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On Hypergeometric Functions of Two Variables of Complexity One

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Abstract. For a series of examples of Horn systems and the Lauricella system for functions of two variables the description of solution of complexity one is given. Several questions are formulated.

Keywords: analytical complexity,hypergeometric functions, Horn system, Lauricella system, differential ring.

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Introduction

The class of hypergeometric functions (HG-functions) of several variables is of considerable interest. It continues to be a subject of great attention [1]. On the other hand, there is the theory of analytical complexity, which is oriented to the study of questions about representability of functions of several variables with the help of superpositions of functions of lesser number of variables. In particular, the questions of representability of functions of two variables with the help of functions of one variable [2]. In the context of this theory, the simplest functions of two variables are the functions of complexity one (the functions of one variable have complexity zero). These are analytic functions of variables (x, y), which can be locally represented as $z(x, y) = c(a(x) + b(y))$ (a, b, c are nonconstant analytic functions of one variable). These functions are of special interest. First, they are the functions, which have the stabilizer of the maximal dimension in the gauge group (the dimension is equal to three) [3]. Second, if we consider $z(x, y)$ as a function of a 3-web on the plane, then such web is equivalent to the hexagonal web if and only if z has the specified form [4].

The set of all such functions is, except for the functions of one variable, the set of analytic functions, which is the set of the solutions of a differential polynomial of order three. This polynomial is exactly the numerator of the following differential fraction: $\left(\ln(z'_x/z'_y)\right)''_{xy}$, i.e., the defining condition for the functions of complexity one has the form:

$$d_1(z) = z'_x z'_y (z'''_{xxy} z''_y - z'''_{xyy} z'_x) + z''_{xy}((z'_x)^2 z''_{yy} - (z'_y)^2 z''_{xx}) = 0, \quad z'_x\, z'_y \neq 0. \tag{1}$$

Note that the class of functions of complexity one includes all four arithmetic operations. If we remove the inequality, which excludes the functions of one variable, we obtain $Cl^1 = \{d_1(z) = 0\}$, which is the class of the functions of complexity not greater than one.

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We think that the theory of hypergeometric functions of several (in particular, of two) variables differs qualitatively from the theory of hypergeometric functions of one variable by the fact that the class of hypergeometric functions of several variables is in a sense too large and the problem of the choice of a narrower class of the most interesting HG-functions arise. Which HG-functions are the most interesting? No doubt it is possible to give different answers to this question. For the functions of two variables we offer the following answer:

Good HG-functions are the HG-functions of complexity one.

We interpret a HG-function of two variables (Examples 1-7) as, following [1], a solution of a Horn system. To define the Horn system for functions of the variables (x, y), we need four polynomials P, Q, R, S in two variables. Let $X = x\,\frac{\partial}{\partial x}$, $Y = y\,\frac{\partial}{\partial y}$ be the homogeneous partial differential operators. Then the Horn system corresponding to the given four polynomials is the system of two linear differential equations with nonconstant coefficients with respect to the function $z(x, y)$ of the form

$$G_x\, z = (x\, P(X, Y) - Q(X, Y))\, z = 0,$$
$$G_y\, z = (y\, R(X, Y) - S(X, Y))\, z = 0. \tag{2}$$

What is the set of all solutions of complexity one of this system? The aim of this paper is to give the explicit description of the solutions of complexity one for a series of examples of systems of the form (2). Almost all examples are from [1]. With the growth of degrees of the defining equations and the number of free parameters the problem of the explicit description of the space of solutions quickly becomes computationally difficult even for polynomials of degree not greater than two. However, one can hope that the consideration of these examples will allow to formulate questions for the further study (some of them are given at the end of the paper).

In the theory of HG-functions there is an established approach, which suggests that an important characteristic of the Horn system is its holonomicity. The holonomicity of the system, in particular, guarantees the finite-dimensionality of the space of solutions. In our considerations we do not require holonomicity.

Since the Horn system is a system of linear differential equations, the set of its solutions is a linear space. Equation (1) is not linear. From the geometric point of view, the set of its solutions is an infinite-dimensional cone. In fact, the transformation $(z(x, y) \to \lambda\, z(x, y))$ maps the solutions to solutions. Hence, we can understand our question as the question about the construction of the intersection of the cone and a linear subspace.

Further we will need the following simple observation. Let two nonconstant functions $a(x)$ and $b(y)$ are given. For the existence of a function $c(t)$ for the function $w(x, y)$, such that in a neighbourhood of a generic point there is a local representation of the form $w = c(a(x) + b(y))$, it is necessary and sufficient that

$$V(w) = \left(\frac{1}{a'(x)}\,\frac{\partial}{\partial x} - \frac{1}{b'(y)}\,\frac{\partial}{\partial y}\right)(w) = 0. \tag{3}$$

The concrete computations were performed using Maple.

1. A set of examples

Example 1. Let

$$P = x^2, \quad Q = q_1\, x + q_2\, y, \quad R = y^2, \quad S = s_1\, x + s_2\, y, \quad q_1 q_2 s_1 s_2 \neq 0.$$

Then the system (2) for $z = c(a(x)+b(y))$ takes the form (low indexes are orders of derivatives)

$$G_x\, z = x^3 {a_1}^2 c_2 + x^3 a_2 c_1 + x^2 a_1 c_1 - q_1 x c_1 a_1 - q_2 y c_1 b_1 = 0,$$
$$G_y\, z = y^3 {b_1}^2 c_2 + y^3 b_2 c_1 - s_1 x c_1 a_1 + y^2 b_1 c_1 - s_2 y c_1 b_1 = 0.$$

We can express the ratio c_2/c_1 from first equation and from second equation. We get two relations: the first is the equality of the both expressions, the second is the result of action of operator V on each of them. Thus,

$$\begin{aligned} e_1 = x^3y^3{a_1}^2b_2 - x^3y^3a_2{b_1}^2 - x^4{a_1}^3s_1 + x^3y^2{a_1}^2b_1 - x^3y{a_1}^2b_1s_2 - \\ -x^2y^3a_1{b_1}^2 + xy^3a_1{b_1}^2q_1 + y^4{b_1}^3q_2 = 0, \end{aligned}$$
$$\begin{aligned} e_2 = -a_3a_1b_1x^4 + 2\,{a_2}^2b_1x^4 + a_2a_1b_1x^3 - a_2a_1b_1x^2q_1 - {a_1}^2b_2xyq_2 - \\ -2\,a_2{b_1}^2xyq_2 + {a_1}^2b_1x^2 - 2\,{a_1}^2b_1xq_1 - {a_1}^2b_1xq_2 - 3\,a_1{b_1}^2yq_2 = 0. \end{aligned}$$

The expressions e_1 and e_2 are linear wrt b_2. The coefficients of b_2 are not idenitcally zero. It is possible to express b_2 from $e_1 = 0$ and from $e_2 = 0$. We get two relations: the first is the equality of the both expressions, the second is the result of the action of operator ∂/∂_x on each of them. Thus,

$$\begin{aligned} e_3 = x^6y^2a_1a_3b_1 - 2\,x^6y^2{a_2}^2b_1 - x^5y^2a_1a_2b_1 + x^4y^2a_1a_2b_1q_1 + \\ +3\,x^3y^3a_2{b_1}^2q_2 - -x^4y^2{a_1}^2b_1 + x^4{a_1}^3q_2s_1 + 2\,x^3y^2{a_1}^2b_1q_1 + \\ x^3y{a_1}^2b_1q_2s_2 + 4\,x^2y^3a_1{b_1}^2q_2 - xy^3a_1{b_1}^2q_1q_2 - y^4{b_1}^3{q_2}^2 = 0, \end{aligned}$$
$$\begin{aligned} e_4 = a_3a_1{b_1}^2x^4y^3 - 2\,{a_2}^2{b_1}^2x^4y^3 + a_2{a_1}^3x^5s_1 - a_2a_1{b_1}^2x^3y^3 + a_2a_1{b_1}^2x^2y^3q_1 + \\ +2\,a_2{b_1}^3xy^4q_2 + {a_1}^4x^4s_1 - {a_1}^2{b_1}^2x^2y^3 + 2\,{a_1}^2{b_1}^2xy^3q_1 + 3\,a_1{b_1}^3y^4q_2 = 0. \end{aligned}$$

The expressions e_3 and e_4 are quadratic wrt b_2. The necessary condition for solvability is the equality to zero of the resultant of e_3 and e_4 with respect to b_1. This resultant has the form

$$x^{11}y^{11}{a_1}^4{q_2}^2s_1\, r(x, y, a_1, a_2, a_3),$$

hence, $r = 0$. And r is the polynomial of degree four with respect to y. The coefficient in r of y^4 is equal to

$$-x^2{a_1}^5{q_2}^3{s_2}^3\,(xa_2 + a_1)\,(2\,xa_2 + 3\,a_1)^2\,.$$

Thus, it is enough to consider two cases:
the first is $(2\,xa_2 + 3\,a_1) = 0$, and second is $(xa_2 + a_1) = 0$. We can solve these differential equations. In the first case we get $a_1 = k/x^{3/2}$ and in the second case we get $a_1 = k/x$. If we substitute the first solution in r, we can see that the equation $r = 0$ is impossible. If we substitute the second solution in r, we have

$$r = -\frac{k^8{q_2}^3s_1\,(q_1s_2 - q_2s_1)}{x^7}\,y.$$

And we see that $r = 0$ iff $(q_1, q_2) = \lambda\,(s_1, s_2)$, $\lambda \neq 0$. We can substitute $a_1 = k/x$ in $e_3 = 0$ and we have $yb_1q_2 + kq_1 = 0$. In such case we have $a(x) + b(y) = k\,\ln(x) - k\,\dfrac{q_1}{q_2}\,\ln(y) + const$ and $z = c(y/x^{\alpha})$, where $\alpha = q_2/q_1 = s_2/s_1$. Then we can substitute such z in $G_x\, z = G_y\, z = 0$ and we get:
$c''(t)\,t + c'(t) = 0$, hence $c(t) = \lambda\,\ln(t) + \mu$. Thus, we have

Proposition 1 *The solutions of the Horn system of complexity one (i.e. of the kind $z = c(a(x)+b(y))$, a, b, c are nonconstant) for Example 1 exist iff $q_2/q_1 = s_2/s_1 = \alpha$. In this case the solutions have the form*

$$z = \lambda \ln\left(\frac{y}{x^\alpha}\right) + \mu, \quad \alpha \neq 0,\ \lambda \neq 0.$$

Example 2. Let

$$P = x + 1, \quad Q = 1, \quad R = y + 1, \quad S = 1.$$

Then the system (2) for $z = c(a(x) + b(y))$ takes the form

$$G_x\, z = x^2 a_1 c_1 + x c_0 - c_0 = 0, \quad G_y\, z = y^2 b_1 c_1 + y c_0 - c_0 = 0.$$

Then we have

$$\frac{c_1}{c_0} = -\frac{x-1}{x^2 a_1} = -\frac{y-1}{y^2 b_1} = \frac{1}{\lambda}.$$

Thus,

$$a_1 = -\frac{(x-1)\,\lambda}{x^2}, \quad b_1 = -\frac{(y-1)\,\lambda}{y^2}.$$

If we substitute this expressions in $G_x = 0$ and $G_y = 0$, we get

$$-\,(x-1)\,(\lambda\, c_1 - c_0) = -\,(y-1)\,(\lambda\, c_1 - c_0) = 0.$$

Hence $c(t) = \exp(-t/\lambda)$ and $z = \mu\,(x\,y\,\exp(1/x + 1/y))^{-1}$.

Proposition 2. *The solutions of the Horn system of complexity one for Example 2 have the form $z = \mu\,(x\,y\,\exp(1/x + 1/y))^{-1}$.*

Example 3. Let

$$P = 1, \quad Q = (x-1) \quad R = 1, \quad S = (y-1).$$

If $z = c(a(x) + b(y))$ we have

$$G_x\, z = -x c_1 a_1 + x c_0 + c_0 = 0, \quad G_y\, z = -y c_1 b_1 + y c_0 + c_0 = 0.$$

After elimination of c we obtain

$$-x a_1 y + y b_1 x - x a_1 + y b_1 = 0, \quad a_2 x^2 + a_2 x + a_1 = 0$$

Hence

$$a(x) = \frac{\lambda}{x} + \lambda\,\ln(x) + \ln(\alpha)\,, \quad b(y) = \lambda\,\ln(y) + \frac{\lambda}{y} + \ln(\beta)\,, \quad \lambda \neq 0.$$

Then we get the equation for the function c. When we solve this equation, we obtain the following proposition.

Proposition 3. *The solutions of the Horn system of complexity one for Example 3 have the form*

$$z = \mu\, xy\, \mathrm{e}^{(x+y)}, \quad \mu \neq 0.$$

Example 4. Let

$$P = x^2\, y + 1, \quad Q = 1 \quad R = y + 1, \quad S = 1.$$

For $z = c(a(x) + b(y))$ we have

$$G_x\, z = x^3 y {a_1}^2 b_1 c_3 + x^3 y a_2 b_1 c_2 + x^2 y a_1 b_1 c_2 + x c_0 - c_0 = 0,$$
$$G_y\, z = y^2 b_1 c_1 + y c_0 - c_0 = 0.$$

From the second equation we get $c_1 = (-b_2y^2 - yb_1 + b_2y + 2\,b_1)\,c_0$. After differentiation of this equation with respect to x we obtain such expressions for c_2 and c_3. After the substitution of these expressions in $G_x z = 0$ we get:

$$l = x^3y^4a_2b_1 - x^3y^3{a_1}^2 - 2\,x^3y^3a_2b_1 + x^2y^4a_1b_1 + xy^5{b_1}^2 + 3\,x^3y^2{a_1}^2 + \\ +x^3a_2y^2b_1 - 2\,x^2y^3a_1b_1 - y^5{b_1}^2 - 3\,x^3y{a_1}^2 + x^2a_1y^2b_1 + x^3{a_1}^2 = 0.$$

From $V(c_1/c_0) = 0$ we get $e = -b_2y^2 - yb_1 + b_2y + 2\,b_1 = 0$. After differentiation of l with respect to y we obtain:

$$(l)'_y = x^3y^4a_2b_2 + 4\,x^3y^3a_2b_1 - 2\,x^3y^3a_2b_2 + x^2y^4a_1b_2 + 2\,xy^5b_1b_2 - 3\,x^3y^2{a_1}^2 - \\ -6\,x^3a_2y^2b_1 + x^3a_2y^2b_2 + 4\,x^2y^3a_1b_1 - 2\,x^2y^3a_1b_2 + 5\,xy^4{b_1}^2 - 2\,y^5b_1b_2 + \\ +6\,x^3y{a_1}^2 + 2\,x^3a_2yb_1 - 6\,x^2a_1y^2b_1 + x^2a_1y^2b_2 - 5\,y^4{b_1}^2 - 3\,x^3{a_1}^2 + 2\,x^2a_1yb_1 = 0.$$

The resultant of $(l)'_y$ and e with respect to b_2 equals

$$r_1 = 3\,x^3y^4a_2b_1 - 3\,x^3y^3{a_1}^2 - 6\,x^3y^3a_2b_1 + 3\,x^2y^4a_1b_1 + 3\,xy^5{b_1}^2 + 9\,x^3y^2{a_1}^2 + \\ +3\,x^3a_2y^2b_1 - 6\,x^2y^3a_1b_1 - xy^4{b_1}^2 - 3\,y^5{b_1}^2 - 9\,x^3y{a_1}^2 + 3\,x^2a_1y^2b_1 + \\ +y^4{b_1}^2 + 3\,x^3{a_1}^2 = 0.$$

The resultant of r_1 and l with respect to b_1 equals

$$r_2 = x^6y^8{a_1}^4\,(y-1)^6\,(x-1)^2 = 0.$$

But r_2 is not equal to zero identically. Thus, we have:

Proposition 4. *The solutions of the Horn system of complexity one for Example 4 do not exist.*

Example 5. Let

$$P = x + y - p, \quad Q = x + q, \quad R = x + y - p, \quad S = y + s.$$

Case $p = 0$.

$$G_x\,z = x^2a_1c_1 + xyb_1c_1 - xa_1c_1 - qc_0 = 0, \\ G_y\,z = xya_1c_1 + y^2b_1c_1 - yb_1c_1 - sc_0 = 0.$$

From $c_1 \neq 0$ we get $q \neq 0$ and $s \neq 0$. After elimination of c we obtain

$$e_1 = qxya_1 + qy^2b_1 - sx^2a_1 - sxyb_1 - qyb_1 + sxa_1 = 0, \\ e_2 = -x^2a_2b_1 + xya_1b_2 - xa_1b_1 + xa_2b_1 - y{b_1}^2 + a_1b_1 = 0.$$

From $e_1 = 0$ we get b_1. b_1 does not depend on x, hence

$$e_3 = -a_2q^2xy^2 + 2\,a_2qsx^2y - a_2s^2x^3 + a_2q^2xy - a_1q^2y^2 - a_2qsx^2 + 2\,a_1qsxy - \\ a_2qsxy - a_1s^2x^2 + a_2s^2x^2 + a_1q^2y - 2\,a_1qsx + a_2qsx - a_1qsy + a_1qs = 0.$$

This expression is quadratic in y. Write that the coefficient of y^2 is equal to zero, we get $q^2\,(xa_2 + a_1) = 0$. Hence $a(x) = \lambda\,\ln(x) + \alpha$ and $s = -q,\ b(y) = -\lambda\,\ln(y) + \beta$. As a result we get $z = \mu\,(y/x)^q$.

Case $p \neq 0$.

We have

$$G_x\, z = x^2 a_1 c_1 + xy b_1 c_1 - pxc_0 - xc_1 a_1 - qc_0 = 0,$$
$$G_y\, z = xya_1 c_1 + y^2 b_1 c_1 - pyc_0 - yc_1 b_1 - sc_0 = 0.$$

By elimination of c we get

$$e_1 = pxya_1 - pxyb_1 + qxya_1 + qy^2 b_1 - sx^2 a_1 - sxyb_1 - qyb_1 + sxa_1 = 0,$$
$$e_2 = -a_2 b_1 px^3 + a_1 b_2 px^2 y + a_2 b_1 px^2 - a_2 b_1 qx^2 + a_1 b_2 qxy - a_1 b_1 qx + a_2 b_1 qx - \\ -{b_1}^2 qy + a_1 b_1 q = 0.$$

Let us express b_2 from $e_2 = 0$ and b_1 from $e_1 = 0$. We have two conditions. The first: $(b_1)'_x = 0$ and the second: $(b_1)'_y = b_2$. After elimination the nonzero factors we can see that both conditions coincide and have the form

$$e_3 = a_2 p^2 x^2 y + a_2 pqx^2 y - a_2 pqxy^2 - a_2 psx^3 + a_2 psx^2 y - a_2 q^2 xy^2 + 2\, a_2 qsx^2 y - \\ -a_2 s^2 x^3 + a_2 pqxy - a_1 pqy^2 - a_1 psx^2 + a_2 psx^2 + a_2 q^2 xy - a_1 q^2 y^2 - a_2 qsx^2 + \\ +2\, a_1 qsxy - a_2 qsxy - a_1 s^2 x^2 + a_2 s^2 x^2 + a_1 pqy + a_1 q^2 y - 2\, a_1 qsx + \\ +a_2 qsx - a_1 qsy + a_1 qs = 0.$$

This expression is quadratic in y. Write that the coefficient of y^2 is equal zero, we get:

$$q\,(xa_2 + a_1)\,(p+q) = 0.$$

Case $p \neq 0,\ q = 0$.

Then e_3 has the form $x^2\,(p+s)\,(pya_2 - sxa_2 - sa_1 + sa_2)$. We have two opportunities for this expression to be equal to zero: either $s = -p$ or $a_2 = s = 0$. The first case is impossible, because in this case $e_1 = pxa_1\,(x + y - 1)$. In the second case $a(x) = \lambda\, x + \alpha,\ b(y) = \lambda\, y + \beta$. Thus, we get:

$$z = \mu\,(1 - (x+y))^p,\ q = s = 0,\ p \neq 0.$$

Case $p \neq 0,\ q = -p,\ q \neq 0$. Then e_3 is divisible by s. But $s = 0$ is impossible, if $e_1 = 0$. Thus, we have $s = -p$ and:

$$z = \mu\left(\frac{(x-1)(y-1)}{xy} - 1\right), \quad s = q = -p.$$

Case $p \neq 0,\ q \neq 0, (p+q) \neq 0\ \ (xa_2 + a_1) = 0$. Then $a(x) = \lambda\, \ln(x) + \alpha$. From $e_2 = 0$ we get $s = -(p+q)$ and from $e_1 = 0$ we get $b_1 = -\lambda(p+q)/qy,\ \ b(y) = -\lambda(p+q)/q\, \ln(y) + \beta$. We have:

$$z = \mu\,\frac{y^{p+q}}{x^q}, \quad s = -(p+q), \quad q \neq 0,\ (p+q) \neq 0.$$

Proposition 5. *The solutions of the Horn system of complexity one for Example 5 exist in three cases only:*

$$(a)\ q \neq 0, \quad p + q \neq 0, \quad s = -(p+q), \quad z = \frac{\mu}{x^q\, y^s},$$
$$(b)\ p \neq 0,\ s = q = 0, \quad z = \mu\,(1 - (x+y))^p\,,$$
$$(c)\ s = q = -p \neq 0, \quad z = \mu\left(\frac{(x-1)(y-1)}{x\,y} - 1\right).$$

Example 6. Let
$$P = x(x+y), \quad Q = x^2, \quad R = y(x+y), \quad S = y^2.$$
For $z = c(a(x) + b(y))$ we have:
$$G_x\, z = x^2 {a_1}^2 c_2 + xya_1b_1c_2 + x^2a_2c_1 - xc_2{a_1}^2 + xa_1c_1 - xc_1a_2 - c_1a_1 = 0,$$
$$G_y\, z = xya_1b_1c_2 + y^2{b_1}^2c_2 + y^2b_2c_1 - yc_2{b_1}^2 + yb_1c_1 - yc_1b_2 - c_1b_1 = 0.$$
After elimination of c we get
$$\begin{aligned} e_1 = {} & -x^3ya_1a_2b_1 + x^2y^2{a_1}^2b_2 - x^2y^2a_2{b_1}^2 + xy^3a_1b_1b_2 - x^2y{a_1}^2b_2 + \\ & + x^2ya_1a_2b_1 + x^2ya_2{b_1}^2 - xy^2{a_1}^2b_2 - xy^2a_1b_1b_2 + xy^2a_2{b_1}^2 - x^2{a_1}^2b_1 + \\ & + xy{a_1}^2b_2 - xya_2{b_1}^2 + y^2a_1{b_1}^2 + x{a_1}^2b_1 - ya_1{b_1}^2 = 0, \\ e_2 = {} & -a_3{a_1}^2b_1x^4 + 2\,{a_2}^2a_1b_1x^4 - a_2{a_1}^2b_2x^3y - a_3a_1{b_1}^2x^3y + {a_2}^2{b_1}^2x^3y + \\ & + 2\,a_3{a_1}^2b_1x^3 - 4\,{a_2}^2a_1b_1x^3 - {a_1}^3b_2x^2y + a_2{a_1}^2b_2x^2y - a_2a_1{b_1}^2x^2y + a_3a_1{b_1}^2x^2y - \\ & - {a_2}^2{b_1}^2x^2y - a_2{a_1}^2b_1x^2 - a_3{a_1}^2b_1x^2 + 2\,{a_2}^2a_1b_1x^2 + {a_1}^3b_2xy - \\ & - {a_1}^3b_1x + a_2{a_1}^2b_1x - {a_1}^2{b_1}^2y + {a_1}^3b_1 = 0. \end{aligned}$$
And we obtain two expression for b_2. Then we have two conditions: equality of both expressions and their independence of x. We obtain $e_3(a_1, a_2, a_3, b_1) = e_4(a_1, a_2, a_3, b_1) = 0$ (e_3 contains 43 monomials, e_4 contains 45 monomials). If r is the resultant of e_3 and e_4 with respect to b_1, we have:
$$r = y^2{a_1}^4\,(y-1)\,(x-1)^2\,(x+y-1)^2\; r_1^2\, r_2\, r_3, \text{ where}$$
$$r_1 = (xa_2 + a_1)\,, \quad r_2 = x^3a_1a_3 - x^3{a_2}^2 + x^2a_1a_2 - x^2a_1a_3 + x^2{a_2}^2 + {a_1}^2, \quad r_3 = r_{30} + y\,r_{31},$$
where
$$\begin{aligned} r_{30} = {} & x^7{a_1}^2{a_3}^2 - 4\,x^7a_1{a_2}^2a_3 + 4\,x^7{a_2}^4 - 2\,x^6{a_1}^2a_2a_3 - 3\,x^6{a_1}^2{a_3}^2 + 4\,x^6a_1{a_2}^3 + \\ & + 12\,x^6a_1{a_2}^2a_3 - 12\,x^6{a_2}^4 - 2\,x^5{a_1}^3a_3 + 5\,x^5{a_1}^2{a_2}^2 + 6\,x^5{a_1}^2a_2a_3 + 3\,x^5{a_1}^2{a_3}^2 - \\ & - 12\,x^5a_1{a_2}^3 - 12\,x^5a_1{a_2}^2a_3 + 12\,x^5{a_2}^4 + 2\,x^4{a_1}^3a_2 + 6\,x^4{a_1}^3a_3 - 15\,x^4{a_1}^2{a_2}^2 - \\ & - 6\,x^4{a_1}^2a_2a_3 - x^4{a_1}^2{a_3}^2 + 12\,x^4a_1{a_2}^3 + 4\,x^4a_1{a_2}^2a_3 - 4\,x^4{a_2}^4 + x^3{a_1}^4 - 6\,x^3{a_1}^3a_2 - \\ & - 6\,x^3{a_1}^3a_3 + 15\,x^3{a_1}^2{a_2}^2 + 2\,x^3{a_1}^2a_2a_3 - 4\,x^3a_1{a_2}^3 - 3\,x^2{a_1}^4 + 6\,x^2{a_1}^3a_2 + \\ & + 2\,x^2{a_1}^3a_3 - 5\,x^2{a_1}^2{a_2}^2 + 3\,x{a_1}^4 - 2\,x{a_1}^3a_2 - {a_1}^4, \\ r_{31} = {} & 4\,x^7a_1{a_2}^2a_3 - 4\,x^7{a_2}^4 + 8\,x^6{a_1}^2a_2a_3 + x^6{a_1}^2{a_3}^2 - 4\,x^6a_1{a_2}^3 - 12\,x^6a_1{a_2}^2a_3 + \\ & + 12\,x^6{a_2}^4 + 4\,x^5{a_1}^3a_3 + 4\,x^5{a_1}^2{a_2}^2 - 16\,x^5{a_1}^2a_2a_3 - 2\,x^5{a_1}^2{a_3}^2 + 12\,x^5a_1{a_2}^3 + \\ & + 12\,x^5a_1{a_2}^2a_3 - 12\,x^5{a_2}^4 + 4\,x^4{a_1}^3a_2 - 8\,x^4{a_1}^3a_3 + 3\,x^4{a_1}^2{a_2}^2 + 10\,x^4{a_1}^2a_2a_3 + \\ & + x^4{a_1}^2{a_3}^2 - 12\,x^4a_1{a_2}^3 - 4\,x^4a_1{a_2}^2a_3 + 4\,x^4{a_2}^4 + 2\,x^3{a_1}^3a_2 + 6\,x^3{a_1}^3a_3 - \\ & - 11\,x^3{a_1}^2{a_2}^2 - 2\,x^3{a_1}^2a_2a_3 + 4\,x^3a_1{a_2}^3 + 3\,x^2{a_1}^4 - 6\,x^2{a_1}^3a_2 - 2\,x^2{a_1}^3a_3 + \\ & + 5\,x^2{a_1}^2{a_2}^2 - 3\,x{a_1}^4 + 2\,x{a_1}^3a_2 + {a_1}^4. \end{aligned}$$
Thus, we need to consider three cases:
Case $r_1 = 0$. In this case $a(x) = \lambda\,\ln(x) + \alpha$. After the substitution of this expression in e_1, we obtain
$$e_1 = \lambda\,(y-1)\,(yb_2 + b_1)\,(xyb_1 + \lambda\,x - \lambda) = 0,$$
hence $(yb_2 + b_1) = 0$ and $b(y) = \mu\,\ln(y) + \beta$. Then we have $c_2 = 0$ and
$$z = \lambda\,\ln(x) + \mu\,\ln(y) + \nu, \quad \lambda\,\mu \neq 0.$$

Case $r_2 = 0,\ r_1 \neq 0$. If we solve $r_2 = 0$ with respect to a_3 and substitute this expression in $e_2 = 0$, we obtain:

$$\frac{b_2}{b_1}\, y = \frac{a_2}{a_1}\,(x-1) = \lambda = const.$$

We solve these differential equations with respect to a and b, then we substitute these expressions in $e_1 = 0$. And we obtain a contradiction.

Case $r_3 = 0,\ r_1 \neq 0,\ r_2 \neq 0$. If $r_3 = 0$, then $r_{30} = r_{31} = 0$. Thus, the resultant of r_{30} and r_{31} with respect to a_3 is equal to zero and we have:

$$(x-1)^6\,(xa_2 + a_1)^4\,x^{10}{a_1}^4\left(2\,x^2a_2 + 2\,a_1x - 2\,xa_2 - a_1\right)^8 = 0.$$

From the equation $\left(2\,x^2a_2 + 2\,a_1x - 2\,xa_2 - a_1\right) = 0$ we get

$$a_2 = -1/2\,\frac{a_1\,(2\,x-1)}{x\,(x-1)},\ \ a_3 = 1/4\,\frac{a_1\left(8\,x^2 - 8\,x + 3\right)}{x^2\,(x-1)^2}.$$

After substitution of these expressions into $e_2 = 0$ we obtain $2\,xya_1b_2 - 2\,y{b_1}^2 + a_1b_1 = 0$. Then we have $a_1 = 2\,\dfrac{y{b_1}^2}{2\,xyb_2 + b_1}$.

Let us substitute expressions for a_1, a_2, a_3 into $e_1 = 0$, then we extract the term without x and equate it to zero. We obtain ${b_1}^2\,(y-1) = 0$. The contradiction. We do not have such solutions. Thus, we have

Proposition 6. *The solutions of the Horn system of complexity one for Example 6 have the form:*

$$z = \lambda\,\ln(x) + \mu\,\ln(y) + \nu, \quad \lambda\,\mu \neq 0.$$

This example is from [1] (n. 8.1.9., p. 304). In this book there is the basis of 4-dimensional space of solutions. It is possible to get our result from this description. Also we can note that in this case the solutions of complexity one is the linear subspace of codimension one in the general solutions space of this Horn system.

Example 7. Let

$$P = (x + 2y + p), \quad Q = (x + y - q),$$
$$R = (x + 2y + p)(x + y + p + 1), \quad S = (x + y - q)(y - s).$$

For $z = c(a(x) + b(y))$ we have:

$$\begin{aligned} G_x\,z &= x^2a_1c_1 + 2\,xyb_1c_1 + pxc_0 + xc_1a_1 + yc_1b_1 - qc_0 = 0,\\ G_y\,z &= x^2y{a_1}^2c_2 + xy^2a_1b_1c_2 + pxya_1c_1 + x^2ya_2c_1 + xya_1b_1c_2 + y^2{b_1}^2c_2 +\\ &+pxa_1c_1 + 3\,pyb_1c_1 + qyb_1c_1 + sxa_1c_1 + syb_1c_1 + 2\,xya_1c_1 +\\ &+y^2b_2c_1 + p^2c_0 - qsc_0 + 3\,yc_1b_1 + pc_0 = 0. \end{aligned}$$

Let $f^{(n)}$ be $(f_1, \dots, f_n)$. After the elimination of c we obtain

$$\begin{aligned} e_1(a^{(2)}, b^{(2)}) &= a_2b_1px^3 - 2\,a_1b_2px^2y - a_1b_1px^2 + a_2b_1px^2 - a_1b_2pxy - a_2b_1qx^2 +\\ &+2\,a_1b_2qxy - b_1a_1px - {b_1}^2py - a_2b_1qx + a_1b_2qy - 2\,{b_1}^2qy = 0,\\ e_2(a^{(2)}, b^{(2)}) &= 0 \quad \text{is the sum of 79 monomials.} \end{aligned}$$

Thus, we obtain two expressions for b_2: the first $b_2 = B_{21}(a^{(2)}, b_1)$ and the second $b_2 = = B_{22}(a^{(2)}, b_1)$. Then we have two new conditions: equality of both expressions and their independence of x. We obtain $e_3(a^{(3)}, b_1) = 0$ (41 monomials) and $e_4(a^{(3)}, b_1) = 0$ (98 monomials). Here e_3 is linear with respect to b_1, and e_4 is quadratic with respect to b_1. If we express b_1 from $e_3 = 0$, then we get $b_1 = B_1(a^{(3)})$. The condition $(B_1)'_x = 0$ has the form $(p\,x - q)\,e_5(a^{(4)}) = 0$ (e_5 is the sum of 55 monomials).

Our calculation is the tree of cases.

Case 1: $(px - q) \neq 0,\ e_5 = 0$.

If we substitute $b_1 = B_1(a^{(3)})$ into $e_4(a^{(3)}, b_1) = 0$ we obtain $ee_4(a^{(3)}) = ee_{40}(a^{(3)}) + y\,ee_{41}(a^{(3)}) = 0$. Thus, we have $ee_{40}(a^{(3)}) = ee_{41}(a^{(3)}) = 0$, where ee_{40} consists of 489 monomials and ee_{41} consists of 215 monomials. Substitution of $b_1 = B_1(a^{(3)})$ in $B_{21}(a^{(2)}, b_1)$ yields $BB_{21}(a^{(3)})$. We can write $(B_1)'_y = BB_2$ and we get $e_6 = e_{61}\,e_{62} = 0$, where

$$
\begin{aligned}
e_{61} &= x^2 a_3 + 4\,a_2 x + x a_3 + 2\,a_1 + 2\,a_2,\\
e_{62} &= 2\,p^2x^5a_1a_3 - 2\,p^2x^5{a_2}^2 + 2\,p^2x^4a_1a_2 + 3\,p^2x^4a_1a_3 - 3\,p^2x^4{a_2}^2 - 4\,pqx^4a_1a_3 +\\
&\quad +4\,pqx^4{a_2}^2 + 2\,p^2x^3a_1a_2 + p^2x^3a_1a_3 - p^2x^3{a_2}^2 - 4\,pqx^3a_1a_2 - 6\,pqx^3a_1a_3 +\\
&\quad +6\,pqx^3{a_2}^2 + 2\,q^2x^3a_1a_3 - 2\,q^2x^3{a_2}^2 + {a_1}^2p^2x^2 + p^2x^2a_1a_2 + 2\,{a_1}^2pqx^2 -\\
&\quad -4\,pqx^2a_1a_2 - 2\,pqx^2a_1a_3 + 2\,pqx^2{a_2}^2 + 2\,q^2x^2a_1a_2 + 3\,q^2x^2a_1a_3 - 3\,q^2x^2{a_2}^2 +\\
&\quad +2\,pqx{a_1}^2 - 2\,pqxa_1a_2 + 2\,q^2xa_1a_2 + q^2xa_1a_3 - q^2x{a_2}^2 + pq{a_1}^2 + q^2a_1a_2.
\end{aligned}
$$

Case 1.1: $e_{61} = 0$, then $a(x) = \lambda\,\ln(x+1) + \mu\,(\ln(x) - \ln(x+1)) + \nu$. Substitution of this a in $e_3 = 0$ yields

$$
\begin{aligned}
&\left(2\,x^4p\lambda + 4\,x^3p\lambda + x^2p\lambda - \lambda\,x^2q + x^2p\mu + 2\,\mu\,x^2q + 2\,\mu\,xq + \mu\,q\right) \times\\
&\qquad\qquad \times (pyb_1 + 2\,qyb_1 + \lambda\,q + p\mu) = 0.
\end{aligned}
$$

The set of coefficients of the first factor has the form:

$$\{\mu\,q, 2\,\lambda\,p, 4\,\lambda\,p, 2\,\mu\,q, \lambda\,p - \lambda\,q + p\mu + 2\,\mu\,q\}\,.$$

All of them can not vanish. Hence the second factor is zero.

Case 1.1.1: $p + 2q \neq 0$. From $(pyb_1 + 2\,qyb_1 + \lambda\,q + p\mu) = 0$ we get

$$b(y) = -\left(\frac{\lambda\,q + \mu p}{p + 2\,q}\right)\ln(y) + \beta.$$

Substitution of this a and b in $e_2 = 0$ yields $ee_2 = 0$, where ee_2 is the polynomial of degree 2 in (x, y), the coefficients of which depend on (p, q, s, λ, μ). One of them equals $\lambda\,(p + q + 1)\,(\lambda - 2\,\mu)$.

Case 1.1.1.1.: $(\lambda - 2\,\mu) = 0$. The analysis of coefficients of ee_2 shows us that $ee_2 = 0$ is impossible in this case.

Case 1.1.1.2.: $p + q + 1 = 0$.

Case 1.1.1.2.1.: $\lambda = \mu$. We have $ee_2 = 0$. Then we have $p = -1,\ q = 0$. The solution has the form $z = \nu\,(y/x)$.

Case 1.1.1.2.2.: $\lambda = -\mu\,p \neq 0$. We have $p = s = \lambda = 0,\ q = -1,\ \mu \neq 0$. The solution has the form $z = \nu(y/x^2)$.

Case 1.1.2: $p = 2q \neq 0$. From $e_4 = 0$ we obtain $b_1 = B_1(a^{(2)})$. After substitution of this expression in B_{21} we get BB_{21}. From $(B_1)'_y = BB_{21}$ we get $g_0(a^{(2)}) + y\, g_1(a^{(2)}) + y^2\, g_2(a^{(2)}) = 0$ and hence $g_0(a^{(2)}) = g_1(a^{(2)}) = g_2(a^{(2)}) = 0$. The resultant of $g_1(a^{(2)})$ and $g_2(a^{(2)})$ with respect to a_2 is some polynomial in x of degree 11. The value of this polynomial for $x = 0$ is $(q-1)^6$ and hence $q = 1$. Then we have

$$\forall s \qquad r = -2\left(2\, sx^2 + sx - 2\, x^2\right)^2 \left(-2\, x^2 - 2\, x\right)^3 x \neq 0.$$

A contradiction. Solutions are absent.

Case 1.2: $e_{62} = 0$. Let us express a_3 from this equation and substitute the result in $e_2 = 0$. We obtain $ee_2 = ee_{20}(a^{(3)}) + y\, ee_{21}(a^{(3)}) = 0$, then

$$g_1 = \left(-psx - qsx + p^2 + qp + xp + p\right) g^3 = 0,$$
$$g_2 = \left(xp^2 + pqx + x^2 p - qp + 2\, xp - q^2 - q\right) g^3 = 0,$$

where $g = \left(2\, px^2 a_2 + 4\, pxa_1 + pxa_2 - 2\, qxa_2 + pa_1 - 2\, qa_1 - qa_2\right)$.

Case 1.2.1.: $g \neq 0$. In this case all coefficients of both factors must vanish. The set of these coefficients is:

$$\{p\,(p+q+1)\,, -ps - qs + p,\, p,\, p\,(p+q+2)\,, -q\,(p+q+1)\}\,.$$

We see that $p = 0$, then $s = 0$ $(q \neq 0)$ and $q = -1$. We have the solution $z = \nu(y/x^2)$ (this solution coincides with the solution of the case 1.1.1.2.2.).

Case 1.2.2.: $g = 0$. Let us express a_2 from this equation and substitute the result in $e_{62} = 0$. We obtain

$$2\, p^2 x^3 + 4\, pqx^3 + 6\, pqx^2 + 6\, pqx + pq - q^2 = 0.$$

The set of coefficients of this polynomial is:

$$\{q\,(p-q)\,, 6\, pq, 2\, p\,(p+2\, q)\}\,.$$

The vanishing of all of them is impossible $(px - q \neq 0)$.

Case 2: $p = q = 0$. The equation $G_x(z) = 0$ takes the form:

$$G_x\, z = x^2 a_1 + 2\, xyb_1 + xa_1 + yb_1 = 0.$$

Hence

$$a_1\, \frac{(x+1)}{(2\, x+1)} = \lambda = b_1\, y, \quad \lambda \neq 0 \text{ is constant.}$$

And then we get

$$a(x) = \lambda(2x - \ln(x)) + \alpha,\; b(y) = \lambda\, \ln(y) + \beta.$$

After elimination of c from

$$G_y\, z = x^2 y {a_1}^2 c_2 + xy^2 a_1 b_1 c_2 + x^2 y a_2 c_1 + xya_1 b_1 c_2 + y^2 {b_1}^2 c_2 +$$
$$+ sxa_1 c_1 + syb_1 c_1 + 2\, xya_1 c_1 + y^2 b_2 c_1 + 3\, yc_1 b_1 = 0$$

for our a and b we get:

$$8\, sx^3 y - 12\, sx^2 y - 8\, x^2 y^2 + 2\, sxy + 4\, xy^2 - 10\, xy - y^2 + 4\, y - 2 = 0.$$

for all (x, y). This is impossible for all s. A contradiction. Thus, we have:

Proposition 7. *The solutions of the Horn system of complexity one for Example 7 exist in two cases only:*

$$(a)\ p=-1,\ q=0, \quad z=\nu\,\frac{y}{x},$$
$$(b)\ p=s=0,\ q=-1, \quad z=\nu\,\frac{y}{x^2}.$$

Example 8. The Lauricella's functions are the subclass of the class of hypergeometric functions [5], [6]. They are the solutions of the Lauricella system (some generalization of the hypergeometric Gauss equation). If the number of the independent variables is two, this system is the system of two equations for the function $z(x, y)$ of the form

$$\begin{aligned} L_x(z) &= x\,(1-x)\,\frac{\partial^2}{\partial x^2} z\,(x,y) + (1-x)\,y\,\frac{\partial^2}{\partial y \partial x} z\,(x,y) + \\ &+ (q-(1+p_1+\rho)\,x)\,\frac{\partial}{\partial x} z\,(x,y) - p_1 y \frac{\partial}{\partial y} z\,(x,y) - p_1\rho\, z\,(x,y) = 0, \\ L_y(z) &= y\,(1-y)\,\frac{\partial^2}{\partial y^2} z\,(x,y) + (1-y)\,x\,\frac{\partial^2}{\partial y \partial x} z\,(x,y) + \\ &+ (q-(1+p_2+\rho)\,y)\,\frac{\partial}{\partial y} z\,(x,y) - p_2 x \frac{\partial}{\partial x} z\,(x,y) - p_2\rho\, z\,(x,y) = 0. \end{aligned}$$

The parameters (p_1, p_2, ρ) are any complex numbers and $q \in \mathbf{C} \setminus \{0, -1, -2, \dots\}$. Our goal is to describe the solutions of the Lauricella system of complexity one (of kind $z = c(a(x) + b(y))$, where (a, b, c) are not constant). In order to simplify our calculation we will assume that $\rho = 0$. Thus, we have three complex parameters only (p_1, p_2, q). The Lauricella system for $\rho = 0$, $z = c(a(x) + b(y))$ has the form:

$$\begin{aligned} L_x(z) &= -x^2 {a_1}^2 c_2 - y c_2 a_1 b_1 x - x^2 a_2 c_1 + x {a_1}^2 c_2 - x a_1 c_1 p_1 + \\ &\quad + y c_2 a_1 b_1 - p_1 y c_1 b_1 + q a_1 c_1 - x a_1 c_1 + x a_2 c_1 = 0, \\ L_y(z) &= -y c_2 a_1 b_1 x - y^2 {b_1}^2 c_2 + x c_2 a_1 b_1 - p_2 x c_1 a_1 - y^2 b_2 c_1 + \\ &\quad + y {b_1}^2 c_2 - y b_1 c_1 p_2 + q b_1 c_1 - y b_1 c_1 + y b_2 c_1 = 0. \end{aligned}$$

Case 1. $x a_1 + y b_1 = 0$. We have

$$a(x) = -\lambda \ln(x) + \alpha, \quad b(y) = \lambda \ln(y) + \beta, \quad \lambda \neq 0$$

and our equations have the form

$$-\frac{\lambda\, c_1\,(q-1)}{x} = \frac{\lambda\, c_1\,(q-1)}{y} = 0.$$

The condition of solvability is $q = 1$ and we obtain $z = c(y/x)$, where $c(t)$ is any analytical function.

Case 2. $x a_1 + y b_1 \neq 0$. We can express c_2/c_1 from both equations. We have

$$c_2/c_1 = LC_{21}(a^{(2)}, b_1) = LC_{22}(a_1, b^{(2)}) = 0.$$

The solvability conditions are: $LC_{21} = LC_{22}$ and $V(LC_{21}) = 0$. This conditions are $e_1(a^{(2)}, b^{(2)}) = 0$ (20 monomials) and $e_2(a^{(3)}, b^{(2)}) = 0$ (39 monomials).

Case 2.1. $-x^2a_2 + qa_1 - xa_1 + xa_2 = 0$. Thus,

$$a_1 = \frac{(x-1)^{q-1}}{x^q}, \quad b_1\, y = -\left(\frac{x-1}{x}\right)^{q-1}.$$

Hence $q = 1$, $a_1 = 1/x$ and $b_1 = -1/y$, and further $xa_1 + yb_1 = 0$. In this case it is impossible.

Case 2.2. $-x^2a_2 + qa_1 - xa_1 + xa_2 \neq 0$. The equations $e_1 = 0$ and $e_2 = 0$ are linear with respect to b_2. We have two expressions for b_2: $b_2 = B_{21}(a^{(2)}, b_1)$ and $b_2 = B_{22}(a^{(3)}, b_1)$. We get two conditions: $B_{21} = B_{22}$ yields $e_3(a^{(3)}, b_1) = 0$ (79 monomials), $(B_{21})'_x = 0$ yields $e_4(a^{(3)}, b_1) = 0$ (36 monomials). e_3 is cubic with respect to b_1, e_4 is quadratic with respect to b_1. We can divide e_3 by e_2 with the remainder (as polynomials with respect to b_1). We obtain that the reminder is zero. Thus, we have:

$$-yb_1 - \frac{\left(x^2a_2 + qa_1 + xa_1 - xa_2 - a_1\right) a_1}{xa_2 + a_1 - a_2} = 0.$$

Hence

$$yb_1 = \lambda = -\frac{\left(x^2a_2 + qa_1 + xa_1 - xa_2 - a_1\right) a_1}{xa_2 + a_1 - a_2}.$$

And further we get

$$b(y) = \lambda \ln(y) + \beta, \quad a_2 = -\frac{a_1\,(qa_1 + xa_1 + \lambda - a_1)}{x^2a_1 + \lambda\, x - xa_1 - \lambda}, \; \lambda \neq 0.$$

After substitution of these expressions in $e_1 = e_2 = e_3 = e_4 = 0$ we get $ee_1(a_1) = ee_2(a_1) = ee_3(a_1) = ee_4(a_1) = 0$. The resultant of ee_1 and ee_2 with respect to a_1 is a polynomial with respect to (x, y). The coefficient of x^6y in this polynomial is $(q-1)^4$, hence $q = 1$. For such q we have $xa_1 + yb_1 = 0$. In this case it's impossible. Thus, we have the following proposition.

Proposition 8. *The solutions of the Lauricella system of complexity one for $\rho = 0$ exist in the case $q = 1$ only. These solutions have the form:*

$$z = c\left(\frac{y}{x}\right), \quad \text{where } c(t) \text{ is any nonconstant analytical function.}$$

It is possible that the additional computational efforts would allow us to free ourselves from the constraint $\rho = 0$.

Conclusion

Both (2) and (1) are equalities to zero of differential polynomials, which are the elements of the differential ring $\mathcal{R}$, the ring of differential polynomials with complex coefficients and generators $(x, y, z, \partial_x, \partial_y)$ (and with obvious relations) [7], to which the field of fractions $\mathcal{F}$ corresponds. The ring $\mathcal{R}$ is a classical object of differential algebra. We can look at the common zeros of a system of differential-polynomial equations, i.e., at the solutions of these equations, from two different points of view. From a quite abstract algebraical point of view they are the elements of the differential-algebraic closure of the field $\mathcal{F}$. From the analytical point of view they are analytic functions, which gives solutions to the system of differential equations. In $\mathcal{R}$ there is the subring $\mathbf{C}[x, y]$, which is the commutative ring of the polynomials in (x, y). The set of the common zeros of a system of polynomials is an affine algebraic subvariety of two-dimensional space. This is the area of responsibility of algebraic geometry. If we move from $\mathbf{C}[x, y]$ to $\mathcal{R}$, then the object arise, which is quite analogous to an algebraic variety: the set of the common zeros of a differential-polynomial system, which is a *differential-algebraic manifold* (DA-manifold). The term is not stable, there are variants, e.g., *diffiety* [8]. From this point of view the discussed above examples are examples of DA-manifolds, which are defined by three differential polynomials (two of them are the Horn system, and the third is the defining equation of the first class). The Horn systems from this point of view are not very interesting, the corresponding DA-manifold is a linear space. By adding the defining equation of the first class, we provide opportunities for a larger diversity. Studying an algebraic variety, one usually pays attention to a series of natural characteristics, namely: irreducible components, stratification of the points on the variety with respect to the dimension of the tangent space and so on. In the study of DA-manifolds these characteristics are also of interest. Nevertheless, there is a certain specifics.

For example, the dimension of the linear space of the solutions of system (2) can be either finite or infinite. In the case when it is infinite the question arise:

Question 9: (a) Under which condition the dimension of the intersection is finite? (b) If the dimension is finite, how to estimate it? (c) How to estimate the number of irreducible components?

DA-manifold defined by the equation $d_1(z) = 0$ is a cone, and DA-manifold defined by a Horn system is a linear space. However, when we speak about conic sections, we mean that the cutting plane does not necessary go through the vertex of the cone, as it is in our examples. We can easily avoid this limitation. Let $z_0(x, y)$ be an analytic function of complexity one, which is a solution of the Horn system, i.e., $G_x(z_0) = G_y(z_0) = 0$. Then we can consider the affine subspace, which consists of the functions of the form $\{z = z_0 + \delta z\}$, where δz is a solution of the Horn system, i.e., $G_x(\delta z) = G_y(\delta z) = 0$, and construct its intersection with the cone Cl^1, which is certainly nonempty (there z_0 lies).

Some of the discussed examples of Horn systems are systems with parameters. This feature can be easily interpreted with the help of differential algebra. In the definition of the differential ring $\mathcal{R}$ we should include these parameters in the field of constants.

Next, note that all our considerations can be adapted to functions of larger number of variables. The functions of complexity one in n variables are analytic functions of the form $z(x_1, \ldots, x_n) = c(a_1(x_1) + \cdots + a_n(x_n))$, where $(a_1, \ldots, a_n, c)$ are functions of one variable. The class of such functions, as in the case of two variables, is defined by a set of differential polynomials.

The consideration of examples with parameters allows us to note that in all discussed situations for the existence of solutions of complexity one there are necessarily restrictions on the parameters. I.e., solutions exist only for a proper algebraic subset of the space of parameters.

Question 10: Do there exist holonomic Horn systems with parameters, such that there are solutions of complexity one for all values of parameters?

Let a Horn system with parameters be given. And let solutions of complexity not greater than a fixed n of this system exist only under some nontrivial analytic conditions (for all natural n). Then it is easy to show that all solutions for generic values of parameters (outside solutions of some enumerable system of analytic equations) have infinite complexity. On the other hand, if we assume that for a Horn system with parameters all solutions are of finite complexity, then there exists a number N, such that *all* solutions for *all* values of parameters have complexity not greater than N.

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О гипергеометрических функциях двух переменных сложности один

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Аннотация. Для серии примеров систем Горна и системы Лауричеллы для функций двух переменных дано описание решений, имеющих аналитическую сложность один. Ставится ряд вопросов.

Ключевые слова: аналитическая сложность, гипергеометрические функции, система Горна, система Лауричеллы, дифференциальное кольцо.