a, a^2), (-a, a^2) \textit{ are roots.}$$

The difference equations system for the product $h(x_1, x_2) = \varphi(x_1, x_2) \cdot \psi(x_1, x_2)$ has the form

$$\begin{cases} h(x_1 + 2, x_2) - a^2 h(x_1, x_2) = 0 \\ h(x_1, x_2 + 1) - a^2 h(x_1, x_2) = 0 \end{cases} \qquad (a, a^2), (-a, a^2).$$

This is a very simple example, it can be complicated.

$$\begin{cases} \varphi(x_1+1,x_2)-\lambda_1\varphi(x_1,x_2)=0 \\ \varphi(x_1,x_2+1)-\lambda_2\varphi(x_1,x_2)=0 \end{cases} \qquad (\lambda_1,\lambda_2) \text{ is the root,}$$

$$\begin{cases} \psi(x_1+2,x_2)-a^2\psi(x_1,x_2)=0 \\ \psi(x_1,x_2+1)-b\psi(x_1,x_2)=0 \end{cases} \qquad (a,b),(-a,b) \text{ are roots.}$$

As a result, the system of difference equations for the product $h(x_1,x_2)=\varphi(x_1,x_2)\cdot\psi(x_1,x_2)$ *has the form*

$$\begin{cases} h(x_1+2,x_2)-h(x_1,x_2+1)-(\lambda_1^2a^2-\lambda_2 b)h(x_1,x_2)=0 \\ h(x_1,x_2+1)-\lambda_2 bh(x_1,x_2)=0 \end{cases}.$$

Another example:

$$\begin{cases} h(x_1+2,x_2)-Ah(x_1,x_2+1)-(\lambda_1^2a^2-A\lambda_2 b)h(x_1,x_2)=0 \\ h(x_1,x_2+1)-\lambda_2 bh(x_1,x_2)=0 \end{cases},$$

where A *is an arbitrary constant, characteristic roots for the system of difference equations* $(\lambda_1 a,\lambda_2 b),(-\lambda_1 a,b)$.

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Разностные уравнения и композиция Адамара кратных степенных рядов

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Аннотация. Приведены достаточные условия на коэффициенты двух кратных степенных рядов, которые обеспечивают рациональность композиции Адамара этих рядов, и при некоторых дополнительных ограничениях доказывается существование системы полиномиальных разностных уравнений, которой удовлетворяют коэффициенты композиции.

Ключевые слова: системы полиномиальных линейных уравнений с постоянными коэффициентами, кратные степенные ряды, композиция Адамара.