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# On the Closedness of Carpets of Additive Subgroups Associated With a Chevalley Group Over a Commutative Ring

**Yakov N. Nuzhin***
Siberian Federal University
Krasnoyarsk, Russian Federation

*Dedicated to the blessed memory of my teacher Vladimir M. Levchuk*

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**Abstract.** Let $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ be a carpet of additive subgroups of type $\Phi$ over an arbitrary commutative ring $K$. A sufficient condition for the carpet $\mathfrak{A}$ to be closed is established. As a corollary, we obtain a positive answer to question 19.63 from the Kourovka notebook and a confirmation of one conjecture by V. M. Levchuk, provided that the type of $\Phi$ is different from $C_l$, $l \geqslant 5$ when the characteristic of the ring $K$ is 0 or $2m$ for some natural number $m > 1$. Also, a partial answer to question 19.62 has been obtained.

**Keywords:** Lie algebra and ring, Chevalley group, commutative ring, carpet of additive subgroups, carpet subgroup.

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## 1. Introduction and preliminaries

Let $\Phi$ be a reduced indecomposable root system, $E(\Phi, K)$ be an elementary Chevalley group of type $\Phi$ over the commutative ring $K$ with unit 1. The groups $E(\Phi, K)$ is generated by the root subgroups $x_r(K) = \{x_r(t) \mid t \in K\}$, $r \in \Phi$. The subgroups $x_r(K)$ are Abelian, and

$$x_r(t)x_r(u) = x_r(t+u), \tag{1}$$

for any $r \in \Phi$ and $t, u \in K$. We call *a carpet of type* $\Phi$ *over* $K$ a collection of additive subgroups $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ of the field $K$ with the condition

$$C_{ij,rs}\mathfrak{A}_r^i\mathfrak{A}_s^j \subseteq \mathfrak{A}_{ir+js}, \quad r, s, ir+js \in \Phi,\ i, j > 0, \tag{2}$$

where $\mathfrak{A}_r^i = \{a^i \mid a \in \mathfrak{A}_r\}$, and constants $C_{ij,rs}$ are equal to $\pm 1$, $\pm 2$ or $\pm 3$. Inclusions (2) come from the Chevalley commutator formula

$$\big[x_s(u), x_r(t)\big] = \prod_{i,j>0} x_{ir+js}\big(C_{ij,rs}(-t)^i u^j\big), \quad r, s, ir+js \in \Phi. \tag{3}$$

*nuzhin2008@rambler.ru

Every carpet $\mathfrak{A}$ defines a *carpet subgroup* $E(\Phi, \mathfrak{A})$ generated by the subgroups $x_r(\mathfrak{A}_r),\ r \in \Phi$. A carpet $\mathfrak{A}$ is called *closed* if its carpet subgroup $E(\Phi, \mathfrak{A})$ has no new root elements, i.e., if

$$E(\Phi, \mathfrak{A}) \cap x_r(K) = x_r(\mathfrak{A}_r).$$

Note that the following fact follows from the condition (2). If $t \in \mathfrak{A}_r$ and $u \in \mathfrak{A}_s$, then each factor from the right-hand side of (3) lies in $E(\Phi, \mathfrak{A})$.

The above definition of a carpet was introduced by V. M. Levchuk, and it was first written down in the next question from the Kourovka notebook [1].

**Question A).** *What are the conditions on the carpet $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ (in terms of $\mathfrak{A}_r$) over a commutative ring $K$ necessary and sufficient for $\mathfrak{A}$ to be closed?* [1, question 7.28, 1980]

The difficulty of this question lies in the fact that the answer to it, in fact, should be the following statement: the defining relations of the carpet subgroup $E(\Phi, \mathfrak{A})$ over an arbitrary commutative ring $K$ are exhausted by the relations (1), (3) (which give rise to the carpet conditions (2)), and relations in the subgroup $\langle x_r(\mathfrak{A}_r),\ x_{-r}(\mathfrak{A}_{-r}\rangle,\ r \in \Phi$. In the case when the carpet subgroup $E(\Phi, \mathfrak{A})$ coincides with the entire Chevalley group $E(\Phi, K)$ over the field $K$, this is the well-known result of R. Steinberg on defining relations in Chevalley groups over the field [2, Sec. 6, Theorem 8] In 1983, the author of question A) himself gave the answer to it for a locally finite field $K$ [3] and reformulated it in the case when the main ring $K$ is a field, in the following form.

**Question B).** *Is it true that the carpet $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ of type $\Phi$ over a field $K$ is closed if and only if the subcarpets $\{\mathfrak{A}_r,\ \mathfrak{A}_{-r}\},\ r \in \Phi$, of rank 1 are closed?* [1, question 15.46, 2002]

The carpet $\{\mathfrak{A}_r,\ \mathfrak{A}_{-r}\},\ r \in \Phi$, of rank 1 corresponds to an elementary matrix carpet of degree 2, and its carpet subgroup is isomorphic to the group generated by opposite elementary transvections $t_{12}(u)$, $u \in \mathfrak{A}_r$ and $t_{21}(u)$, $u \in \mathfrak{A}_{-r}$ .

Lemma 14 is noted in [4] without proof, however, after more than 40 years, its proof has not appeared. Therefore, we formulate the assertion of this lemma as a conjecture.

**Conjecture C).** (V. M. Levchuk, 1982) *Inclusions $\mathfrak{A}_r\mathfrak{A}_{-r}\mathfrak{A}_r \subseteq \mathfrak{A}_r$, $r \in \Phi$, are sufficient for the carpet of additive subgroups $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ over a commutative ring $K$ to be closed.*

The next question of the author of this note is a strengthening of conjecture C), and in the case when the characteristic of the main coefficient ring is odd, they are equivalent.

**Question D).** *Are the inclusions $\mathfrak{A}_r^2\mathfrak{A}_{-r} \subseteq \mathfrak{A}_r$, $r \in \Phi$ , sufficient for the carpet of additive subgroups $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ over a commutative ring $K$ to be closed?* [1, Question 19.63]

In the articles [5,6] give examples of irreducible (if all additive subgroups are nonzero) carpets of any type $\Phi$ over rings of any even characteristic, for which the inclusions $\mathfrak{A}_r^2\mathfrak{A}_{-r} \subseteq \mathfrak{A}_r$, $r \in \Phi$, are valid, but the inclusions from hypothesis C) are not satisfied. Question D) was first written in 2012 in [7, p. 199] and this is due to the fact that the inclusions $\mathfrak{A}_r^2\mathfrak{A}_{-r} \subseteq \mathfrak{A}_r$, $r \in \Phi$, are necessary and sufficient conditions for the carpet subring $L(\Phi, \mathfrak{A})$ to be invariant under the carpet subgroup $E(\Phi, \mathfrak{A})$ according to the following definitions.

Let $\Pi$ be the fundamental root system for $\Phi$. The structure constants in the Chevalley basis $\{e_r,\ r \in \Phi;\ \ h_s,\ s \in \Pi\}$ of the simple complex Lie algebra $L(\Phi, \mathbb{C})$ are integers , so one can define a Lie ring (algebra) $L(\Phi, K)$ with a Chevalley basis over an arbitrary commutative ring

$K$ (see, for example, [8, p. 62]). By definition, we assume that the subring $L(\Phi,\mathfrak{A})$ is generated (with respect to both operations) by all the sets $\mathfrak{A}_r e_r,\ \ r\in\Phi$. We will call $L(\Phi,\mathfrak{A})$ a *carpet* Lie subring. Note that the basis elements $e_r,\ h_s$ do not have to lie in $L(\Phi,\mathfrak{A})$. A carpet $\mathfrak{A}$ is called *L-closed* if $L(\Phi,\mathfrak{A})\cap Ke_r=\mathfrak{A}_r e_r,\ \ r\in\Phi$. The elementary Chevalley group $E(\Phi,K)$ acts on the Lie ring $L(\Phi,K)$ as an automorphism group. We will say that the subring $R\subseteq L(\Phi,K)$ is *invariant* under $G\subseteq E(\Phi,K)$ if $gr\in R$ for any $g\in G$ and $r\in R$. Any carpet subring $L(\Phi,\mathfrak{A})$ that is invariant under the corresponding carpet subgroup $E(\Phi,\mathfrak{A})$ is $L$-closed [7, p. 199].

For any $r\in\Phi$, the automorphism $x_r(t)$ of the Lie ring $L(\Phi,K)$ acts on the Chevalley basis as follows:

$$\begin{aligned} e_r &\to e_r, \\ e_{-r} &\to e_{-r}+th_r-t^2e_r, \\ h_s &\to h_s-tA_{sr}e_r, \qquad s\in\Pi, \\ e_s &\to \sum_{i=0}^{q} C_{i1,rs}t^i e_{ir+s}, \qquad s\in\Phi\setminus\{\pm r\}, \end{aligned} \tag{4}$$

where $A_{rs}=\dfrac{2(r,s)}{(r,r)}$, $q=q(r,s)$ is the largest non-negative integer such that $s+qr\in\Phi$ and by definition $C_{01,rs}=1$. Therefore, if $s\in\Phi\setminus\{\pm r\}$ and $t\in\mathfrak{A}_r,\ u\in\mathfrak{A}_s$, then under the automorphism $x_r(t)$, the vector $ue_s$, goes into a linear combination $\sum\limits_{i=0}^{q} C_{i1,rs}t^i ue_{ir+s}$, each term of which, by virtue of the carpet condition, lies in the subring $L(\Phi,\mathfrak{A})$.

The main result of the article is

**Theorem 1.** *The inclusions $\mathfrak{A}_r^2\mathfrak{A}_{-r}\subseteq\mathfrak{A}_r$, $r\in\Phi$, are sufficient for closedness of the carpet of additive subgroups $\mathfrak{A}=\{\mathfrak{A}_r\mid r\in\Phi\}$ over an arbitrary commutative ring $K$ provided that the type of $\Phi$ is different from $C_l$, $l\geqslant 5$, when the characteristic of the ring $K$ is $0$ or $2m$ for some natural number $m>1$.*

## 2. Proof of Theorem 1

Let $x_p(t)\in E(\Phi,\mathfrak{A})$ for fixed $p\in\Phi$ and some $t\in K$. Then, for suitable $r_i\in\Phi$ and $t_i\in\mathfrak{A}_{r_i}$ we have the equality

$$x_p(t)=x_{r_1}(t_1)x_{r_2}(t_2)\dots x_{r_n}(t_n). \tag{5}$$

Our task is to establish the inclusion $t\in\mathfrak{A}_p$. By virtue of the third equality from (4), for any root $q\in\Phi$ the following equality holds

$$x_p(t)h_q=h_q-A_{qp}te_p. \tag{6}$$

Consider the integer $E(\Phi,\mathfrak{A})$-submodule $M$ of the ring $L(\Phi,K)$ generated by the element $h_q$ and show the inclusion

$$M\leqslant\mathbb{Z}h_q+L(\Phi,\mathfrak{A}). \tag{7}$$

For any $r\in\Phi$ we put $\mathfrak{H}_r=\{u\in K\mid uh_r\in M\}$. By virtue of (4), the the following inclusions hold:

$$\mathfrak{H}_q\leqslant\mathbb{Z}+\sum\mathfrak{A}_q\mathfrak{A}_{-q}, \tag{8}$$

$$\mathfrak{H}_s\leqslant\sum\mathfrak{A}_s\mathfrak{A}_{-s},\quad s\neq q, \tag{9}$$

where by definition $\sum \mathfrak{A}_r\mathfrak{A}_{-r}$ consists of sums of the form $u_1v_1+\cdots+u_kv_k$ for $u_i \in \mathfrak{A}_r$, $v_i \in \mathfrak{A}_{-r}$, $i=1,\dots,k$, for any natural $k$. Indeed, the generation of the module $M$ begins with the equality

$$x_r(\mathfrak{A}_r)h_q = h_q - A_{qr}\mathfrak{A}_r e_r.$$

Thus, there is a base of induction. By virtue of (4), we have the equalities:

$$x_r(\mathfrak{A}_r)\mathfrak{A}_s e_s = \sum_{i=0}^{q} C_{i1,rs}\mathfrak{A}_r^i\mathfrak{A}_s e_{ir+s},\ s \neq -r, \tag{10}$$

$$x_r(\mathfrak{A}_r)\mathfrak{A}_{-r}e_{-r} = \mathfrak{A}_{-r}e_{-r} + \mathfrak{A}_r\mathfrak{A}_{-r}h_r - \mathfrak{A}_r^2\mathfrak{A}_{-r}e_r, \tag{11}$$

$$x_r(\mathfrak{A}_r)\mathfrak{H}_s h_s = \mathfrak{H}_s h_s - A_{sr}\mathfrak{A}_r\mathfrak{H}_s e_r. \tag{12}$$

Further, by the definition of the $M$, these equalities are applied in any order. By Theorem 3.1 in [7] the subring $L(\Phi,\mathfrak{A})$ is invariant under the carpet subgroup $E(\Phi,\mathfrak{A})$ if and only if $\mathfrak{A}_r^2\mathfrak{A}_{-r} \subseteq \mathfrak{A}_r$, $r \in \Phi$. Therefore $Ke_r \cap L(\Phi,\mathfrak{A}) = \mathfrak{A}_r e_r$ for all $r \in \Phi$ [7, p. 199]. This gives us the inclusions (8) and (9), since $\mathfrak{A}_r\mathfrak{A}_{-r}h_r \in L(\Phi,\mathfrak{A})$, $r \in \Phi$, due to (11). Thus, the inclusion (7) also holds.

In what follows, we need one lemma, the assertion of which for systems all of whose roots have the same length is obvious. For a system of type $F_4$ it follows from the fact that any of its root lies in a subsystem of type $A_2$. For root systems of types $B_l$, $C_l$ and $G_2$ it can be verified directly.

**Lemma 1.** *For any root $p \in \Phi$ there exists a root $q \in \Phi$ such that*

$$A_{qp} = \begin{cases} \pm 2 & \text{if } \Phi \text{ of type } C_l,\ l \geqslant 1, \text{ and } p \text{ is a long root,} \\ \pm 1 & \text{otherwise.} \end{cases} \tag{13}$$

So now, due to (5), (6) and (7) we have $A_{qp}t \in \mathfrak{A}_p$. Hence, by Lemma 1, we obtain the desired inclusion $t \in \mathfrak{A}_p$ if $\Phi$ is different from $C_l$, $l \geqslant 1$, in the case of an even characteristic of the ring $K$. The constraint "$\Phi$ is different from $C_l$, $l \geqslant 1$" can be relaxed to "$\Phi$ is different from $C_l$, $l \geqslant 5$" for an ring $K$ of even characteristic due to the following equalities and inclusions for root systems $A_1 = C_1 \subset C_2 = B_2 \subset C_3 \subset C_4 \subset F_4$. Indeed, we embed a carpet $\mathfrak{A}$ of type $C_l$ for $l = 1,2,3,4$ into a carpet of type $F_4$ such that $\mathfrak{A}_r = 0$ for all $r \in F_4 \setminus C_l$. Since Lemma 1 is true for a root system of type $F_4$, the above proof also holds in this case. We only note that in this case the root $q$ in Lemma 1 must be taken from the difference $F_4 \setminus C_l$.

It remains to consider the case when $\Phi$ is of type $C_l$, $l \geqslant 5$, and the characteristic of the ring $K$ is equal to 2. In this case, there exists a homomorphism $\varphi$ of the group $E(C_l,K)$ into the group $E(B_l,K)$ such that

$$\varphi(x_r(u)) = \begin{cases} x_{r/2}(u) & \text{if } r \text{ is a long root,} \\ x_r(u^2) & \text{if } r \text{ is a short root,} \end{cases} \tag{14}$$

for any $u \in K$ (see, for example, [2, Theorem 28] and [9, Sec. 3]). Since the carpet conditions come from the Chevalley commutator formula, $\varphi$ induces a homomorphism of the carpet subgroup $E(C_l,\mathfrak{A})$ onto the carpet subgroup $E(B_l,\mathfrak{A}')$, where the carpet $\mathfrak{A}' = \{\mathfrak{A}'_r \mid r \in B_l\}$ according to (14) is defined as

$$\mathfrak{A}'_r = \begin{cases} \mathfrak{A}_{2r} & \text{if } r \text{ is a short root of root system of type } B_l, \\ \mathfrak{A}_r^2 & \text{if } r \text{ is a long root of root system of type } B_l. \end{cases} \tag{15}$$

The correspondence (15) between additive subgroups $\mathfrak{A}'_r$ and $\mathfrak{A}_s$ is given by the bijection $p_i \leftrightarrow q_i$, $i = 1, 2, \ldots, l$ , between fundamental root systems of type $C_l$ and $B_l$, where $p_1$ is a long root for type $C_l$, and $q_1$ is a short root for type $B_l$, and the sum $p_i + p_{i+1}$ and $q_i + q_{i+1}$ are roots for all $i = 1, 2, \ldots, l-1$. The inclusions ${\mathfrak{A}'}^2_r \mathfrak{A}'_{-r} \subseteq \mathfrak{A}'_r$, $r \in B_l$, remain valid, since from $S^2T \subseteq S$ follows $(S^2)^2T^2 \subseteq S^2$ for any subsets $S$ and $T$ of the ring $K$. For $\Phi$ of type $C_l$, only the answer to the following question was not known above. Does the inclusion $x_p(t) \in E(C_l, \mathfrak{A})$, when $p$ is a long root and the characteristic of the ring $K$ is even, imply the inclusion $t \in \mathfrak{A}_p$? Since $\varphi(x_p(t)) = x_{p/2}(t)$ by definition, then $\mathfrak{A}'_{p/2} = \mathfrak{A}_p$. The inclusion $t \in \mathfrak{A}'_{p/2}$ is fulfilled, since we are dealing with a carpet of type $B_l$. Hence $t \in \mathfrak{A}_p$. Which is what needed to be shown.

The Theorem 1 is proved. □

## 3. Corollaries from Theorem 1

Since the inclusions $\mathfrak{A}_r \mathfrak{A}_{-r} \mathfrak{A}_r \subseteq \mathfrak{A}_r$, $r \in \Phi$, entails the inclusions $\mathfrak{A}_r^2 \mathfrak{A}_{-r} \subseteq \mathfrak{A}_r$, $r \in \Phi$, then Theorem 1 implies

**Corollary 1.** *Conjecture C) is true if the type of $\Phi$ is different from $C_l$, $l \geqslant 5$, when the characteristic of the ring $K$ is $0$ or $2m$ for some natural numbers $m > 1$.*

Given an elementary carpet $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ of type $\Phi$ of rank $l \geqslant 2$, we define a collection of additive subgroups $\mathfrak{B}_p = \sum C_{ij,rs} \mathfrak{A}_r^i \mathfrak{A}_s^j, \quad p \in \Phi$, where the sum is over all natural numbers $i$, $j$ and roots $r, s \in \Phi$ for which $ir + js = p$. The collection $\mathfrak{B} = \{\mathfrak{B}_p \mid p \in \Phi\}$ is a carpet and is called the *derived* of $\mathfrak{A}$ [10]. It is known that for $\Phi = A_l$ the carpet $\mathfrak{B}$ is closed [11, Proposition 1]. The following question was first written in [10, p. 534, 2011].

**Question E).** *Is every derived carpet of type $\Phi$ over a commutative ring closed?* [1, Question 19.62, 2018]

For each root system $\Phi$ we define the number $m = m(\Phi) = \max\limits_{r,s \in \Phi} \dfrac{(r,r)}{(s,s)}$. In fact,

$$m = \begin{cases} 1 & \text{if} \quad \Phi = A_l, D_l, E_l, \\ 2 & \text{if} \quad \Phi = B_l, C_l, F_4, \\ 3 & \text{if} \quad \Phi = G_2. \end{cases}$$

In [10, Theorem 1] it is proved that any derived carpet $\mathfrak{B}$ of type $\Phi$ of rank $l \geqslant 2$ satisfies the inclusions $m! \mathfrak{B}_p \mathfrak{B}_{-p} \mathfrak{B}_p \subseteq \mathfrak{B}_p$, $p \in \Phi$. Combining this result with the assertion of Theorem 1, we obtain

**Corollary 2.** *A derived carpet of type $\Phi$ of rank $l \geqslant 2$ over a commutative ring $K$ of characteristic $p$ is closed if $gcd(p, 2) = 1$ for $\Phi$ of type $B_l$, $C_l$ or $F_4$, and if $gcd(p, 6) = 1$ for $\Phi$ of type $G_2$.*

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# О замкнутости ковров аддитивных подгрупп, ассоциированных с группой Шевалле над коммутативным кольцом

**Яков Н. Нужин**
Сибирский федеральный университет
Красноярск, Российская Федерация

**Аннотация.** Установлено достаточное условие замкнутости ковра аддитивных подгрупп $\mathfrak{A} = \{\mathfrak{A}_r \mid r \in \Phi\}$ типа $\Phi$ над произвольным коммутативным кольцом $K$. В качестве следствий получаем положительный ответ на вопрос 19.63 из Коуровской тетради и подтверждение одной гипотезы В. М. Левчука при условии, что тип $\Phi$ отличен от $C_l$, $l \geqslant 5$, когда характеристика кольца $K$ есть 0 или $2m$ для некоторого натурального числа $m > 1$. Также получен частичный ответ на вопрос 19.62.

**Ключевые слова:** алгебра и кольцо Ли, группа Шевалле, коммутативное кольцо, ковер аддитивных подгрупп, ковровая подгруппа.