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Chebyshev Polynomials with Zeros outside the Open Arc Segment

Natalia N. Rybakova*
Siberian Federal University
Krasnoyarsk, Russian Federation

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Abstract. The problem of unitary polynomials of degree n with real coefficients least deviating from zero on an arbitrary fixed arc of a circle with a zero set outside an open segment of the same arc is considered. The description of the extremal polynomials of the solution of this problem is given and their norm depending on the degree of the polynomial and the arc length is obtained.

Keywords: Chebyshev polynomials, polynomials least deviating from zero, zero set, polynomials with real coefficients.

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Let K be some compact complex plane, $\mathfrak{A}_n$ be a class of unitary polynomials of degree n. For any polynomial $P_n(z)$ from $\mathfrak{A}_n$ norm on a compact K, we define the current: $\|P_n\|_K = \max_{z\in K} |P_n(z)|$, with the symbol $Z(P_n)$ denote the set of all its zeros.

The problem of finding the polynomials least deviating from zero by K with a zero set on some fixed subset of $D \subset \mathbb{C}$ is posed as follows: find the number $E_n(K, D) = \inf\{\|P_n\|_K \mid P_n \in \mathfrak{A}_n,\ Z(P_n) \subset D\}$ and a polynomial P_n^* such that $\|P_n^*\|_K = E_n(K, D)$, $Z(P_n^*) \subset D$, which is called the extremal or Chebyshev polynomial. Finding such polynomials and other similar problems have been considered by many mathematicians (see, for example, [1–7]).

Chebyshev polynomials considered on the arc of a circle, without restriction on the location of zeros in the complex plane, were studied N. I. Akhiezer and many others (see, for example, [8]). For a narrower problem, with zeros on the arc of a circle, L. S. Maergoiz et al. [9] determined the extremal polynomials and their norm.

This article is devoted to finding the norm of Chebyshev polynomials with real coefficients and a zero set outside an open arc segment.

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1. Two problems of finding extreme polynomials

Consider the arc $\Gamma_\alpha = \{z = e^{i\varphi} \in \mathbb{C} : |\varphi| \leqslant \alpha\}$, where $0 < \alpha < \pi$, interval $G_\alpha = \{z = \cos\alpha + iy : |y| < \sin\alpha\}$, U_α — open arc segment Γ_α.

We define two classes of polynomials: unitary polynomials with real coefficients with a zero set outside the open segment of the arc are

$$\beta_n = \{P_n \in \mathfrak{A}_n : Z(P_n) \subset \mathbb{C} \setminus U_\alpha\}$$

*nrybakova@sfu-kras.ru

and unitary polynomials with real coefficients with zero set on the boundary of the arc segment are

$$\widetilde{\beta_n} = \{P_n \in \mathfrak{A}_n : Z(P_n) \subset \Gamma_\alpha \cup G_\alpha\}$$

with norm

$$\|P_n\| = \max_{z\in\Gamma_\alpha} |P_n(z)| = \max_{|\varphi|\leqslant\alpha} |P_n(e^{i\varphi})|.$$

Task A: Find the number $E_{n,\alpha} = \inf_{P_n\in\beta_n} \|P_n\| = \min_{P_n\in\beta_n} \|P_n\|$ and the polynomial P_n^* with the norm $\|P_n^*\| = E_{n,\alpha}$, which is called the extremal or Chebyshev polynomial.

Task B: Find the number $E_{n,\alpha} = \inf_{P_n\in\widetilde{\beta_n}} \|P_n\| = \min_{P_n\in\widetilde{\beta_n}} \|P_n\|$ and the polynomial P_n^* with the norm $\|P_n^*\| = E_{n,\alpha}$.

The main result of the work:

Theorem 1. *An extreme polynomial vacation abroad that can be seen*

$$T_n(z) = S_n^{(r)} := \prod_{k=1}^{r}(z^2 - 2a_k^{(2r)}z + 1) \text{ for even } n = 2r,$$
$$T_n(z) = (z-1)S_n^{(r)} \text{ for odd } n = 2r+1,$$

where $a_k^{(n)} = 1 - 2\cos^2\dfrac{\pi(2k-1)}{2n}\sin^2(\alpha/2)$ $k = 1,\ldots,r,$ $\|T_n\| = 2\sin^n(\alpha/2)$.

Remark. For $n = 2$ and $\cos\alpha \geqslant 0$, the solution of the problem is not unique: in addition to the polynomial T_2, there is another extreme polynomial $F_2(z) = z^2 - 2z\cos\alpha + \cos\alpha$.

The polynomials from Theorem 1 and their norm were first written out in the article [9] in connection with solving the problem of finding extreme polynomials on the arc of a unit circle with a zero set coinciding with this arc.

The proof of the theorem relies on a number of lemmas.

Lemma 1. *Tasks A and B have the same solution.*

Proof. If the polynomial $P_n(z)$ belongs to the class β_n, but does not belong to the class $\widetilde{\beta_n}$, then there is at least one root of it that does not belong to $\Gamma_\alpha \cup G_\alpha$.

Let $b_0 \notin \Gamma_\alpha \cup G_\alpha$ be such a root of the polynomial $P_n(z)$. Consider the polynomial $\widetilde{P_n}(z)$ such that $Z(\widetilde{P_n}) = Z(P_n)/b_0 \cup \widehat{b}_0$ and $\|\widetilde{P_n}\| < \|P_n\|$. There are three possible cases:

1. $|b_0| = |r\cdot e^{i\gamma}| > 1$ и $Re(b_0) \geqslant \cos\alpha$; $\widehat{b}_0 = e^{i\gamma} \in \Gamma_\alpha$.
2. $Re(b_0) < \cos\alpha$ и $Im(b_0) \leqslant \sin\alpha$; $\widehat{b_0} = \cos\alpha + iIm(b_0) \in G_\alpha$.
3. $Re(b_0) < \cos\alpha$ и $Im(b_0) > \sin\alpha$; $\widehat{b_0} = \cos\alpha + i\sin\alpha \in \Gamma_\alpha$. □

2. The case of an even number of roots on G_α

Consider the following result.

Lemma 2. *For polynomials*

$$L_2(z) = (z-\cos\alpha - iy)(z-\cos\alpha + iy),$$

where $0 \leqslant y < \sin\alpha$, *and*

$$P_2(z) = (z-e^{i\psi})(z-e^{-i\psi}),$$

where ψ *is determined from the equality* $\cos\psi = (\sin^2\alpha - y^2)/2 + \cos\alpha$, *the inequality* $|L_2(z)| \geqslant |P_2(z)|$ *is valid for all* $z \in \Gamma_\alpha$, *equality is achieved only at points* 1 *and* $e^{\pm i\alpha}$.

Proof. Denote $|L_2(e^{i\varphi})|^2 := q(\cos\varphi)$, $|P_2(e^{i\varphi})|^2 := p(\cos\varphi)$. Note that $p(\cos\alpha) = q(\cos\alpha)$ and $p(1) = q(1)$, by this $|L_2(z)| = |P_2(z)|$ for $z \in \{1; e^{\pi\alpha}\}$.

The inequality $p(t) - q(t) < 0$ holds for all t from the interval $(\cos\alpha; 1)$, because $\cos\alpha$ and 1 are the roots of a square trinomial

$$p(t) - q(t) = 4(\sin^2\alpha - y^2)(t^2 - t(\cos\alpha + 1) + \cos\alpha),$$

and the multiplier $\sin^2\alpha - y^2 > 0$ according to the lemma, all this indicates that the control is unmanageable $|L_2(z)| > |P_2(z)|$ for everything $z \in \Gamma_\alpha \setminus \{1; e^{\pi\alpha}\}$. □

Let's consider a few lemmas-consequences.

Lemma 3. *For the norm of polynomials of the form*

$$L_2(z) = (z - \cos\alpha - iy)(z - \cos\alpha + iy),$$

where $0 \leqslant y < \sin\alpha$, *the inequality is valid* $|L_2(z)| \geqslant 2\sin^2\alpha/2$, *and the equality holds only for polynomials* $F_2(z) = z^2 - 2z\cos\alpha + \cos\alpha$ *and only in the case of* $0 \leqslant \alpha \leqslant \pi/2$.

Proof. The norm of the polynomial P_2 described in Lemma 2 is equal to $2\sin^2\alpha/2$, if and only if it coincides with the polynomial T_2 (see Theorem 1), and in other cases is greater than this value. We define the polynomial L_2 of Lemma 2 when $P_2 \equiv T_2$. Equating $\cos\psi$ and $a_k^{(2)}$, described in Lemma 2 and Theorem 1, respectively, we get $\cos\psi = \dfrac{1 + \cos\alpha}{2}$, hence $y^2 = \cos\alpha - \cos^2\alpha \geqslant 0$, y^2 is non-negative only for $0 \leqslant \alpha \leqslant \dfrac{\pi}{2}$. The desired polynomial is — $L_2(z) = z^2 - 2z\cos\alpha + \cos\alpha$. It is not difficult to check that its norm is $2\sin^2\alpha/2$. □

Lemma 4. *For any polynomial* $P_n \in \widetilde{\beta_n}$, *where* $n > 2$, *having an even number of roots on* G_α *(taking into account multiplicity) and at least two roots, the inequality* $\|P_n\| > 2\sin^n\alpha/2$ *is valid.*

Proof. If we consider any polynomial $P_n \in \widetilde{\beta_n}$, where $n \geqslant 2$, having an even number of roots on G_α (taking into account multiplicity), then the inequality $\|P_n\| \geqslant 2\sin^n\alpha/2$ is valid for it, because if we replace each pair of complex-conjugate roots from G_α of the polynomial $P_n \in \widetilde{\beta_n}$ with a pair of roots from Γ_α described in Lemma 2, we get the polynomial $\widetilde{P_n}(z)$.

Because if $Z(\widetilde{P_n}) \in \Gamma_\alpha$, then $\|\widetilde{P_n}\| \geqslant 2\sin^n\alpha/2$ (see [9]). By lemma 2 $|\widetilde{P_n}(z)| \leqslant |P_n(z)|$ for all $z \in \Gamma_\alpha$. Consequently, $\|P_n\| \geqslant \|\widetilde{P_n}\| \geqslant 2\sin^n\alpha/2$.

If $\widetilde{P_n} \equiv T_n$, what exists means ξ such that $\xi \neq 1$, $\xi \neq e^{\pm i\alpha}$ and $|T_n(\xi)| = 2\sin^n\alpha/2$ (the paper [10] describes the properties of such polynomials, in particular, such as: there is $n + 1$ points on Γ_α such that at these points the module of the polynomial T_n is equal to its norm). This is possible because $n > 2$. Therefore, the inequality

$$\|P_n\| \geqslant |P_n(\xi)| > |\widetilde{P_n}(\xi)| = |T_n(\xi)| = 2\sin^n\alpha/2$$

is valid.

If the polynomials $\widetilde{P_n}(z)$ and $T_n(z)$ are not identically equal, then the inequality holds for the norm of the polynomial

$$\|P_n\| \geqslant \|\widetilde{P_n}\| > \|T_n\| = 2\sin^n\alpha/2.$$

□

3. Properties of Chebyshev polynomials on a segment

To solve our problem, we need several well-known facts about polynomials with real coefficients, including weight functions, on the segment $[-1;1]$ of the real axis.

Let's write down a special case of the theorem [11, Chapter II, page 66].

Lemma 5. *Let $s(x)$ be a weight function and a family of polynomials $Q_k(x) = \prod\limits_{j=1}^{k}(x - r_j)$, then for an extremal polynomial $Q_k^*(x)$, such that*

$$\|s(x) \cdot Q_k^*(x)\|_{[-1;1]} = \min_{Q_k(x)} \|s(x) \cdot Q_k(x)\|_{[-1;1]},$$

the statement is true: there are $k+1$ points $\lambda_j, j = 1, \ldots, k+1$, such that

$$|s(\lambda_j) \cdot Q_k^*(\lambda_j)| = \|s(x) \cdot Q_k^*(x)\|_{[-1;1]}, j = 1, \ldots, k+1.$$

Properties of Chebyshev polynomials $t_n(\chi) = \dfrac{1}{2^{n-1}}\cos(n \arccos \chi) = \prod\limits_{j=1}^{n}(\chi - \gamma_j)$ on the segment $[-1;1]$ can be viewed in [12], in particular, $\|t_n\|_{[-1,1]} = \dfrac{1}{2^{n-1}}$, $\sum\limits_{j=1}^{n}\gamma_j = 0$ if η_j, $j = 1, \ldots, n-1$ are roots of the derivative of the Chebyshev polynomial, then $\sum\limits_{j=1}^{n-1}\eta_j = 0$.

Let $-1 = \eta_0$, $1 = \eta_n$, the Chebyshev polynomial deviates most from zero on the segment $[-1;1]$ at the points η_j, $j = 0, \ldots, n$, and if n is odd, then $t_n(1) = t_n(\eta_{n-2}) = \ldots = t_n(\eta_1) = \dfrac{1}{2^{n-1}}$ and $t_n(-1) = t_n(\eta_2) = \ldots = t_n(\eta_{n-1}) = -\dfrac{1}{2^{n-1}}$; if n is even, then $t_n(\eta_1) = t_n(\eta_3) = \ldots = t_n(\eta_{n-1}) = -\dfrac{1}{2^{n-1}}$ and $t_n(-1) = t_n(\eta_2) = \ldots = t_n(1) = \dfrac{1}{2^{n-1}}$.

Lemma 6. *If the norm of the polynomial is $f(\chi)$ on the segment $[-1;1]$ is not greater than the norm of the polynomial $t_n(\chi)$, then there are at least n points p_j (taking into account multiplicity) such that*

$$-1 \leqslant p_1 \leqslant \eta_1 \leqslant p_2 \leqslant \eta_2 \leqslant \ldots \leqslant \eta_{n-1} \leqslant p_n \leqslant 1,$$

$t_n(p_j) - f(p_j) = 0$, $j = 1, \ldots, n$, at the same time , the inequality $-1 \leqslant \sum\limits_{j=1}^{n} p_j \leqslant 1$ is fulfilled.

Lemma 7. *The unitary polynomial least deviating from zero on the segment $[a,b]$ has a norm equal to $\left(\dfrac{b-a}{2}\right)^n \cdot \dfrac{1}{2^{n-1}}$.*

4. The case of an odd number of roots on G_α

In the case of an odd number of roots on G_α, to solve problem B it is enough to consider polynomials with no multiple roots (see lemma 2 and the proof of property 4 of [10] is easily transferred to this case too)

$$P_n(z) = (z - \cos\alpha)\prod_{j=1}^{k}(z - e^{i\varphi_j})(z - e^{-i\varphi_j}), \tag{1}$$

where $|\varphi_j| \leqslant \alpha, j = 1, \dots, k$ для $n = 2k+1$, $k \in \mathbb{N}$ and

$$P_n(z) = (z - \cos\alpha)(z-1)\prod_{j=1}^{k}(z - e^{i\varphi_j})(z - e^{-i\varphi_j}), \tag{2}$$

for $n = 2k+2$, $k \in \mathbb{N}$. It is easy to check that the polynomial (2) is not extremal at $k = 0$.

Lemma 8. *If* $\cos\alpha = 0$, $n \geqslant 2$, *then for all polynomials of the form*

$$P_n(z) = (z - \cos\alpha)\prod_{j=1}^{n-1}(z - e^{i\varphi_j}), \tag{3}$$

where $|\varphi_j| \leqslant \alpha$, $j = 1, \dots, n-1$, *in particular, for (1) and (2), the inequality* $\|P_n\| > 2\sin^n\alpha/2$.

Proof. For the norm of the polynomial (3) the inequality is true $\|P_n(z)\| \geqslant \min_{z\in\Gamma_\alpha}|z - \cos\alpha| \cdot \|T_{n-1}(z)\|$, where $T_{n-1}(z)$ is an extremal polynomial (see Theorem 1) of degree $n-1$.

$\|P_n(z)\| \geqslant \min\limits_{z\in\Gamma_\alpha}|z - \cos\alpha| \cdot 2 \cdot \sin^{n-1}(\alpha/2) > 2\sin^n\alpha/2.$ □

Find the moduli of the polynomials (1) and (2) if $\alpha \neq \pi/2$, for $z = e^{i\varphi} \in \Gamma_\alpha$:

$$|P_n(e^{i\varphi})| = \sqrt{1 + \cos^2\alpha - 2\cos\alpha\cos\varphi} \cdot \prod_{j=1}^{k}|2(\cos\varphi_j - \cos\varphi)|,$$

$$|P_n(e^{i\varphi})| = \sqrt{1 + \cos^2\alpha - 2\cos\alpha\cos\varphi} \cdot \sqrt{2 - 2\cos\varphi} \cdot \prod_{j=1}^{k}|2(\cos\varphi_j - cos\varphi)|.$$

Applying the transformation $\chi(\varphi) = sign(\cos\alpha)\Big(2\left(\frac{\sin\varphi/2}{\sin\alpha/2}\right)^2 - 1\Big)$, we obtain $|P_n(e^{i\varphi})| = = 2(\sin\alpha/2)^n \cdot 2^{(n-2)/2}|l_n(\chi)|$, where for polynomials (1) and (2) respectively

$$l_n(\chi) = \sqrt{a\chi + 1} \cdot \prod_{j=1}^{k}(\chi - \nu_j) \tag{4}$$

and

$$l_n(\chi) = \sqrt{a\chi + 1} \cdot \sqrt{(1 + sign(\cos\alpha)\chi)} \cdot \prod_{j=1}^{k}(\chi - \nu_j), \tag{5}$$

$a = |\cos\alpha|$, $\nu_j = \chi(\varphi_j)$, $j = 1, \dots, r$. If the norm of polynomials is defined on Γ_α, then the norm of functions $l_n(\chi)$ is on the segment $[-1; 1]$ of the real axis.

Lemma 9. *If all roots of the derivative of the function* $l_n(\chi)$, *(for (4) and (5))* $n \geqslant 2$ *belong to the segment* $[-1; 1]$, *then the inequality*

$$\|l_n\| = \max_{|\chi|\leqslant 1}|l_n(\chi)| > \frac{1}{2^{\frac{n-2}{2}}} \tag{6}$$

is true for the norm of the function.

Proof. If all roots of the derivative of the function $l_n(\chi)$, $n \geqslant 2$ and $a \neq 0$ belong to the segment $[-1;1]$, then the norm of the function $\|l_n\| = \max_{|\chi|\leqslant 1}|l_n(\chi)|$ on the segment $[-1;1]$ coincides with the norm of the function on the segment $[-1;1]$ with the norm of the function on the segment $\left[-\frac{1}{a};1\right]$. If $r_n(\chi) = l_n^2(\chi) - \frac{1}{2^{n-1}}$, then by Lemma 7 $\|r_n\|\|_{[-1/a;1]} \geqslant a\cdot\left(\frac{1+1/a}{2}\right)^n\cdot\frac{1}{2^{n-1}}$. Notice that $0 < a < 1$. The function $m(a) = a\cdot\left(\frac{1+1/a}{2}\right)^n$ in the interval $(0;1)$ is decreasing for $n \geqslant 2$. Thus $m(1) = 1$, hence, $\|r_n\|_{[-1/a;1]} > \frac{1}{2^{n-1}}$. The lemma is proved. □

Lemma 10. *For the norm of the function (4) for which* $\sum\limits_{j=1}^{k}\nu_j < \frac{1}{2}$, *the inequality (6) holds.*

Proof. Consider the family of functions $l_n(\chi) = \sqrt{a\chi+1}\cdot\prod\limits_{j=1}^{k}(\chi-\nu_j), |\nu_j| \leqslant 1, j = 1,\ldots,k$, for which the inequality $\||l_n\| \leqslant \frac{1}{2^{\frac{n-2}{2}}2}$ is satisfied. If $r_n(\chi) = l_n^2(\chi) - \frac{1}{2^{n-1}}$, then by assumption $\|r_n(\chi)\|_{[-1;1]} = \frac{1}{2^{n-1}}$.

Consider the Chebyshev polynomial on the segment $[-1;1]$

$$t_n(\chi) = \frac{1}{2^{n-1}}\cos(n\arccos\chi) = \prod_{j=1}^{n}(\chi-\gamma_j).$$

The roots of the derivative of the Chebyshev polynomial are $\eta_j (j = 1,\ldots,n-1)$. Since, by assumption, $\||r_n\| = \|t_n\|$, then Lemma 6 is valid.

Thus, the identity

$$t_n(\chi) - r_n(\chi) = (1-a)\prod_{j=1}^{n}(\chi-p_j)$$

is valid for p_j, $j = 1,\ldots,n$ are described in Lemma 6:

$$\prod_{j=1}^{n}(\chi-\gamma_j) - (a\chi+1)\prod_{j=1}^{k}(\chi-\nu_j)^2 + \frac{1}{2^{n-1}} = (1-a)\prod_{j=1}^{n}(\chi-p_j).$$

Let us equate the coefficients of the polynomials at χ^{n-1} of the right and left parts of the previous equality:

$$-\sum_{j=1}^{n}\gamma_j - \left(-2a\sum_{j=1}^{k}\nu_j + 1\right) = -(1-a)\sum_{j=1}^{n}p_j.$$

Using the inequality of Lemma 7 and properties of Chebyshev polynomials, we obtain

$$-1+a \leqslant -2a\sum_{j=1}^{k}\nu_j + 1 \leqslant 1-a$$

или

$$\frac{1}{2} \leqslant \sum_{j=1}^{k}\nu_j \leqslant \frac{2-a}{2a}. \qquad (7)$$

Hence, if for the coefficients $\nu_j, j = 1, \dots, k$ of the function $l_n(\chi) = \sqrt{a\chi + 1} \cdot \prod_{j=1}^{k} (\chi - \nu_j)$ the inequality (7) is not satisfied, then for the norm of this function the inequality (6) holds for the norm of this function. □

The following two lemmas are proved in the same way.

Lemma 11. *For the norm of the function (5) at* $\cos\alpha > 0$*, for which* $\sum_{j=1}^{k} \nu_j \notin \left[1; \frac{1}{a}\right]$*, the inequality (6) holds.*

Lemma 12. *For the norm of the function (5) at* $\cos\alpha < 0$ *for which* $\sum_{j=1}^{k} \nu_j \notin \left[0; \frac{1-a}{a}\right]$*, the inequality (6) holds.*

Lemma 13. *For a function* $l_n(\chi)$ *(4) for which there exists* $k+1$ *points* $\lambda_j \in [-1; 1]$ *such that* $|l_n(\lambda_j)| = \|l_n\|, j = 1, \dots, k+1$ *and having the largest deviation from zero on the segment* $[-1; 1]$ *at the roots of the derivative located between the points* $\nu_j, j = 1, \dots, k$ *and at the ends of the segment, the inequality (6) holds.*

Proof. Let the conditions of the lemma be satisfied. This is possible if and only if, at the ends of the segment and in the roots of the derivative belonging to the segment $[-1; 1]$, the modulus of the function coincides with its norm. Find some positive number L such that $\|l_n(\chi)\|_{[-1,1]} = \sqrt{\frac{L}{2^{2k-1}}}$. Consider the transformation $r_n(\chi) = l_n^2(\chi) - \frac{L}{2^{n-1}}$, where $n = 2k+1$. For the function $b_n(\chi) = t_n(\chi) - r_n(\chi)/L$ Lemma 7 holds, $b_n(p_j) = 0$, where p_j, $j = 1, \dots, n$, are described in Lemma 6. Note that $p_n = 1$, the remaining arrangement of zeros of the polynomial b_n is possible if $\sum_{j=1}^{k} \nu_j < \sum_{j=1}^{k} \eta_{2j}$. Indeed, let $\nu_k \geqslant \eta_{2k}$, $(2k = n-1)$, then the function b_n on the segment $[\eta_{2k}; 1]$ has at least two roots (if the root is at η_{2k}, it has multiplicity two). Hence, the total zeros of the function b_n are more than n, then $b_n \equiv 0$. This is impossible because of the difference between the polynomials $t_n(\chi)$ and $r_n(\chi)/L$. Similarly consider the inequality $\nu_{k-1} \geqslant \eta_{2k-2}$, etc. The equality $\sum_{j=1}^{k} \eta_{2j} = 1/2$ follows easily from the properties of the Chebyshev polynomial. The inequality (6) follows from Lemma 10. □

The proof of the next two lemmas differs slightly from the proof of the previous lemma.

Lemma 14. *For a function* $l_n(\chi) = \sqrt{a\chi+1} \cdot \sqrt{(1-\chi)} \cdot \prod_{j=1}^{k}(\chi - \nu_j)$*,* $a \neq 0$*, for which there exist* $k+1$ *points* $\lambda_j \in [-1; 1]$ *such that* $|l_n(\lambda_j)| = \|l_n\|, j = 1, \dots, k+1$ *and having the largest deviation from zero on the interval* $[-1; 1]$ *in the roots of the derivative located on the interval* $[-1; 1]$ *and at the point -1, the inequality (6) holds.*

Lemma 15. *For a function* $l_n(\chi) = \sqrt{a\chi+1} \cdot \sqrt{(\chi+1)} \cdot \prod_{j=1}^{k}(\chi - \nu_j)$*,* $a \neq 0$*, for which there exist* $k+1$ *points* $\lambda_j \in [-1; 1]$ *such that* $|l_n(\lambda_j)| = \|l_n\|, j = 1, \dots, k+1$ *and having the largest deviation from zero on the interval* $[-1; 1]$ *in the roots of the derivative located on the interval* $[-1; 1]$ *and at the point 1, the inequality (6) holds.*

Lemma 16. *Let* $n \geqslant 2$*. The norms of the polynomials (1) and (2) are greater than* $2\sin^n \alpha/2$*.*

Proof. Let $a \neq 0$. If P_n^* is an extremal polynomial among (1) and (2), then a function $l_n^*(\chi)$ of the form (4) or (5), respectively, associated with the extremal polynomial. The function $l_n^*(\chi)$ falls under the conditions of Lemma 5: $\sqrt{a\chi+1}$ and $\sqrt{a\chi+1} \cdot \sqrt{(1 + sign(\cos\alpha)\chi)}$ are weight functions. Hence, there are $k+1$ points $\lambda_j \in [-1;1]$ such that $|l_n^*(\lambda_j)| = \|l_n^*\|, j = 1, \ldots, k+1$ (see Lemma 5). The derivative of the function $l_n^*(\chi)$ has k roots. There are two possible cases: either all roots of the derivative function $l_n^*(\chi)$ lie on the segment $[-1;1]$, or one root does not belong to the segment $[-1;1]$, and in both the first and second cases $k-1$ roots are located between the points $\widetilde{\nu}_j, j = 1, \ldots, k$. Moreover, in the second case, the function $l_n^*(\chi)$ has the largest deviation from zero on the segment $[-1;1]$ at the roots of the derivative located between the points $\widetilde{\nu}_j, j = 1, \ldots, k$ and at one or two ends of the segment, depending on the structure of this function.

All possible variants have been considered above. If $a = 0$, then Lemma 8 is valid. □

Theorem 1 is proved.

The main result was reported at the conference [13].

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Многочлены Чебышева с нулями вне открытого сегмента дуги

Наталья Н. Рыбакова
Сибирский федеральный университет
Красноярск, Российская Федерация

Аннотация. Рассмотрена задача об унитарных многочленах степени n с вещественными коэффициентами, наименее уклоняющихся от нуля на произвольной фиксированной дуге окружности, с нулевым множеством вне открытого сегмента той же самой дуги. Дано описание экстремальных многочленов решения этой задачи и получена их норма, зависящая от степени полинома и длины дуги.

Ключевые слова: многочлены Чебышева, многочлены, наименее уклоняющиеся от нуля, нулевое множество, многочлены с вещественными коэффициентами.