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A Note on Two General Reduction Formulas for the Srivastava-Daoust Double Hypergeometric Functions

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Abstract. The aim of this note is to provide two new and general reduction formulas for the Srivastava-Doust double hypergeometric functions. A few interesting special cases have also been given.

Keywords: hypergeometric functions, Humbert double hypergeometric functions, Appell functions, Srivastava & Daoust double hypergeometric function, beta integral method.

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1. Introduction and results required

We start with the definition of generalized hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters as [11]

$$ {}_pF_q\left[\begin{array}{ccc} a_1, & \dots, & a_p \\ b_1, & \dots, & b_q \end{array}; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \tag{1.1} $$

where $(a)_n$ denotes the well-known Pochhammer symbol (or the raised or the shifted factorial, since $(1)_n = n!$) defined for any complex number a by

$$ (a)_n = \begin{cases} a(a+1)\dots(a+n-1), & n \in \mathbb{N} \\ 1, & n = 0 \end{cases} = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{1.2} $$

where Γ is the well-known Gamma function. For convergence conditions (including absolute convergence) and properties of this function, we refer standard texts [2, 11].

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It is not out of place to mention here that the vast popularity and immense usefulness of the hypergeometric function ${}_2F_1$ (which is a special case of ${}_pF_q$ for $p = 2$ and $q = 1$) and the generalized hypergeometric function ${}_pF_q$ in one variable have inspired and stimulated a large number of researchers to study hypergeometric functions of two and more variables. In this contexts, serious and very significant study of the functions of two variables initiated by Appell [1] who introduced the so-called functions F_1, F_2, F_3 and F_4 named in the literature, the Appell functions which are the natural generalizations of ${}_2F_1$ and ${}_pF_q$. Also their confluent forms were studied by Humbert [16, 17]. A complete list of these functions can be seen in the standard text [7] and also in [3, 4].

Later on, the Appell functions F_1, F_2, F_3 and F_4 and their confluent forms were further generalized by Kampé de Fériet [1], who introduced a more general function in two variables. The notation for this function was subsequently abbreviated by Burchnall and Chaundy [5, 6]. However, in our present investigations, we recall here the definition of a more general function in two variables (than the one defined by Kampé de Fériet) in a slightly modified notation which is due to Srivastava and Panda [18, Equ.(26), p.123] defined as follows

$$F^{H:A;B}_{G:C;D}\left[\begin{matrix}(h_H): & (a_A); & (b_B); \\ (g_G): & (c_C); & (d_D);\end{matrix}\, x,\, y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((h_H))_{m+n}((a_A))_m((b_B))_n}{((g_G))_{m+n}((c_C))_m((d_D))_n}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.3}$$

where (h_H) denotes the sequence of parameters $(h_1, h_2, \cdots, h_H)$ and for $n \in \mathbb{N}_0$, define the Pochhammer symbol

$$((h_H))_n := (h_1)_n \cdots (h_H)_n,$$

where, when $n = 0$, the product is understood to reduce to unity. The symbol (h) is a convenient contraction for the sequence of parameters $h_1, h_2, \cdots, h_H$ and the Pochhammer symbol $(h)_n$ is the same as defined in (1.2). For details about the convergence for this function, we refer to [16].

The Srivastava-Daoust generalized Kampé de Fériet hypergeometric function of two variables initially introduced in [13, 14] will be defined and represented in the following manner:

$$S^{A:B;B'}_{C:D;D'}\left[\begin{matrix}x\\y\end{matrix}\right] \equiv S^{A:B;B'}_{C:D;D'}\left[\begin{matrix}[(a):\theta,\phi]: & [(b):\psi]; & [(b'):\psi']; \\ [(c):\delta,\epsilon]: & [(d):\eta]; & [(d'):\eta'];\end{matrix}\, x,\, y\right] = \tag{1.4}$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{A}\Gamma(a_j + m\theta_j + n\phi_j)\prod_{j=1}^{B}\Gamma(b_j + m\psi_j)\prod_{j=1}^{B'}\Gamma(b'_j + n\psi'_j)}{\prod_{j=1}^{C}\Gamma(c_j + m\delta_j + n\epsilon_j)\prod_{j=1}^{D}\Gamma(d_j + m\eta_j)\prod_{j=1}^{D'}\Gamma(d'_j + m\eta'_j)}\frac{x^m}{m!}\frac{y^n}{n!},$$

where, for convergence

$$\begin{cases} 1 + \sum\limits_{j=1}^{C}\delta_j + \sum\limits_{j=1}^{D}\eta_j - \sum\limits_{j=1}^{A}\theta_j - \sum\limits_{j=1}^{B}\psi_j > 0, \\ 1 + \sum\limits_{j=1}^{C}\epsilon_j + \sum\limits_{j=1}^{D'}\eta'_j - \sum\limits_{j=1}^{A}\phi_j - \sum\limits_{j=1}^{B'}\psi'_j > 0. \end{cases}$$

A detailed account of the above function can be found in the research paper [15] and in the text [17].

We also give below the definitions of the Humbert function Φ_3 (confluent forms of the Appell functions) in the following manner ([3–6, 8, 10, 17]).

$$\Phi_3(b;\ c;\ w,\ z) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(b)_k}{(c)_{k+m}}\frac{w^k\, z^m}{k!\, m!} \tag{1.5}$$

which converge absolutely at any $w,\ z \in \mathbb{C}$.

By employing the following generalizations of the Kummer's summation theorem due to Rakha and Rathie [12] viz.

$$ {}_2F_1\left[\begin{array}{c} a, \quad b \\ a-b+n+1 \end{array}; -1\right] = \frac{2^{n-2b}\Gamma(b-n)\Gamma(a-b+n+1)}{\Gamma(b)\Gamma(a-2b+n+1)} \times \\ \times \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma\left(\frac{a+k+n+1}{2}-b\right)}{\Gamma\left(\frac{a+k-n+1}{2}\right)} \tag{1.6} $$

and

$$ {}_2F_1\left[\begin{array}{c} a, \quad b \\ a-b-n+1 \end{array}; -1\right] = \frac{2^{-2b-n}\Gamma(a-b-n+1)}{\Gamma(a-2b-n+1)} \times \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma\left(\frac{a+k-n+1}{2}-b\right)}{\Gamma\left(\frac{a+k-n+1}{2}\right)}. \tag{1.7} $$

Very recently, Brychkov *et al.* [4] established the following general reduction formulas for the Humbert functions Φ_3 viz.

(a) For $n = 0, 1, 2, \ldots$, the following result holds true:

$$ \Phi_3\left(b; \frac{b+n}{2}+1; z, -z^2\right) = \frac{(-2)^n}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \times \\ \times \left\{ \frac{\Gamma\left(\frac{b+n+2k+2}{4}\right)}{\Gamma\left(\frac{b-3n+2k+2}{4}\right)} {}_3F_4\left[\begin{array}{c} 1, \frac{b+n+2k+2}{4}, \frac{2-2k-b+3n}{4} \\ \frac{n+1}{2}, \frac{n}{2}+1, \frac{b+n+2}{4}, \frac{b+n}{4}+1 \end{array}; -z^2\right] + \right. \\ \left. + \frac{8z}{(n+1)(b+n+2)} \frac{\Gamma\left(\frac{b+n+2k}{4}+1\right)}{\Gamma\left(\frac{b-3n+2k}{4}\right)} {}_3F_4\left[\begin{array}{c} 1, \frac{b+n+2k}{4}+1, 1-\frac{b-3n+2k}{4} \\ \frac{n}{2}+1, \frac{n+3}{2}, \frac{b+n}{4}+1, \frac{b+n+6}{4} \end{array}; -z^2\right] \right\}. \tag{1.8} $$

(b) For $n = 0, 1, 2, \ldots$, the following result holds true:

$$ \Phi_3\left(b; \frac{b-n}{2}+1; z, -z^2\right) = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \left\{ {}_2F_3\left[\begin{array}{c} \frac{b-n+2k+2}{4}, \frac{2-2k-b+n}{4} \\ \frac{1}{2}, \frac{b-n+2}{4}, \frac{b-n}{4}+1 \end{array}; -z^2\right] + \right. \\ \left. + \frac{2(b-n+2k)\,z}{b-n+2} {}_2F_3\left[\begin{array}{c} \frac{b-n+2k}{4}+1, 1-\frac{b-n+2k}{4} \\ \frac{3}{2}, \frac{b-n}{4}+1, \frac{b-n+6}{4} \end{array}; -z^2\right] \right\}. \tag{1.9} $$

In addition to this, the Beta function $B(a, b)$ is defined by the first integral and known to be evaluated as the second one as follows:

$$ B(a,b) = \begin{cases} \int_0^1 z^{a-1}(1-z)^{b-1}dz, & (\mathrm{Re}(a) > 0,\ \mathrm{Re}(b) > 0) \\ \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, & (a,\, b \in \mathbb{C} \setminus \mathbb{Z}_0^-) \end{cases}. \tag{1.10} $$

Recently, Krattenthaler and Rao [9] made a systematic use of the so-called *Beta integral method*, a method of deriving new hypergeometric identities from *old* ones by mainly using the beta integral in (1.10) based on the Mathematica Package HYP, to illustrate several interesting identities for the hypergeometric function and Kamp*é* de F*é*riet functions.

Thus, the aim of this note is to provide certain new and general transformation formulas for the Srivastava–Daoust double hypergeometric functions with the help of the general reduction formulas (1.8) to (1.9). The results are established by employing the beta integral method. A few results obtained earlier by Wei *et al.* [19] follow special cases of our main findings.

The results presented in this note are simple, interesting, easily established and may be useful (potentially).

2. Transformation formulas

The new and general transformation formulas for the Kampé de Fériet functions to be established in this note are given in the following theorems.

Theorem 2.1.

$$\begin{aligned}
&S^{1:1;0}_{2:0;0}\left[\begin{matrix}[e:1,2] & : & [b:1]; & - \\ [d:1,2], \quad [\frac{b+n}{2}+1:1,1]: & & -; & - \end{matrix}\middle|\begin{matrix} u \\ -u^2\end{matrix}\right] = \frac{(-2)^n}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\times \\
&\times\left\{{}_5F_6\left[\begin{matrix}1,\ \frac{b+n+2k+2}{4},\ \frac{2-2k-b+3n}{4},\ \frac{e}{2},\ \frac{e+1}{2} \\ \frac{n+1}{2},\ \frac{n}{2}+1,\ \frac{b+n+2}{4},\ \frac{b+n}{4}+1,\ \frac{d}{2},\ \frac{d+1}{2}\end{matrix};-u^2\right]+\right. \\
&+\frac{8ue\,\Gamma\left(\frac{b+n+2k}{4}+1\right)}{d(n+1)(b+n+2)\,\Gamma\left(\frac{b-3n+2k}{4}\right)}\times \\
&\left.\times\,{}_5F_6\left[\begin{matrix}1,\ \frac{b+n+2k}{4}+1,\ 1-\frac{b-3n+2k}{4},\ \frac{e+1}{2},\ \frac{e}{2}+1 \\ \frac{n}{2}+1,\ \frac{n+3}{2},\ \frac{b+n}{4}+1,\ \frac{b+n+6}{4},\ \frac{d+1}{2},\ \frac{d}{2}+1\end{matrix};-u^2\right]\right\}.
\end{aligned}\tag{2.1}$$

Theorem 2.2.

$$\begin{aligned}
&S^{1;1;0}_{2:0;0}\left[\begin{matrix}[e:1,2] & : & [b:1]; & - \\ [d:1,2], \quad [\frac{b-n}{2}+1:1,1]: & & -; & - \end{matrix}\middle|\begin{matrix} u \\ -u^2\end{matrix}\right] = 2^{-n}\sum_{k=0}^{n}\binom{n}{k}\times \\
&\times\left\{{}_4F_5\left[\begin{matrix}\frac{b-n+2k+2}{4},\ \frac{2-2k-b+n}{4},\ \frac{e}{2},\ \frac{e+1}{2} \\ \frac{1}{2},\ \frac{b-n+2}{4},\ \frac{b-n}{4}+1,\ \frac{d}{2},\ \frac{d+1}{2}\end{matrix};-u^2\right]+\right. \\
&\left.+\frac{2ue(b-n+2k)}{d(b-n+2)}\,{}_4F_5\left[\begin{matrix}\frac{b-n+2k}{4}+1,1-\frac{b-n+2k}{4},\ \frac{e+1}{2},\ \frac{e}{2}+1 \\ \frac{3}{2},\ \frac{b-n}{4}+1,\ \frac{b-n+6}{4},\ \frac{d+1}{2},\ \frac{d}{2}+1\end{matrix};-u^2\right]\right\}.
\end{aligned}\tag{2.2}$$

Proof. For the derivation of the result (2.1) asserted in the theorem 2.1, we proceed as follows. First of all replacing z by zu in the reduction formula (1.8), we have

$$\begin{aligned}
&\Phi_3\left(b;\frac{b+n}{2}+1;uz,-(uz)^2\right) = \frac{(-2)^n}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\times \\
&\times\left\{\frac{\Gamma\left(\frac{b+n+2k+2}{4}\right)}{\Gamma\left(\frac{b-3n+2k+2}{4}\right)}\ {}_3F_4\left[\begin{matrix}1,\ \frac{b+n+2k+2}{4},\ \frac{2-2k-b+3n}{4} \\ \frac{n+1}{2},\ \frac{n}{2}+1,\ \frac{b+n+2}{4},\ \frac{b+n}{4}+1\end{matrix};-(uz)^2\right]+\right. \\
&\left.+\frac{8uz}{(n+1)(b+n+2)}\ \frac{\Gamma\left(\frac{b+n+2k}{4}+1\right)}{\Gamma\left(\frac{b-3n+2k}{4}\right)}\times{}_3F_4\left[\begin{matrix}1,\ \frac{b+n+2k}{4}+1,1-\frac{b-3n+2k}{4} \\ \frac{n}{2}+1,\ \frac{n+3}{2},\ \frac{b+n}{4}+1,\ \frac{b+n+6}{4}\end{matrix};-(uz)^2\right]\right\}.
\end{aligned}\tag{2.3}$$

Now, multiply the left-hand side of (2.3) by $z^{e-1}(1-z)^{d-e-1}$ and integrating with respect to z over the interval $[0,1]$, we have

$$\text{L.H.S} = \int_0^1 z^{e-1}(1-z)^{d-e-1}\Phi_3(b;\frac{b+n}{2}+1;zu,-(zu)^2)dz.$$

Expressing the Humbert function Φ_3 as a double series with the aid of its definition (1.5), change the order of integration and summation (which is easily seen to be justified), we have

$$\text{L.H.S} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(b)_k(-1)^m u^{k+2m}}{\left(\frac{b+n}{2}+1\right)_{k+m}k!\,m!}\int_0^1 z^{e+k+2m-1}(1-z)^{d-e-1}dz.$$

Evaluating the beta integral and using the result

$$\frac{\Gamma(a+k+2m)}{\Gamma(a)}=(a)_{k+2m}$$

and after some simplification, summing up the series by interpreting with the help of (1.4), we have

$$\text{L.H.S}=\frac{\Gamma(e)\Gamma(d-e)}{\Gamma(d)}S^{1:1;0}_{2:0;0}\left[\begin{matrix}[e:1,2] & : & [b:1]; & - \\ [d:1,2], \quad [\frac{b+n}{2}+1:1,1]: & & -; & -\end{matrix}\left|\begin{matrix}u\\ -u^2\end{matrix}\right.\right]. \tag{2.4}$$

Similarly, multiply the right-hand side of (2.3) by $z^{e-1}(1-z)^{d-e-1}$, and integrating with respect to z over the interval $[0,1]$ and proceeding as above, we have

$$\begin{aligned}\text{R.H.S}=&\frac{\Gamma(e)\Gamma(d-e)(-2)^n}{\Gamma(d)n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\times\\ &\times\Bigg\{{}_5F_6\left[\begin{matrix}1,\ \frac{b+n+2k+2}{4},\ \frac{2-2k-b+3n}{4},\ \frac{e}{2},\ \frac{e+1}{2}\\ \frac{n+1}{2},\ \frac{n}{2}+1,\ \frac{b+n+2}{4},\ \frac{b+n}{4}+1,\ \frac{d}{2},\ \frac{d+1}{2}\end{matrix};-u^2\right]+\\ &+\frac{8ue\,\Gamma\left(\frac{b+n+2k}{4}+1\right)}{d(n+1)(b+n+2)\Gamma\left(\frac{b-3n+2k}{4}\right)}\times\\ &\times\ {}_5F_6\left[\begin{matrix}1,\ \frac{b+n+2k}{4}+1, 1-\frac{b-3n+2k}{4},\ \frac{e+1}{2},\ \frac{e}{2}+1\\ \frac{n}{2}+1,\ \frac{n+3}{2},\ \frac{b+n}{4}+1,\ \frac{b+n+6}{4},\ \frac{d+1}{2},\ \frac{d}{2}+1\end{matrix};-u^2\right]\Bigg\}.\end{aligned} \tag{2.5}$$

Finally, equating the equations (2.4) and (2.5), we arrive at the result (2.1) asserted in the Theorem 2.1. this completes the proof of the result (2.1).

In exactly the same manner, the result (2.2) asseerted in the Theorem 2.2 can be established.

□

3. Corollaries

In this section, we shall mention some of very interesting results of our main findings.

Corollary 3.1. *In Theorem 2.1 or 2.2, if we take $n=0$, we get the following result:*

$$\begin{aligned}&S^{1:1;0}_{2:0;0}\left[\begin{matrix}[e:1,2] & : & [b:1]; & - \\ [d:1,2], \quad [\frac{b}{2}+1:1,1]: & & -; & -\end{matrix}\left|\begin{matrix}u\\ -u^2\end{matrix}\right.\right]=\\ &={}_3F_4\left[\begin{matrix}\frac{2-b}{4},\ \frac{e}{2},\ \frac{e+1}{2};\\ \frac{1}{2},\ \frac{b}{4}+1,\ \frac{d}{2},\ \frac{d+1}{2}\end{matrix};-u^2\right]+\frac{2ueb}{d(b+2)}{}_3F_4\left[\begin{matrix}1-\frac{b}{4},\ \frac{e+1}{2},\ \frac{e}{2}+1;\\ \frac{3}{2},\ \frac{b+6}{4},\ \frac{d+1}{2},\ \frac{d}{2}+1\end{matrix};-u^2\right],\end{aligned} \tag{3.1}$$

which is also believed to be a new result.

Corollary 3.2. *In Theorem 2.1, if we take $n=1$, we get the following result:*

$$\begin{aligned}&S^{1:1;0}_{2:0;0}\left[\begin{matrix}[e:1,2] & : & [b:1]; & - \\ [d:1,2], \quad [\frac{b+3}{2}:1,1]: & & -; & -\end{matrix}\left|\begin{matrix}u\\ -u^2\end{matrix}\right.\right]=\\ &=-2\Bigg\{{}_3F_4\left[\begin{matrix}\frac{5-b}{4},\ \frac{e}{2},\ \frac{e+1}{2};\\ \frac{3}{2},\ \frac{b+5}{4},\ \frac{d}{2},\ \frac{d+1}{2}\end{matrix};-u^2\right]+\frac{4ue\Gamma(\frac{b+5}{4})}{d(b+3)\Gamma(\frac{b-3}{4})}{}_4F_5\left[\begin{matrix}1,\ \frac{7-b}{4},\ \frac{e+1}{2},\ \frac{e}{2}+1;\\ \frac{3}{2},\ 2,\ \frac{b+7}{4},\ \frac{d+1}{2},\ \frac{d}{2}+1\end{matrix};-u^2\right]-\\ &-{}_3F_4\left[\begin{matrix}\frac{3-b}{4},\ \frac{e}{2},\ \frac{e+1}{2};\\ \frac{3}{2},\ \frac{b+3}{4},\ \frac{d}{2},\ \frac{d+1}{2}\end{matrix};-u^2\right]-\frac{4ue\Gamma(\frac{b+7}{4})}{d(b+3)\Gamma(\frac{b-1}{4})}{}_4F_5\left[\begin{matrix}1,\ \frac{5-b}{4},\ \frac{e+1}{2},\ \frac{e}{2}+1;\\ \frac{3}{2},\ 2,\ \frac{b+5}{4},\ \frac{d+1}{2},\ \frac{d}{2}+1\end{matrix};-u^2\right]\Bigg\},\end{aligned} \tag{3.2}$$

which is also believed to be a new result.

Corollary 3.3. *In Theorem 2.2, if we take $n = 1$, we get the following result:*

$$S^{1:1;0}_{2:0;0}\left[\begin{matrix}[e:1,2] & & : & [b:1]; & - \\ [d:1,2], & [\frac{b+1}{2}:1,1]: & & -; & -\end{matrix}\middle| \begin{matrix} u \\ -u^2 \end{matrix}\right] =$$

$$= \frac{1}{2}\left\{ {}_3F_4\left[\begin{matrix}\frac{3-b}{4}, \frac{e}{2}, \frac{e+1}{2}; \\ \frac{1}{2}, \frac{b+3}{4}, \frac{d}{2}, \frac{d+1}{2}\end{matrix}; -u^2\right] + \frac{2ue(b-1)}{d(b+1)}\,{}_3F_4\left[\begin{matrix}\frac{5-b}{4}, \frac{e+1}{2}, \frac{e}{2}+1; \\ \frac{3}{2}, \frac{b+5}{4}, \frac{d+1}{2}, \frac{d}{2}+1\end{matrix}; -u^2\right] + \right.$$

$$\left. + \,{}_3F_4\left[\begin{matrix}\frac{1-b}{4}, \frac{e}{2}, \frac{e+1}{2}; \\ \frac{1}{2}, \frac{b+1}{4}, \frac{d}{2}, \frac{d+1}{2}\end{matrix}; -u^2\right] + \frac{2ue}{d}\,{}_3F_4\left[\begin{matrix}\frac{3-b}{4}, \frac{e+1}{2}, \frac{e}{2}+1; \\ \frac{3}{2}, \frac{b+3}{4}, \frac{d+1}{2}, \frac{d}{2}+1\end{matrix}; -u^2\right]\right\}, \quad (3.3)$$

which is also believed to be a new result.

Similarly, other results can be obtained by giving values to n. However, we prefer to omit the details.

Concluding Remark

In this present note, we have provided certain new transformation formulas in the most general case for any $n \in \mathbb{N}_0$ for the Srivastavas-Doust double hypergeometric functions with the help of certain general reduction formulas for Humbert's functions. The results are derived by employing the well-known Beta integral method. Interested reader can develop further new and interesting formulas by employing beta integral method.

Competing interests. The authors declare that they have no competing interests.

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Замечание о двух общих формулах приведения для двойных гипергеометрических функций Шриваставы-Дауста

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Аннотация. Цель этой заметки — предоставить две новые общие формулы приведения для двойных гипергеометрических функций Шриваставы–Дауста. Также приведено несколько интересных частных случаев.

Ключевые слова: гипергеометрические функции, двойные гипергеометрические функции Гумберта, функции Аппеля, двойная гипергеометрическая функция Шриваставы и Дауста, метод бета-интеграла.