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Coincidence and Common Fixed Point Theorems for Hybrid Mappings Via C-class Function

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Abstract. In this paper, we prove common fixed point theorems for two pairs of hybrid mappings in metric spaces using the concept of C-class function and T-weak commutativity. Our Theorems generalize some well-know results.

Keywords: metric space, hybrid Mappings, C-class function.

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1. Introduction and preliminaries

Recently, many authors have provided new fixed point results for multivalued mappings in the literature by taking into account different conditions on metric spaces (see [4, 7, 8, 11, 17]).

In the present article, we prove a coincidence and common fixed points of multivalued maps via C-class functions with a self map are taken into account with ageneralized form of contraction condition.

Let (X, d) be a metric space. For $x \in X$ and $A \subset X$,we denote

$$D(x, A) = \inf\{d(x, y), y \in A\}.$$

Let $CB(X)$ be the set of all nonempty closed and bounded subsets of X.

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Let H be the Hausdorff-Pompeiu metric with respect to d defined by

$$H(A,B)=\max\left\{\sup_{a\in A}D(a,B),\sup_{b\in B}D(A,b)\right\},$$

for every $A,B\in CB(X)$.

It is well known that $(CB(X),H)$ is a metric space and if (X,d) is complete, then $(CB(X),H)$ is also complete

Let $f:X\to X$ be a single-valued mapping and $T:X\to CB(X)$ be a multi-valued mapping.

(i) A point $x\in X$ is a fixed point of f (resp. T) if $fx=x$ (resp. $x\in Tx$).

(ii) A point $x\in X$ is a coincidence point of f and T if $fx\in Tx$.

(iii) A point $x\in X$ is a common fixed point of f and T if $x=fx\in Tx$.

Lemma 1 ([12]). *If $A,B\in CB(X)$ and $k>1$, then for each $a\in A$, there exists $b\in B$ such that*

$$d(a,b)\leqslant kH(A,B). \tag{1}$$

Let $f:X\to X$ be a single-valued mapping and $T:X\to CB(X)$ be a multi-valued mapping

Definition 1 ([12]). *1) A point $x\in X$ is said to be a coincidence point of f and T if $fx\in Tx$. We denote by $C(f,T)$ the set of all coincidence points of f and T.*

2) A point $x\in X$ is a fixed point of T if $x\in Tx$.

Definition 2 ([5]). *f and T are said to be commuting in X if for all $x\in X$,*

$$fTx\in Tfx.$$

Definition 3 ([15]). *f and T are said to be weakly commuting on X if for all $x\in X$, $fTx\in CB(X)$ and*

$$H(fTx,Tfx)\leqslant D(fx,Tx).$$

Definition 4 ([13]). *f and T are said to be R-weakly commuting at $x\in X$, if*

$$fTx\in CB(X)$$

and there exists an $R>0$ such that

$$H(fTx,Tfx)\leqslant RD(fx,Tx). \tag{2}$$

Remark 1.1 ([6]). *Commuting implies weakly commuting, but the converce is not true in general.*

We defined that f and T are said to be pointwise R-weakly commuting on X if for all $x\in X$, $fTx\in CB(X)$ and (2) holds for some $R>0$.

Definition 5 ([16]). *1) f and T are said to be (IT)-commuting at $x\in X$ if*

$$fTx\subset Tfx.$$

2) A pointwise R-weakly commuting hybrid pair is not weakly compatible in general.

3) IT-commutativity of f and T at a coincidence point is more general than their weak compatibility at the same point.

4) A pointwise R-weak commutativity at a coincidence point is equivalent to (IT) commutativity at this point.

Definition 6 ([9]). *1) f is T-weakly commuting at $x \in X$ if $ffx \in Tfx$.*

2) For a hybrid pair (f,T), (IT) commuting at coincidence points implies that f is T-weakly commuting at these points.

Lemma 2. *a) If f is T-weakly commuting at $x \in X$, then $fx \in C(f,T)$.*

b) If f is T-weakly commuting at $x \in X$ and $fx = ffx$, then fx is a common fixed point of f and T.

In 2014, A. H. Ansari [2] introduced the concept of a C-class functions which covers a large class of contractive conditions.

Definition 7 ([2]). *A continuous function $F : [0,+\infty)^2 \to \mathbb{R}$ is called C-class function if for any $s,t \in [0,+\infty)^2$; the following conditions hold*

c1 $F(s,t) \leqslant s$,

c2 $F(s,t) = s$ implies that either $s = 0$ or $t = 0$.

An extra condition on F that $F(0,0) = 0$ could be imposed in some cases if required. The letter C will denote the class of all C-functions.

Example 1. *The following examples shows that the class C is nonempty:*

1. $F(s,t) = s - t$.

2. $F(s,t) = ms$, for some $m \in (0,1)$.

3. $F(s,t) = \dfrac{s}{(1+t)^r}$, for some $r \in (0,1)$.

4.$F(s,t) = \dfrac{\log(t+a^s)}{(1+t)}$, for some $a > 1$.

Let Φ denote the class of the functions $\varphi : [0,+\infty) \to [0,+\infty)$ which satisfy the following conditions:

a) φ is continuous ;

b) $\varphi(t) > 0$, $t > 0$ and $\varphi(0) \geqslant 0$.

Definition 8 ([10]). *A function $\psi : [0,+\infty) \to [0,+\infty)$ is called an altering distance function if the following properties are satisfied:*

i) ψ is non-decreasing and continuous;

ii) $\psi(t) = 0$ if and only if $t = 0$.

Let us suppose that Ψ denote the class of the altering distance functions.

Definition 9. *A tripled (ψ,φ,F) where $\psi \in \Psi$; $\varphi \in \Phi_u$ and $F \in C$ is said to be a monotone if for any $x,y \in [0,+\infty)$;*

$$x \leqslant y \text{ implies } F(\psi(x),\varphi(x)) \leqslant F(\psi(y),\varphi(y)).$$

Example 2. *Let $F(s,t) = s - t$, $\varphi(x) = \sqrt{x}$*

$$\psi(x) = \begin{cases} \sqrt{x} & if \quad 0 \leqslant x \leqslant 1 \\ x^2 & if \quad x > 1 \end{cases},$$

then (ψ,φ,F) is monotone.

Lemma 3 ([14]). *Let (X,d) be a metric space and let $\{y_n\}$ be a sequence in X such that $d(y_n,y_{n+1}) = 0$ is nonincreasing and*

$$\lim_{n\to+\infty} d(y_n,y_{n+1}) = 0.$$

If $\{y_{2n}\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following sequences tend to ε when $k \to +\infty$

$$d(x_{2n_k},x_{2m_k}), d(x_{2n_k+1},x_{2m_k}), d(x_{2n_k},x_{2m_k-1}, d(x_{2n_k+1},x_{2m_k-1}), d(x_{2n_k+1},x_{2m_k+1}), \ldots \quad (3)$$

2. Main results

In the following theorem we obtain the coincidence and common fixed point for a hybrid pair of mappings via C-class function

Theorem 2.1. *Let (X, d) be a metric space, $S, T : X \to X$ and $K, G : X \to CB(X)$ satisfying*

$$K(X) \subset T(X) \quad \text{and} \quad G(X) \subset S(X) \tag{4}$$

$$\psi\left(rH(Kx, Gy)\right) \leqslant F\left(\psi\left(M\left(x, y\right)\right), \varphi\left(M\left(x, y\right)\right)\right) \tag{5}$$

where $r \geqslant 1$, $F : [0, +\infty)^2 \to \mathbb{R}$ is C-class function, $\psi : [0, +\infty) \to [0, +\infty)$ is an altering distance function, $\varphi : [0, +\infty) \to [0, +\infty)$ is an ultra altering distance function and

$$M\left(x, y\right) = \max\left\{d(Sx, Ty), D(Sx, Kx), D(Ty, Gy), \frac{D(Sx, Gy) + D(Kx, Ty)}{2}\right\}$$

for all $x, y \in X$, $D(Sx, Gy) + D(Kx, Ty) \neq 0$ and $H(Kx, Gy) = 0$ whenever $D(Sx, Gy) + D(Kx, Ty) = 0$. Suppose that one of $S(X)$ or $T(X)$ is complete. Then

a) there exists $p, q \in X$ such that $Sp \in Kp$ and $Tq \in Gq$.

Further, if S is K-weakly commuting and T is G-weakly commuting at their coincidence points, therefore

b) There exists $z \in X$ such that $Sz \in Kz$ and $Tz \in Gz$.

c) In the case (b), if $Sz = Tz$, then $Sz = Tz \in Kz \cap Gz$.

d) In the case (c), if $Sz = Tz = z$, then z is a common fixed point of S, T, K and G.

Proof. First, assume that there exists $p, q \in X$ such that

$$D(Sp, Gq) + D(Kp, Tq) = 0.$$

So, $D(Sp, Gq) = 0$ and $D(Kp, Tq) = 0$ which implies that $Sp \in Gq$ and $Tq \in Kp$. Since $H(Kp, Gq) = 0$, it follows that

$$D(Sp, Kp) \leqslant H(Kp, Gq) = 0.$$

Hence $Sp \in Kp$.

In a similar manner, we get $Tq \in Gq$.

Now, assume that

$$D(Sx, Gy) + D(Kx, Ty) \neq 0 \ \text{ for all } \ x, y \in X.$$

Let $x_0 \in X$ be an arbitrary point. By (4) and (1), we define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Sx_{2n} \in Gx_{2n-1}, \quad y_{2n+1} = Tx_{2n+1} \in Kx_{2n}$$

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &\leqslant kH(Kx_{2n}, Gx_{2n-1}), \\ d(y_{2n+1}, y_{2n+2}) &\leqslant kH(Kx_{2n}, Gx_{2n+1}), \quad \text{for } \ n = 1, 2, \ldots \end{aligned}$$

Using (5) we have

$$\begin{aligned}
\psi\left(rH(Kx_{2n},Gx_{2n-1})\right) &\leqslant F\big(\psi\left(M\left(x_{2n},x_{2n-1}\right)\right),\varphi\left(M\left(x_{2n},x_{2n-1}\right)\right)\big) \leqslant \\
&\leqslant F\Big(\psi\Big(\max\big\{d(y_{2n},y_{2n-1}),D(y_{2n},Kx_{2n}),D(y_{2n-1},Gx_{2n-1}) \\
&\quad \frac{D(y_{2n},Gx_{2n-1})+D(Kx_{2n},y_{2n-1})}{2}\big\}\Big), \\
&\quad \varphi\Big(\max\big\{d(y_{2n},y_{2n-1}),D(y_{2n},Kx_{2n}),D(y_{2n-1},Gx_{2n-1}) \\
&\quad \frac{D(y_{2n},Gx_{2n-1})+D(Kx_{2n},y_{2n-1})}{2}\big\}\Big)\Big) \leqslant \\
&\leqslant F\Big(\psi\Big(\max\big\{d(y_{2n},y_{2n-1}),D(y_{2n},y_{2n+1}),D(y_{2n-1},y_{2n}) \\
&\quad \frac{0+D(y_{2n+1},y_{2n-1})}{2}\big\}\Big), \\
&\quad \varphi\Big(\max\big\{d(y_{2n},y_{2n-1}),D(y_{2n},y_{2n+1}),D(y_{2n-1},y_{2n}) \\
&\quad \frac{0+D(y_{2n+1},y_{2n-1})}{2}\big\}\Big)\Big) \leqslant \\
&\leqslant F\Big(\psi\left(\max\left\{d(y_{2n},y_{2n-1}),d(y_{2n},y_{2n+1}),\frac{d(y_{2n+1},y_{2n-1})}{2}\right\}\right), \\
&\quad \varphi\left(\max\left\{d(y_{2n},y_{2n-1}),d(y_{2n},y_{2n+1}),\frac{d(y_{2n+1},y_{2n-1})}{2}\right\}\right)\Big) \leqslant \\
&\leqslant F\Big(\psi\Big(\max\big\{d(y_{2n},y_{2n-1}),d(y_{2n},y_{2n+1}) \\
&\quad \frac{d(y_{2n+1},y_{2n})+d(y_{2n},y_{2n-1})}{2}\big\}, \\
&\quad \varphi\Big(\max\big\{d(y_{2n},y_{2n-1}),d(y_{2n},y_{2n+1}) \\
&\quad \frac{d(y_{2n+1},y_{2n})+d(y_{2n},y_{2n-1})}{2}\big\}\Big)\Big)\Big) \leqslant \\
&\leqslant F\big(\psi\left(\max\left\{d(y_{2n},y_{2n-1}),d(y_{2n},y_{2n+1})\right\}\right), \\
&\quad \varphi\left(\max\left\{d(y_{2n},y_{2n-1}),d(y_{2n},y_{2n+1})\right\}\right)\big). \qquad (6)
\end{aligned}$$

Therefore, we obtain

$$d(y_{2n},y_{2n+1}) \leqslant rH(Kx_{2n},Gx_{2n-1}).$$

By the increasing of ψ, we get

$$\psi\left(d(y_{2n},y_{2n+1})\right) \leqslant \psi\left(rH(Kx_{2n},Gx_{2n-1})\right). \qquad (7)$$

Appling (7) in (6)and the nondecreasing property of ψ that

$$\begin{aligned}
\psi\left(d(y_{2n},y_{2n+1})\right) &\leqslant F\big(\psi\left(\max\left\{d(y_{2n},y_{2n-1}),d(y_{2n},y_{2n+1})\right\}\right), \\
&\quad \varphi\left(\max\left\{d(y_{2n},y_{2n-1}),d(y_{2n},y_{2n+1})\right\}\right)\big) \leqslant \\
&\leqslant \psi\left(\max\left\{d(y_{2n},y_{2n-1}),d(y_{2n},y_{2n+1})\right\}\right) \leqslant \\
&\leqslant \psi\left(d(y_{2n},y_{2n-1})\right).
\end{aligned}$$

Analogously, we can show that

$$\begin{aligned}
\psi\left(d(y_n,y_{n+1})\right) &\leqslant F\big(\psi\left(\max\left\{d(y_n,y_{n-1}),d(y_n,y_{n+1})\right\}\right),\varphi\left(\max\left\{d(y_n,y_{n-1}),d(y_n,y_{n+1})\right\}\right)\big) \leqslant \\
&\leqslant \psi\left(\max\left\{d(y_n,y_{n-1}),d(y_n,y_{n+1})\right\}\right) \leqslant \\
&\leqslant \psi\left(d(y_n,y_{n-1})\right). \qquad (8)
\end{aligned}$$

Then the sequence $[d(y_n, y_{n+1}) \downarrow 0]$ is bounded below and non-increasing, hence there exist $r \geqslant 0$ such that

$$\lim_{n\to+\infty} d(y_n, y_{n+1}) = 0.$$

By taking $n \to +\infty$ in (8) and using continuity of φ and ψ, we deduce that

$$\psi(r) \leqslant F(\psi(r), \varphi(r)) \leqslant \psi(r).$$

So, $\varphi(r) = 0$ or $\psi(r) = 0$. It follows that $r = 0$.

Now, we prove that the sequence $\{y_{2n}\}$ is a Cauchy in the metric space (X, d). Suppose that the sequence $\{y_{2n}\}$ is not a Cauchy sequence in (X, d), then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ as in Lemma 1.10 such that all sequences in (3) are tend to $\varepsilon > 0$, when $k \to +\infty$. Now, for $x = x_{2n(k)}$ and $y = x_{2m(k)+1}$ in equation (5), we get

$$\begin{aligned}
\psi\left(rH(Kx_{2n(k)}, Gx_{2m(k)+1})\right) \;&\leqslant\; F\left(\psi\left(M\left(x_{2n(k)}, x_{2m(k)+1}\right)\right), \varphi M\left(x_{2n(k)}, x_{2m(k)+1}\right)\right) \leqslant \\
&\leqslant\; F\Big(\psi\Big(\max\Big\{d(Sx_{2n(k)}, Tx_{2m(k)+1}), D(Sx_{2n(k)}, Kx_{2n(k)}), \\
&\quad D(Tx_{2m(k)+1}, Gx_{2m(k)+1}), \\
&\quad \frac{D(Sx_{2n(k)}, Gx_{2m(k)+1}) + D(Kx_{2n(k)}, Tx_{2m(k)+1})}{2}\Big\}\Big) \\
&\quad \varphi\Big(\max\Big\{d(Sx_{2n(k)}, Tx_{2m(k)+1}), D(Sx_{2n(k)}, Kx_{2n(k)}), \\
&\quad D(Tx_{2m(k)+1}, Gx_{2m(k)+1}), \\
&\quad \frac{D(Sx_{2n(k)}, Gx_{2m(k)+1}) + D(Kx_{2n(k)}, Tx_{2m(k)+1})}{2}\Big\}\Big)\Big) \leqslant \\
&\leqslant\; F\Big(\psi\Big(\max\Big\{d(y_{2n(k)-1}, y_{2m(k)}), D(y_{2n(k)-1}, y_{2n(k)}), \\
&\quad D(y_{2m(k)}, y_{2m(k)+1}), \\
&\quad \frac{D(y_{2n(k)-1}, y_{2m(k)+1}) + D(y_{2n(k)}, y_{2m(k)})}{2}\Big\}\Big) \\
&\quad \varphi\Big(\Big\{\max d(y_{2n(k)-1}, y_{2m(k)}), D(y_{2n(k)-1}, y_{2n(k)}), \\
&\quad D(y_{2m(k)}, y_{2m(k)+1}), \\
&\quad \frac{D(y_{2n(k)-1}, y_{2m(k)+1}) + D(y_{2n(k)}, y_{2m(k)})}{2}\Big\}\Big)\Big)
\end{aligned} \tag{9}$$

Therefore, taking $k \to +\infty$ in inequality (9) and using the properties of F we get

$$\psi(\varepsilon) \leqslant F(\psi(\varepsilon), \varphi(\varepsilon)) \leqslant \psi(\varepsilon).$$

So, $\psi(\varepsilon) = 0$ or $\varphi(\varepsilon) = 0$, hence we get $\varepsilon = 0$ which contradiction with $\varepsilon > 0$. Thus $\{y_{2n}\}$ is a Cauchy sequence in (X, d), hence by (3) we deduce that the sequence $\{y_n\}$ is Cauchy sequence in X. As $S(X)$ is complete, it converges to $z \in S(X)$ and so there exists $p \in X$ such that $z = Sp$.

Using (5)

$$\begin{aligned}
\psi(H(Kp, Gx_{2n-1})) \;&\leqslant\; F\Big(\psi\Big(\max\Big\{d(Sp, Tx_{2n-1}), D(Sp, Kp), D(Tx_{2n-1}, Gx_{2n-1}), \\
&\quad \frac{D(Sp, Gx_{2n-1}) + D(Kp, Tx_{2n-1})}{2}\Big\}\Big), \\
&\quad \varphi\Big(\max\Big\{d(Sp, Tx_{2n-1}), D(Sp, Kp), D(Tx_{2n-1}, Gx_{2n-1}), \\
&\quad \frac{D(Sp, Gx_{2n-1}) + D(Kp, Tx_{2n-1})}{2}\Big\}\Big)\Big).
\end{aligned}$$

So

$$\begin{aligned}\psi\left(D(Kp,y_{2n})\right) &\leqslant F\Big(\psi\Big(\max\Big\{d(Sp,y_{2n-1}),D(Sp,Kp),d(y_{2n-1},y_{2n}),\\ &\qquad \frac{d(Sp,y_{2n})+D(Kp,y_{2n-1})}{2}\Big\}\Big)\\ &\qquad \varphi\Big(\max\Big\{d(Sp,y_{2n-1}),D(Sp,Kp),d(y_{2n-1},y_{2n}),\\ &\qquad \frac{d(Sp,y_{2n})+D(Kp,y_{2n-1})}{2}\Big\}\Big)\Big).\end{aligned}$$

Letting n tend to infinity, we get

$$\begin{aligned}\psi\left(D(Kp,Sp)\right) &\leqslant F(\psi\left(\max\left\{0,D(Sp,Kp),0,\frac{0+D(Kp,Sp)}{2}\right\}\right)\\ &\qquad \varphi\left(\max\left\{0,D(Sp,Kp),0,\frac{0+D(Kp,Sp)}{2}\right\}\right)\leqslant\\ &\leqslant F(\psi\left(D(Sp,Kp);\varphi\left(D(Sp,Kp)\right)\leqslant \psi D(Sp,Kp).\end{aligned}$$

Thus, we hold $\psi\left(D(Sp,Kp)\right)=0$ or $\varphi\left(D(Sp,Kp)=0\right.$, then $D(Sp,Kp)=0$, with implie $Sp\in Kp$.

Similarly, as $K(X)\subset T(X)$, there exists $q\in X$ such that $z=Sp=Tq$. Applying (5) and letting $n\to+\infty$, bu the same calculate, we can find $Tq\in Gq$.

Since S is F-weakly commuting at $p\in C(S,T)$ and T is G-weakly commuting at $q\in C(G,T)$ it follows that $z=Sp\in C(K,T)$ and $z=Tq\in C(G,T)$. Hence, $Sz\in Kz$ and $Tz\in Gz$. If $Sz=Tz$, then $Sz=Tz\in Kz\cap Gz$ and if $Sz=Tz=z$, then$z$ is a common fixed point of S,T,K and G.

Corollary 1. *Let (X,d) be a metric space, $S,T:X\to X$ and $K,G:X\to CB(X)$ satisfying*

$$K(X)\subset T(X)\quad and\quad G(X)\subset S(X)$$

$$rH(Kx,Gy)\leqslant M\left(x,y\right)\beta\left(M\left(x,y\right)\right)$$

where

$$M\left(x,y\right)=\max\left\{d(Sx,Ty),D(Sx,Kx),D(Ty,Gy),\frac{D(Sx,Gy)+D(Kx,Ty)}{2}\right\}$$

for all $x,y\in X$, $D(Sx,Gy)+D(Kx,Ty)\neq 0$ and $H(Kx,Gy)=0$ whenever $D(Sx,Gy)+D(Kx,Ty)=0$. Suppose that one of $S(X)$ or $T(X)$ is complete. Then

a) there exists $p,q\in X$ such that $Sp\in Kp$ and $Tq\in Gq$.

Further, if S is K-weakly commuting and T is G-weakly commuting at their coincidence points, therefore

b) There exists $z\in X$ such that $Sz\in Kz$ and $Tz\in Gz$.

c) In the case (b), if $Sz=Tz$, then $Sz=Tz\in Kz\cap Gz$.

d) In the case (c), if $Sz=Tz=z$, then z is a common fixed point of S,T,K and G.

Proof. Set $\psi(t)=t$, $F(s,t)=s\beta(s)$ in Theorem (2.1) , $\beta:[0,1)\to[0,+\infty)$. □

Corollary 2. *Let (X,d) be a metric space, $S,T:X\to X$ and $K,G:X\to CB(X)$ satisfying*

$$K(X)\subset T(X)\quad and\quad G(X)\subset S(X)$$

$$rH(Kx,Gy)\leqslant mM\left(x,y\right)$$

where $r \geqslant 1$ *and*

$$M(x,y) = \max\left\{d(Sx,Ty), D(Sx,Kx), D(Ty,Gy), \frac{D(Sx,Gy)+D(Kx,Ty)}{2}\right\}$$

for all $x,y \in X$, $D(Sx,Gy)+D(Kx,Ty) \neq 0$ *and* $H(Kx,Gy)=0$ *whenever* $D(Sx,Gy)+D(Kx,Ty)=0$. *Suppose that one of* $S(X)$ *or* $T(X)$ *is complete. Then*

a) there exists $p,q \in X$ *such that* $Sp \in Kp$ *and* $Tq \in Gq$.

Further, if S *is* K*-weakly commuting and* T *is* G*-weakly commuting at their coincidence points, therefore*

b) There exists $z \in X$ *such that* $Sz \in Kz$ *and* $Tz \in Gz$.

c) In the case (b), if $Sz = Tz$, *then* $Sz = Tz \in Kz \cap Gz$.

d) In the case (c), if $Sz = Tz = z$, *then* z *is a common fixed point of* S, T, K *and* G.

Proof. Set $\psi(t)=t$, $F(s,t)=ms$ in Theorem (2.1), $m \in (0,1)$. □

Now we present some examples to support our Theorem

Example 3. *Define* $\psi, \varphi : [0,+\infty) \to [0,+\infty)$ *by* $\psi(t) = \dfrac{t}{15}$; $\varphi(t) = 2t$ *and* $F(s,t) = ks$ *for* $k \in (0,1)$.

Let $X = [0,1]$ *be endowed with the Euclidean metric* d. *Let* $Gx = \left[0,x^2\right]$, *for all* $x,y \in X$, *we have*

$d(x,y) = |x-y|$, $D(x,Gx) = \inf\left(d(x,b), b \in \left[0,x^2\right]\right)$, $D(y,Gy) = \inf\left(d(y,c), c \in \left[0,y^2\right]\right)$

$$\begin{aligned} H(Gx,Gy) &= H(\left[0,x^2\right],\left[0,y^2\right]) = \\ &= \left|x^2-y^2\right| = (x+y)\,|x-y| \leqslant kd(x,y), \quad k \in (0,1) \quad \text{with } x,y \in [0,1]. \end{aligned}$$

Consequently, these mappings are satisfy all conditions of theorem, then they have a fixed point in X.

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Теоремы о совпадении и общих неподвижных точках для гибридных отображений с помощью функции C-класса

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Аннотация. В данной статье мы доказываем общие теоремы о неподвижной точке для двух пар гибридных отображений в метрических пространствах, используя концепцию функции C-класса и T-слабую коммутативность. Наши теоремы обобщают некоторые хорошо известные результаты.

Ключевые слова: метрическое пространство, гибридные отображения, функция C-класса.